Output contingent securities and efficient investment by firms

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Abstract

We study competitive economies in which firms make risky investments and markets allow decision makers to fully insure against aggregate outcome uncertainty—but not necessarily against all primitive states of nature. It is well-known that the ability to contract upon a complete description of states of nature is unnecessary for achieving an efficient allocation of resources across consumers. The same is not immediate for the productive sector because the map between primitive states and aggregate output levels depends on endogenous investment decisions. We show that if each firm computes its value using "competitive beliefs" about how out-of-equilibrium input decisions affect the probability distribution of its output, then competitive markets lead profit-maximizing firms and utility-maximizing consumers to achieve a Pareto optimal allocation of resources.

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1 Introduction

Following the work by Leon Walras in the 19th century, the general equilibrium literature focused on understanding how anonymous markets coordinate the production and consumption of goods in competitive economies. In this setting, firms' productive decisions and agents' consumption choices are taken independently, and market prices are the only instruments available to coordinate different wishes. Hayek (1945) supported the view that competitive prices have the capacity of aggregating the necessary social knowledge to induce efficient self-interested individual behaviors. This idea was rigorously formulated and independently proven by Kenneth J. Arrow, Gerard Debreu and Lionel W. McKenzie during the 1950's. They listed conditions for existence of a competitive equilibrium and proved that, in absence of externalities and other market frictions, competitive markets lead profit-maximizing firms and utility-maximizing agents to achieve a Pareto optimal allocation of resources. The information embedded in market prices should be sufficient to promote an efficient social coordination across decision makers.

Arrow (1953) and Debreu (1959) extended the general equilibrium analysis to economies in which random states of nature affect productivity. They showed that the standard analysis carries over to environments with uncertainty whenever decision makers are able to trade a complete set of contingent claims—each of them promising to deliver goods in the future conditional on the verification of a given state of nature. However, verification of a state of nature is not a simple matter, and most securities traded in modern financial markets are contingent on observed output instead of primitive states of nature.

We follow the recent contributions by Magill and Quinzii (2009, 2010) and analyze financial economies in which asset payoffs depend on firms' endogenous production and allow full insurance against aggregate outcome uncertainty. In a sequence of recent works, Magill and Quinzii (2009, 2010) and Magill, Quinzii and Rochet (2011) showed that this market feature could potentially matter in production economies. Their works illustrate how subtle is the issue of defining an objective function for firms in order to generate Pareto optimality in equilibrium. They claim that maximizing the present value of firms' profit does not lead to an efficient outcome. In that respect, when financial contracts available for trade only depend on firms' outcomes, it seems that market prices have lost their fundamental role of conveying all the requisite information to coordinate consumption choices and firms' investment decisions. Following a normative analysis, they propose an alternative criterion in which firms should maximize their contribution to expected social welfare, taking into account the impact of their investment choices on the utility of all participants of the economy.

We propose to show that profit maximization can still be socially justified as a decision criterion. We prove the Pareto optimality property of the competitive equilibrium when firms maximize the present value of their profits, computed according to what seems to us to be the natural extension of the concept to this framework. Our main result shows that the prices of promises that are contingent on the aggregate output convey the necessary information for firms and consumers making decisions under uncertainty.

A key contribution of our analysis consists in defining the appropriate stochastic discount factor to compute firms' net present values. In the standard Arrow–Debreu approach, all relevant information for production and market transactions are embedded in the underlying state of nature. In the output-contingent framing, however, financial markets do not distinguish across states of nature that lead to the same aggregate output. This distinction is still relevant for the productive sector as endogenous investment decisions affect the relationship between aggregate output and primitive states of nature. We show that this issue can be taken into account through beliefs that link the investment of each firm to a conditional probability measure over this firm's output given the economy's aggregate production. In equilibrium, these beliefs will be correct. When evaluating possible equilibrium deviations, firms will be assumed to believe that their actions affect the probability measure over their own production but not the probability of different aggregate outputs. In our view, this is the natural way to accommodate the competitive price-taking paradigm to this environment.

With this tool in hands, we carry the analysis and show that the market incompleteness caused by the absence of state-contingent claims is not only irrelevant from the exchange perspective but also for the production decisions. Indeed, if firms have "competitive beliefs" about the conditional distribution of their profit given the economy's aggregate output, then maximizing the present value of expected profit using prices for aggregate output leads to Pareto efficiency.

The remaining of this paper is organized as follows. We describe the standard state-of-nature approach in Section 2 and argue in Section 3 that consumers share the aggregate output risk in any Pareto efficient allocation.[1] In Section 4, we recall the reader that competitive equilibria are always Pareto efficient when utility-maximizing agents and profit-maximizing firms are allowed to trade state-contingent claims. Next, we introduce in Section 5 a financial economy in which agents only trade output-contingent securities that provide insurance against aggregate output risks (allowing for market incompleteness with respect to the underlying primitive states of nature). This is where our main contribution is placed. In our view, competitive firms should not take into account the effect of their investment decisions over aggregate variables. We define beliefs about how changes in the investment level of a given firm would affect the probability distribution of its own production given the aggregate output. We show that competitive equilibria are Pareto optimal in this output-contingent environment. To conclude the paper, we use Section 6 to explicitly compare our work with the existing related literature. Appendix A is reserved for a technical proof.

2 The Economic Environment

Consider an economy with two periods $t \in \{0, 1\}$, a single good, a finite set I of consumers and a finite set K of firms. In the initial date ($t = 0$), each firm k makes an investment a^k in a set $A^k \subset \mathbb{R}_+$. Next, at $t = 1$, they are exposed to exogenous shocks ω from a finite set Ω. States in Ω represent primitive causes which likelihood is given by an exogenous probability measure P. This probability is independent of consumers' and firms' actions. Moreover, without any loss of generality, we assume that every $\omega \in \Omega$ is drawn with strictly positive probability.[2]

Technology. Investments and shocks lead each firm to produce an output y^k in a finite set $Y^k \subset \mathbb{R}_+$ at $t = 1$. The production possibilities are represented by a family $f \equiv (f^k)_{k \in K}$ of random production functions, where

$$f^k(\omega, \cdot) : A^k \longmapsto Y^k.$$

Denote by $Y \equiv \prod_{k \in K} Y^k$ the set of output profiles. We can derive a transition

[1] In other words, from the consumers perspective, consumption choices can be represented by random variables that are contingent on the aggregate output, and the commodity space does not need to be the space of vectors contingent on primitive states of nature.

[2] The probability of an event $A \subset \Omega$ is denoted by $P(A)$. For singleton events $\{\omega\}$, we write $P(\omega) \equiv P(\{\omega\})$.

probability $a \mapsto Q(a)$ on Y by posing

$$Q(y, a) \equiv P(\{f(a) = y\}).^3$$

Since we will frequently consider the summation over firms' index, for every profile $y = (y^k)_{k \in K}$, we define

$$\sigma y \equiv \sum_{k \in K} y^k.$$

The random aggregate production is represented by the function $\omega \mapsto \sigma f(\omega, a)$. We let $Z \equiv \sum_{k \in K} Y^k$ denote the set of all possible aggregate outputs and derive the transition probability $a \mapsto \mu(a)$ on Z by posing

$$\mu(z, a) \equiv \sum_{y \in Y(z)} Q(y, a), \tag{2.1}$$

where $Y(z) \equiv \{y \in Y \ : \ \sigma y = z\}$. Expression (2.1) can also be written in terms of primitive states since $\mu(z, a) = P(\{\sigma f(a) = z\})$.

Preferences. Each agent i has initial resources consisting of an amount $e_0^i \geqslant 0$ of income at date $t = 0$ and the ownership share $\delta_k^i \in [0, 1]$ of each firm k, where $\sum_{i \in I} \delta_k^i = 1$. Agents have no initial endowment at $t = 1$, so that all consumption in that period comes from the firms' output.

Agent i's preferences are represented by a utility function U^i which is separable across time and has the expected utility form for future risky consumption. Formally, if $x_0^i \geqslant 0$ denotes agent i's consumption at $t = 0$ and φ_1^i is a probability measure on $\mathbb{R}_+$ that represents random consumption at $t = 1$, then

$$U^i(x_0^i, \varphi_1^i) \equiv u_0^i(x_0^i) + \int_{\mathbb{R}_+} u_1^i(x_1) \varphi_1^i(dx_1),$$

where $u_t^i : \mathbb{R}_+ \to [-\infty, \infty)$ is increasing, differentiable, strictly concave and satisfies the Inada condition.[4]

[3] We use $\{H = h\}$ for the set $\{\omega \in \Omega \ : \ H(\omega) = h\}$, where H is an arbitrary random function and h a point in its image.

[4] That is, $\lim_{x \to 0_+} (u_t^i(x) - u_t^i(0))/x = \infty$.

3 Optimal risk sharing

The tradition in the general equilibrium literature is to represent agent i's consumption possibilities by a pair (x_0^i, x_1^i), where $x_0^i \geqslant 0$ is consumption at $t = 0$ and $x_1^i \equiv (x_1^i(\omega))_{\omega\in\Omega}$ is a state-contingent vector of consumption for $t = 1$. The associated inter-temporal utility is given by

$$U_\Omega^i(x_0^i, x_1^i) \equiv u_0^i(x_0^i) + \sum_{\omega\in\Omega} u_1^i(x_1^i(\omega))P(\omega).$$

An **Ω-allocation** is a family $((x_0, x_1), a)$, where $(x_0, x_1) \equiv (x_0^i, x_1^i)_{i\in I}$ is the consumption allocation and $a \equiv (a^k)_{k\in K}$ is the investment vector. It is said to be **Ω-feasible** if markets clear at $t = 0$ and in each date-1 state of nature, i.e.,

$$\sum_{i\in I} x_0^i + \sum_{k\in K} a^k = \sum_{i\in I} e_0^i,$$

and

$$\forall\omega \in \Omega, \quad \sum_{i\in I} x_1^i(\omega) = \sigma f(\omega, a).$$

An Ω-feasible allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$ is **Pareto optimal** (or **efficient**) if there is no other Ω-feasible allocation $((x_0, x_1), a)$ such that the associated consumption allocation (x_0, x_1) Pareto dominates $(\bar{x}_0, \bar{x}_1)$.[5]

Since Bernoulli functions u_t^i satisfy Inada's condition, every efficient Ω-allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$ displays $\bar{x}_0^i > 0$ and $\bar{x}_1^i(\omega) > 0$, for all ω. It follows then that there exists a stochastic discount factor (or vector of state price deflators) $\bar{m} = (\bar{m}(\omega))_{\omega\in\Omega}$ such that

$$\forall i \in I, \quad \frac{\partial u_1^i(\bar{x}_1^i(\omega))}{\partial u_0^i(\bar{x}_0^i)} = \bar{m}(\omega).$$

The random vector $\bar{m}$ is called the Ω-sdf associated with the efficient Ω-allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$.

Consider the problem of a social planner who simultaneously chooses consumption and investment levels. At date 1, for each possible aggregate output $z \in Z$, the planner's task is the same for every exogenous state ω in $\{f(a) = z\}$: namely, he has to redistribute the aggregate output z among consumers. It is then natural to expect the random vector $\bar{x}_1^i$ to vary only with aggregate output (or equivalently

[5] In the sense that $U_\Omega^i(x_0^i, x_1^i) \geqslant U_\Omega^i(\bar{x}_0^i, \bar{x}_1^i)$, for all i, with strict inequality for at least one i.

to be measurable with respect to the random variable $\sigma f(a)$). Actually, since the Bernoulli functions u_1^i are assumed to be strictly concave, it follows from Jensen's inequality that there exists $\bar{z}_1^i : Z \to \mathbb{R}_+$ such that

$$\bar{x}_1^i(\omega) = \bar{z}_1^i(\sigma f(\omega, \bar{a})), \tag{3.1}$$

for every exogenous state ω.[6]

This leads us to introduce a different representation of consumption allocations. Let a Z**-allocation** be a family $((x_0, z_1), a) \equiv ((x_0^i, z_1^i)_{i\in I}, (a^k)_{k\in K})$ such that the date-1 consumption only varies with the aggregate output, i.e., $z_1^i \equiv (z_1^i(z))_{z\in Z}$. Such an allocation is said to be Z**-feasible** if markets clear at $t = 0$ and $(z_1^i)_{i\in I}$ defines a sharing rule at $t = 1$, i.e.,

$$\forall z \in Z, \quad \sum_{i\in I} z_1^i(z) = z.$$

The inter-temporal utility of (x_0^i, z_1^i) is given by

$$U_Z^i(x_0^i, z_1^i, a) \equiv u_0^i(x_0^i) + \sum_{z\in Z} u_1^i(z_1^i(z))\mu(z, a). \tag{3.2}$$

We can adapt in a straightforward manner the definition of Pareto optimality to Z-allocations. In particular, given an efficient Ω-allocation $((\bar{x}_0, \bar{x}_1), \bar{a})$, there exists an efficient Z-allocation $((\bar{x}_0, \bar{z}_1), \bar{a})$ such that $\bar{x}_1^i = \bar{z}_1^i(\sigma f(\bar{a}))$, for each i. If we denote by $\bar{\chi} \equiv (\bar{\chi}(z))_{z\in Z}$ the associated stochastic discount factor (hereafter Z-sdf) satisfying

$$\forall i \in I, \quad \frac{\partial u_1^i(\bar{z}_1^i(z))}{\partial u_0^i(\bar{x}_0^i)} = \bar{\chi}(z), \tag{3.3}$$

we obtain the following relation between the Ω-sdf and the Z-sdf:

$$\forall \omega \in \Omega, \quad \bar{m}(\omega) = \bar{\chi}(\sigma f(\omega, \bar{a})).$$

We will use next section to recall the well-known result that the Ω-sdf $\bar{m}$ is a sufficient statistics to efficiently coordinate decisions in competitive environments with complete state-contingent financial markets. A key contribution of this paper

[6] If u_1^i were assumed to be concave (instead of strictly concave) then, for each Pareto optimal allocation, there would exist an allocation satisfying Eq. (3.1) that generates the same profile of utilities and marginal utilities.

is to show that the Z-sdf $\bar{\chi}$ can play a similar role when there is a complete set of aggregate-output contingent contracts. In particular, we will illustrate the role played by the relation $\bar{m} = \bar{\chi}(\sigma f(\bar{a}))$ when we construct the "suitable" out-of-equilibrium market value of firms.

4 Prices for primitive causes

The traditional approach introduced by Arrow (1953) and Debreu (1959) assumes that markets are complete with respect to exogenous states of nature. For each ω, there is a claim traded at $t = 0$ which delivers one unit of consumption in $t = 1$, contingent on the realization of ω. These state-contingent claims are also called Arrow securities.

Let $p \equiv (p(\omega))_{\omega \in \Omega}$ be the price vector of Arrow securities traded at date $t = 0$. Given an investment level a, we denote by $B^i_\Omega(p, a)$ the set of all Ω-consumption plans (x^i_0, x^i_1) satisfying

$$x^i_0 + \sum_{\omega \in \Omega} p(\omega) x^i_1(\omega) \leqslant e^i_0 + \sum_{k \in K} \delta^i_k \left[-a_k + \sum_{\omega \in \Omega} p(\omega) f^k(\omega, a_k) \right].$$

The array $(\bar{p}, (\bar{x}_0, \bar{x}_1))$ is an **Ω-competitive equilibrium** associated with the investment vector $\bar{a}$ if $((\bar{x}_0, \bar{x}_1), \bar{a})$ is Ω-feasible and, for every i, the choice $(\bar{x}^i_0, \bar{x}^i_1)$ maximizes U^i_Ω in the budget set $B^i_\Omega(\bar{p}, \bar{a})$.

In this environment, efficient investment levels are achieved when firms maximize their net present value, evaluated at the equilibrium prices $\bar{p}$. We let $V^k_\Omega : A^k \longrightarrow \mathbb{R}$ be the value function defined by

$$V^k_\Omega(a^k) \equiv -a^k + \sum_{\omega \in \Omega} \bar{p}(\omega) f^k(\omega, a^k).$$

We omit the simple proof of the following standard result.

Theorem 4.1. *Let $(\bar{p}, (\bar{x}_0, \bar{x}_1))$ be an Ω-competitive equilibrium associated with the investment vector $\bar{a} \equiv (\bar{a}^k)_{k \in K}$. Assume that each firm k has chosen $\bar{a}^k$ to maximize V^k_Ω. Then, the Ω-allocation $((\bar{x}_0, \bar{x}_1), \bar{a}))$ is Pareto optimal.*

The objective V^k_Ω of firm k only depends on the firm's technology f^k and equilibrium prices $\bar{p}$. The prices of the state-contingent claims convey all relevant information about the available resources in the society. Firm k does not need to anticipate the investments made by other firms or the agents' consumption and utility functions. In addition to that, when choosing $\bar{a}^k$, firm k does not take the equilibrium effect of $\bar{a}^k$ over $\bar{p}$ into account. The same is true for agents when making their consumption choices. These are behavioral assumptions that are inherent to the notion of a competitive equilibrium and are intended to capture the idea that agents and firms have no market power.

5 Prices for aggregate production

Markets based on primitive states ω as introduced in Section 4 are not common since they are difficult to operate. Indeed, the primitive states of nature used to model production risks are too complex for writing and enforcing of contracts contingent on them. Modern financial markets trade assets whose payoffs depend on the observable profits of the firms.

Let us now recall the concept of a stock market equilibrium when agents only trade contracts based on firms' outcomes. We subsequently show that if the market structure allows insurance against aggregate production risks, then competitive markets efficiently redistribute resources among consumers. The last part of this section is the core of the paper. There we show that if firms maximize profit, suitably computed using aggregate output prices and "competitive beliefs", then the equilibrium outcome is efficient.

5.1 Stock market equilibrium

In the spirit of Magill and Quinzii (2002), we consider two types of assets: the equity contracts of the firms, indexed by the set K, which are in positive net supply, and a set J of securities in zero net supply representing bonds and derivatives. Security j's payoff is characterized by an output contingent function $R_j : Y \to \mathbb{R}$ describing the way the payoff at $t = 1$ of contract j depends on the realized output of the firms in the economy. The price at $t = 0$ of security j is denoted by q_j, and we let $q \equiv (q_j)_{j \in J}$ denote the vector of security prices. The payoff of firm k's equity is defined by the function $y \mapsto y^k$, and its price at $t = 0$ is denoted by E^k.

At date 0, for a given investment vector $a \equiv (a^k)_{k \in K}$ and asset prices (E, q), each agent i chooses consumption $x_0^i \in \mathbb{R}_+$, new equity holdings $\eta^i \in \mathbb{R}^K$ and portfolios $\theta^i \in \mathbb{R}^J$ such that

$$x_0^i + q \cdot \theta^i + E \cdot \eta^i \leqslant e_0^i + (E - a) \cdot \delta^i. \tag{5.1}$$

At date 1, contingent to each output vector y, agent i consumes

$$y_1^i(y) \equiv R(y) \cdot \theta^i + y \cdot \eta^i \geqslant 0. \tag{5.2}$$

We denote by $B_Y^i(E, q)$ the set of consumption plans (x_0^i, y_1^i) for which there is a portfolio (η^i, θ^i) satisfying the budget constraints (5.1) and (5.2). The utility $U_Y^i(x_0^i, y_1^i, a)$ of a consumption plan financed by (η^i, θ^i) is given by

$$\begin{aligned} U_Y^i(x_0^i, y_1^i, a) &\equiv u_0^i(x_0^i) + \sum_{y \in Y} u_1^i(y_1^i(y)) Q(y, a) \\ &= u_0^i(x_0^i) + \sum_{y \in Y} u_1^i(R(y) \cdot \theta^i + y \cdot \eta^i) Q(y, a). \end{aligned}$$

A Y**-allocation** is a family $((x_0, y_1), a) \equiv ((x_0^i, y_1^i)_{i \in I}, (a^k)_{k \in K})$ where the date-1 consumption only varies with firms' output, i.e., $y_1^i \equiv (y_1^i(y))_{y \in Y}$. It is said to be Y**-feasible** if markets clear in all decision nodes, i.e.,

$$\sum_{i \in I} x_0^i + \sum_{k \in K} a^k = \sum_{i \in I} e_0^i$$

and

$$\forall y \in Y, \quad \sum_{i \in I} y_1^i(y) = \sigma y.$$

Notice that, for any Y-feasible allocation, $(y_1^i)_{i \in I}$ defines a sharing rule of the aggregate output.

A **stock market equilibrium** associated with $\bar{a}$ is an array $((\bar{E}, \bar{q}), (\bar{x}_0, \bar{y}_1))$ such that $((\bar{x}_0, \bar{y}_1), \bar{a})$ is Y-feasible and, for each investor i, the consumption plan $(\bar{x}_0^i, \bar{y}_1^i)$ is optimal in $B_Y^i(\bar{E}, \bar{q})$.

Remark 5.1. There is no need to require market clearing for every output profile $y \in Y$. Given an equilibrium investment vector $\bar{a}$, we could restrict attention to market clearing for output levels y occurring with positive probability in equilibrium. Alternatively, we could replace the Y-feasibility condition by a market clearing condition on portfolios, i.e.,

$$\forall (k, j) \in K \times J, \quad \sum_{i \in I} (\theta_j^i, \eta_k^i) = (0, 1).$$

5.2 Optimal distribution of resources

As in Magill and Quinzii (2009, 2010), we focus our attention on the objective of firms and therefore assume that, given any investment decision by firms, markets assure an optimal distribution of resources among consumers. We propose to show that this can be achieved if markets are complete with respect to aggregate output represented by the set Z.

Definition 5.1. Markets are said to be complete with respect to aggregate output, or Z**-complete**, if for every $z \in Z$, there exists a portfolio (η^z, θ^z) such that the associated payoff function satisfies

$$\forall y \in Y, \quad R(y) \cdot \theta^z + y \cdot \eta^z = \begin{cases} 1, & \text{if } \sigma y = z; \\ 0, & \text{elsewhere.} \end{cases}$$

Security markets have undergone a remarkable development in the last thirty years with the introduction of more and more derivative contracts. Magill and Quinzii (2009) used this observation to justify the assumption that the markets are sufficiently rich to span the uncertainty in the outcomes of the firm, i.e., with respect to Y. Completeness with respect to aggregate output is a weaker requirement. The next result shows that Z-completeness is sufficient to get an optimal distribution of resources among consumers (the details of the proof are postponed to Appendix A).

Proposition 5.1. Consider a stock market equilibrium $((\bar{E}, \bar{q}), (\bar{x}_0, \bar{y}_1))$ given an investment vector $\bar{a} \equiv (\bar{a}^k)_{k \in K}$. Assume that markets are complete with respect to aggregate output. First, for each agent i, there exists $\bar{z}_1^i \equiv (\bar{z}_1^i(z))_{z \in Z}$ such that

$$\forall y \in Y, \quad \bar{y}_1^i(y) = \bar{z}_1^i(\sigma y).$$

Second, there exists $\bar{\rho} = (\bar{\rho}(z))_{z \in Z}$ such that

$$\forall i \in I, \quad \bar{\rho}(z) = \mu(z, \bar{a}) \frac{\partial u_1^i(\bar{z}_1^i(z))}{\partial u_0^i(\bar{x}_0^i)}. \tag{5.3}$$

Third, the consumption allocation $(\bar{x}_0, \bar{y}_1)$ (or equivalently $(\bar{x}_0, \bar{z}_1)$) is Pareto optimal,[7] and the equilibrium equity $\bar{E}^k$ of firm k satisfies

$$\bar{E}^k = \sum_{z\in Z} \bar{\rho}(z) \sum_{y^k \in Y^k} y^k \overline{Q}^k(y^k|z),$$

where $\overline{Q}^k(y^k|z)$ is the marginal probability

$$\overline{Q}^k(y^k|z) \equiv \sum_{y^{-k}\in Y^{-k}} \overline{Q}(y|z)$$

of the equilibrium conditional probability

$$\overline{Q}(y|z) \equiv \begin{cases} Q(y,\bar{a})/\mu(z,\bar{a}), & \text{if } \sigma y = z; \\ 0, & \text{elsewhere.} \end{cases} \tag{5.4}$$

Notice that $\bar{\rho}(z)$ is the cost of the portfolio that replicates the Arrow security contingent on z. In particular, the consumption plan $(\bar{x}_0^i, \bar{z}_1^i)$ satisfies the following budget restriction

$$\bar{x}_0^i + \sum_{z\in Z} \bar{\rho}(z)\bar{z}_1^i(z) \leqslant e_0^i + \sum_{k\in K} \delta_k^i \left[-\bar{a}^k + \bar{E}^k\right]. \tag{5.5}$$

We can show that, for any given investment vector $\bar{a}$, the array $((\bar{E},\bar{\rho}),(\bar{x}_0,\bar{y}_1))$ is a stock market equilibrium if and only if $((\bar{\rho},\bar{q}),(\bar{x}_0,\bar{z}_1))$ is a **reduced form equilibrium**—in the sense that markets clear, i.e.,

$$\sum_{i\in I} \bar{x}_0^i + \sum_{k\in K} \bar{a}^k = \sum_{i\in I} e_0^i \quad \text{and} \quad \sum_{i\in I} z_1^i(z) = z, \quad \forall z \in Z;$$

and for each i, the consumption plan $(\bar{x}_0^i, \bar{z}_1^i)$ is optimal in the budget set $B_Z^i(\bar{\rho},\bar{E},\bar{a})$ of all consumption plans (x_0^i, z_1^i) satisfying Eq. (5.5). This result is standard in the general equilibrium literature (see Magill and Quinzii (1996)).

Remark 5.2. In the definition of a reduced form equilibrium given an investment vector $\bar{a}$, we could restrict attention to market clearing for every aggregate output z occuring with positive probability—in the sense that $\mu(z,\bar{a}) > 0$. In particular, if

[7] Here, investment is arbitrarily fixed. Pareto optimality of the consumption allocation $(\bar{x}_0, \bar{y}_1)$ given an investment vector $\bar{a}$ means that we cannot find another consumption allocation (x_0, y_1) satisfying market clearing with the same investment vector $\bar{a}$ and such that $U_Y^i(x_0, y_1, \bar{a}) \geqslant U_Y^i(\bar{x}_0, \bar{y}_1, \bar{a})$, for every i, with a strict inequality for at least one i.

agents understand at $t = 0$ that an aggregate output level $\hat{z}$ is not possible at equilibrium, i.e., $\mu(\hat{z}, \bar{a}) = 0$, then the portfolio replicating the Arrow security contingent on $\hat{z}$ will not be traded, and we will have $\bar{\rho}(\hat{z}) = 0$. The results of Proposition 5.1 are still valid if we replace "Z-completeness" by "equilibrium completeness" in the sense that for every possible aggregate output z with $\mu(z, \bar{a}) > 0$, there exists a portfolio (η^z, θ^z) that replicates the Arrow security contingent on z.

5.3 Objective of the firms

Consider an agent i that is a shareholder of firm k, i.e., $\delta^i_k > 0$. He would like to set $\bar{a}^k$ in order to maximize his welfare. Given the equivalence between a stock market equilibrium and a reduced form equilibrium, agents would like that firm k chose $\bar{a}^k$ in order to maximize

$$\widetilde{V}^k_Z(a^k) = -a^k + \widetilde{E}^k(a^k),$$

where $\widetilde{E}^k(a^k)$ is agents' perceptions about the way the new investment decision a^k affected the price that the "market" would pay for the equity.

A minimal requirement is that agents have *correct expectations at equilibrium*, i.e.,

$$\widetilde{E}^k(\bar{a}^k) = \bar{E}^k = \sum_{z \in Z} \bar{\rho}(z) \sum_{y^k \in Y^k} y^k \overline{Q}^k(y^k|z).$$

We should now define *out-of-equilibrium* price perceptions. We make the traditional behavioral assumption that agents conceive that each firm k has no market power and does not affect market prices for aggregate output. This leads to the following price perception formula

$$\widetilde{E}^k(a^k) = \sum_{z \in Z} \bar{\rho}(z) \sum_{y^k \in Y^k} y^k \widetilde{Q}^k(y^k, a^k|z),$$

where $\widetilde{Q}^k(y^k, a^k|z)$ is the conditional probability (perceived by agents) that firm k's output is y^k when it chooses the investment a^k given the aggregate output z. The amount

$$\sum_{y^k \in Y^k} y^k \widetilde{Q}^k(y^k, a^k|z)$$

represents the conditional expected production of firm k across primitive states in which the equilibrium aggregated output is z.

We make the additional behavioral assumption that agents have *competitive beliefs*: they are convinced that a change in the investment of firm k will not affect the likelihood of each aggregate output z. In our opinion, this a reasonable assumption to justify that agents (and firms) take prices for aggregate output as given. Therefore, we say that agents form *competitive beliefs* when

$$\widetilde{Q}^k(y^k, a^k|z) = P(\{f^k(a^k) = y^k\}|\{\sigma f(\bar{a}) = z\}). \tag{5.6}$$

The interpretation for this is straightforward. Agents understand that if firm k chooses the investment a^k, then the risky output of the firm is represented by the random variable $\omega \mapsto f^k(\omega, a^k)$. When the aggregate production is z, agents consider that firm k's decision does not affect aggregate production and infer that every state of nature ω in $\{\sigma f(\bar{a}) = z\}$ is consistent with aggregate output z, implying that conditional on z, firm k's output belongs to the set $\{f^k(\omega, a^k) \ : \ \sigma f(\omega, \bar{a}) = z\}$.

Remark 5.3. Actors in the economy act as if firm k's investment choices did not affect the aggregate output. This is the analogue of the classical price-taking assumption for this environment. Economists sometimes use the metaphor of continuum sets of agents and firms to illustrate a scenario in which independent actions do not impact aggregate variables. In this case, $\widetilde{Q}^k(y^k, a^k|z)$ should be interpreted as the conditional output distribution of a given firm which invests a^k while all other infinite firms invest $(\bar{a}^{k'})_{k' \neq k}$ and the average aggregate output remains z.

Theorem 5.1. *Let $((\bar{E}, \bar{q}), (\bar{x}_0, \bar{y}_1))$ be a stock market equilibrium associated with the investment vector $\bar{a} \equiv (\bar{a}^k)_{k \in K}$. Assume that*

- *markets are complete with respect to aggregate production;*
- *each firm k has chosen $\bar{a}^k$ to maximize the market value*

$$V_Z^k(a^k) = -a^k + \sum_{z \in Z} \bar{\rho}(z) \sum_{y^k \in Y^k} y^k \widetilde{Q}^k(y^k, a^k|z),$$

  where beliefs $\widetilde{Q}^k(\cdot, a^k|z)$ are competitive in the sense of Eq. (5.6) and $\bar{\rho}(z)$ is the price of aggregate output z.[8]

Then the Y-allocation $((\bar{x}_0, \bar{y}_1), \bar{a})$ is efficient.

[8]This is the market price of the portfolio that replicates the Arrow security contingent on z.

Proof of Theorem 5.1. Consider an alternative Y-feasible allocation $((x_0, y_1), a)$. For every i, we denote by z_1^i the sharing rule $(z_1^i(z))_{z\in Z}$ such that for every z with $\mu(z,a) > 0$, we have

$$z_1^i(z) = \frac{1}{\mu(z,a)} \sum_{y\in Y(z)} y_1^i(y) Q(y,a),$$

where we recall that $Y(z)$ is the set of production vectors y consistent with aggregate outcome z, i.e., $Y(z) \equiv \{y \in Y \,:\, \sigma y = z\}$. For z satisfying $\mu(z,a) = 0$, we can select any arbitrary sharing rule, for instance, $z_1^i(z) = z/(\#I)$. Notice that $((x_0, z_1), a)$ is a Z-feasible allocation and, thanks to (strict) concavity of u^i, it satisfies

$$\forall i \in I, \quad U_Z^i(x_0^i, z_1^i, a) \geqslant U_Y^i(x_0^i, y_1^i, a).$$

Following Proposition 5.1, for each i, there exists $\bar{z}_1^i \equiv (\bar{z}_1^i(z))_{z\in Z}$ such that $\bar{y}_1^i(y) = \bar{z}_1^i(\sigma y)$. Therefore, to prove that $((\bar{x}_0, \bar{y}_1), \bar{a})$ is Pareto optimal, it is sufficient to show that

$$\sum_{i\in I} \frac{1}{\partial u_0^i(\bar{x}_0^i)} U_Z^i(x_0^i, z_1^i, a) \leqslant \sum_{i\in I} \frac{1}{\partial u_0^i(\bar{x}_0^i)} U_Z^i(\bar{x}_0^i, \bar{z}_1^i, \bar{a}),$$

for all Z-feasible allocation $((x_0, z_1), a)$.

Let us first write agents' utilities in terms of competitive beliefs, as given in Eq. (5.6). Recall from Eq. (3.2) that

$$U_Z^i(x_0^i, z_1^i, a) \equiv u_0^i(x_0^i) + \sum_{z'\in Z} u_1^i(z_1^i(z'))\mu(z', a).$$

Define

$$\widetilde{Q}(y, a|z) = P(\{f(a) = y\}|\{\sigma f(\bar{a}) = z\})$$

and notice that, for every $z' \in Z$,

$$\sum_{y'\in Y(z')} \sum_{z\in Z} \widetilde{Q}(y', a|z)\mu(z, \bar{a}) = \mu(z', a), \tag{5.7}$$

where $Y(z') \equiv \{y' \in Y \,:\, \sigma y' = z'\}$. We can then write

$$U_Z^i(x_0^i, z_1^i, a) = u_0^i(x_0^i) + \sum_{z'\in Z} u_1^i(z_1^i(z')) \sum_{y'\in Y(z')} \sum_{z\in Z} \widetilde{Q}(y', a|z)\mu(z, \bar{a}).$$

It follows then from Eq. (2.1) that

$$\begin{aligned} U_Z^i(x_0^i, z_1^i, a) &= u_0^i(x_0^i) + \sum_{y' \in Y} u_1^i(z_1^i(\sigma y')) \sum_{z \in Z} \widetilde{Q}(y', a|z) \sum_{y \in Y(z)} Q(y, \bar{a}) \\ &= u_0^i(x_0^i) + \sum_{y' \in Y} \sum_{y \in Y} u_1^i(z_1^i(\sigma y')) \widetilde{Q}(y', a|\sigma y) Q(y, \bar{a}). \end{aligned} \tag{5.8}$$

Similarly, for $(\bar{x}_0^i, \bar{z}_1^i, \bar{a})$, we have

$$\begin{aligned} U_Z^i(\bar{x}_0^i, \bar{z}_1^i, \bar{a}) &= u_0^i(\bar{x}_0^i) + \sum_{z \in Z} u_1^i(\bar{z}_1^i(z)) \mu(z, \bar{a}) \\ &= u_0^i(\bar{x}_0^i) + \sum_{y \in Y} u_1^i(\bar{z}_1^i(\sigma y)) Q(y, \bar{a}). \end{aligned}$$

Moreover, since $\sum_{y' \in Y} \widetilde{Q}(y', a|\sigma y) = 1$, we can write

$$U_Z^i(\bar{x}_0^i, \bar{z}_1^i, \bar{a}) = u_0^i(\bar{x}_0^i) + \sum_{y \in Y} \sum_{y' \in Y} u_1^i(\bar{z}_1^i(\sigma y)) \widetilde{Q}(y', a|\sigma y) Q(y, \bar{a}). \tag{5.9}$$

We can now compare the utilities in Eq. (5.8) and (5.9). Consider the following notation:

$$\Delta U^i \equiv U_Z^i(x_0^i, z_1^i, a) - U_Z^i(\bar{x}_0^i, \bar{z}_1^i, \bar{a}) \quad \text{and} \quad \Delta x_0^i = x_0^i - \bar{x}_0^i.$$

It follows from Eq. (5.8) and (5.9) together with our assumptions on Bernoulli functions that

$$\Delta U^i \leqslant \partial u_0^i(\bar{x}_0^i) \Delta x_0^i + \sum_{y \in Y} \sum_{y' \in Y} \partial u_1^i(\bar{z}_1^i)[z_1^i(\sigma y') - \bar{z}_1^i(\sigma y)] \widetilde{Q}(y', a|\sigma y) Q(y, \bar{a}).$$

Since markets clear, we have

$$\sum_{i \in I} [z_1^i(\sigma y') - \bar{z}_1^i(\sigma y)] = \sigma y' - \sigma y \quad \text{and} \quad \sum_{i \in I} \Delta x_0^i = -[\sigma a - \sigma \bar{a}].$$

Therefore,

$$\sum_{i \in I} \frac{1}{\partial u_0^i(\bar{x}_0^i)} \Delta U^i \leqslant -[\sigma a - \sigma \bar{a}] + \sum_{z \in Z} \sum_{y' \in Y} \bar{\rho}(z)[\sigma y' - z] \widetilde{Q}(y', a|z),$$

where $\bar{\rho}(z) = \mu(z, \bar{a}) \partial u_1^i(\bar{z}_1^i)/\partial u_0^i(\bar{x}_0^i)$, as proved in Proposition 5.1.

Define

$$\Gamma \equiv -\sigma a + \sum_{z \in Z} \sum_{y' \in Y} \bar{\rho}(z) \sigma y' \widetilde{Q}(y', a|z) \quad \text{and} \quad \bar{\Gamma} \equiv -\sigma \bar{a} + \sum_{z \in Z} \sum_{y' \in Y} \bar{\rho}(z) z \widetilde{Q}(y', a|z).$$

To conclude the proof, we have to show that

$$\Gamma = \sum_{k \in K} V_Z^k(a^k) \quad \text{and} \quad \bar{\Gamma} = \sum_{k \in K} V_Z^k(\bar{a}^k).$$

Notice that $\Gamma = \sum_{k \in K} \Gamma^k$, where

$$\begin{aligned} \Gamma^k &= -a^k + \sum_{z \in Z} \bar{\rho}(z) \sum_{y \in Y} y^k \widetilde{Q}(y, a|z) \\ &= -a^k + \sum_{z \in Z} \bar{\rho}(z) \sum_{y^k \in Y^k} y^k \sum_{y^{-k} \in Y^{-k}} \widetilde{Q}(y, a|z). \end{aligned}$$

The desired result, $\Gamma^k = V_Z^k(a^k)$, follows from the the absence of externalities—that is, for each z, firm k's output likelihood is not affected by other firms' investment decisions:

$$\widetilde{Q}^k(y^k, a^k|z) = \sum_{y^{-k} \in Y^{-k}} \widetilde{Q}(y, a|z).$$

Similarly, we shall also prove that $\bar{\Gamma} = \sum_{k \in K} V_Z^k(\bar{a}^k)$. Since $\sum_{y' \in Y} \widetilde{Q}(y', a|z) = 1$, we have

$$\begin{aligned} \bar{\Gamma} &= -\sigma \bar{a} + \sum_{z \in Z} \bar{\rho}(z) z \\ &= -\sigma \bar{a} + \sum_{z \in Z} \bar{\rho}(z) \sum_{y \in Y(z)} \sigma y \overline{Q}(y|z). \end{aligned}$$

Notice from Equations (5.4) and (5.6) that competitive beliefs $\widetilde{Q}(y, \bar{a}|z)$ at the equilibrium investment vector is correct in the sense that it coincides with $\overline{Q}(y|z)$. Therefore:

$$\bar{\Gamma} = -\sigma \bar{a} + \sum_{z \in Z} \bar{\rho}(z) \sum_{y \in Y} \sigma y \widetilde{Q}(y, \bar{a}|z).$$

In particular, we have $\bar{\Gamma} = \sum_{k\in K} \bar{\Gamma}^k$, where

$$\bar{\Gamma}^k = -\bar{a}^k + \sum_{z\in Z} \bar{\rho}(z) \sum_{y^k \in Y^k} y^k \sum_{y^{-k}\in Y^{-k}} \widetilde{Q}(y, \bar{a}|z).$$

The desired result, $\bar{\Gamma}^k = V_Z^k(\bar{a}^k)$, follows again from the absence of externalities. $\square$

Remark 5.4. We do not address in this paper the issue of existence of a stock market equilibrium with investment levels that maximize the market value defined in Theorem 5.1. There is only one technical difficulty that deserves attention: the convexity of the firms' decision problem. One may assume, as in Magill and Quinzii (2009), that the set A^k of investment possibilites is a convex (and closed) subset of $\mathbb{R}_+$. For simplicity, we have made the assumption that the set Y^k of possible outcomes for each firm k is finite.[9] Allowing for a continuum of possible outcome levels, one may follow the standard general equilibrium literature by assuming that each production function $a^k \mapsto f^k(\omega, a^k)$ is continuous and concave. This would imply that firm k's objective function

$$a^k \longmapsto V_Z^k(a^k) = -a^k + \sum_{\omega\in\Omega} \bar{\chi}(\sigma f(\omega, \bar{a}))f^k(\omega, a^k)P(\omega)$$

is concave. Existence would follow from standard arguments.[10] Alternatively, if one prefer to work with finite production sets $(Y^k)_{k\in K}$, one may model the production sector with a continuum of firms and attempt to apply Lyapunov's theorem to overcome the convexity issue.[11] This approach would involve serious mathematical complications and checking its validity is beyond the scope of this paper.

[9] The assumption that Y^k is finite is only used in this paper to simplify the notation (allowing us to use summations instead of integrals).

[10] In equilibrium, the set of possible aggregate production z will be finite given that the set of primitive states of nature Ω is finite. This occurs regardless of the properties of $(Y^k)_{k\in K}$, since $\mu(z, \bar{a}) > 0$ if and only if $\sigma f(\omega, \bar{a}) = z$ for some ω. Therefore, the assumption of equilibrium-completeness (see Remark 5.2) would only require a finite number of assets being traded in equilibrium.

[11] Considering a continuum of firms does not necessarily mean that we have identical firms. We may replace the finite set K by a non-atomic measure space $(K, \mathcal{K}, \kappa)$, where $(K, \mathcal{K})$ is a measurable set of firms' characteristics and κ is a distribution measure on $\mathcal{K}$.

6 Relation with the literature

We have proved that, even when markets are complete only with respect to aggregate output, it is possible to find a profit-maximization criterion for firms' investment decisions that implies Pareto optimality. Equilibrium prices for aggregate output play a crucial role by conveying part of the socially relevant information, but they are not sufficient. The incompleteness of markets with respect to exogenous states of nature can be subsumed if agents and firms have competitive beliefs about the impact of new investments over the conditional distribution of each firm's output given the equilibrium aggregate output. More precisely, Theorem 5.1 states that a criterion firm k should maximize to obtain a Pareto optimal equilibrium is

$$V_Z^k(a^k, \bar{\rho}) = -a^k + \sum_{z \in Z} \bar{\rho}(z) \mathbb{E}^P[f^k(a^k) | \sigma f(\bar{a}) = z].$$

Recall from Eq. (5.3) that the price of aggregate output z satisfies $\bar{\rho}(z) = \mu(z, \bar{a})\bar{\chi}(z)$, where $\bar{\chi}$ is the Z-sdf defined in Eq. (3.3). We then get the following expression for the firm's objective

$$V_Z^k(a^k, \bar{\rho}) = -a^k + \sum_{\omega \in \Omega} \bar{\chi}(\sigma f(\omega, \bar{a})) f^k(\omega, a^k) P(\omega). \tag{6.1}$$

We will now compare this objective with some others proposed in the literature.

6.1 Standard valuation with prices for primitive causes

When markets are Ω-complete, the firm can use the market price $\bar{p}(\omega)$ of each primitive cause ω to compute the net present value of its investment

$$V_\Omega^k(a^k, \bar{p}) = -a^k + \sum_{\omega \in \Omega} \bar{p}(\omega) f^k(\omega, a^k).$$

We know that, if $(\bar{p}, (\bar{x}_0, \bar{x}_1))$ is an Ω-competitive equilibrium where the investment vector $(\bar{a}^k)_{k \in K}$ is chosen by firms to maximize the objective $V_\Omega^k(\cdot, \bar{p})$, then the allocation of goods and investment is efficient.

We show that the standard objective $V_\Omega^k(\cdot, \bar{p})$ coincides with our objective $V_Z^k(\cdot, \bar{\rho})$ under the Ω-completeness assumption. Recall that $\bar{p}(\omega) = \bar{m}(\omega)P(\omega)$ in any Ω-competitive equilibrium, where $\bar{m}(\omega) = \partial u_1^i(\bar{x}_1^i(\omega))/\partial u_0^i(\bar{x}_0^i)$ is the Ω-sdf. Since $(\bar{x}_0, \bar{x}_1)$ is Pareto optimal, we have seen in Section 3 that $\bar{x}_1$ is measurable with respect to $\sigma f(\bar{a})$, so that we can write

$$\bar{p}(\omega) = \bar{m}(\omega)P(\omega) = \bar{\chi}(\sigma f(\omega, \bar{a}))P(\omega) = \bar{\rho}(\sigma f(\omega, \bar{a})), \tag{6.2}$$

where $\bar{\chi}$ and $\bar{\rho}$—respectively defined by Eq. (3.3) and (5.3)—are the Z-sdf and the prices of aggregate output associated with the Z-representation $(\bar{x}_0, \bar{z}_1)$ of $(\bar{x}_0, \bar{x}_1)$. It then follows from Eq. (6.1) that the standard objective $V_\Omega^k(a^k, \bar{p})$ coincides with our objective $V_Z^k(a^k, \bar{\rho})$.

When financial markets are Z-complete but not Ω-complete, they do not explicitly price every primitive cause ω. Our contribution consists in showing that every firm can recover the stochastic discount factor $\bar{m}(\omega)$ using the prices of traded assets. Equation (6.2) tells us that all firms should set $\bar{m}(\omega) = \bar{\chi}(z)$, where $\bar{\chi}(z)$ is the sdf inferred from market prices at the aggregate production z consistent (in equilibrium) with the primitive cause ω, i.e., $z = \sigma f(\omega, \bar{a})$. We stress that, by doing so, firm k does not take into account the fact that the new investment a^k affects aggregate output in state ω.

6.2 Firms valuation with prices for aggregate output uncertainty

Magill and Quinzii (2008) and Magill, Quinzii and Rochet (2011) argue that if firms maximize the market-value criterion when financial contracts are written on outcomes then the resulting allocation is (generically) not Pareto optimal. This statement seems to contradict our main result Theorem 5.1. The difference stems from the definition of market value. To illustrate our point, consider a stock market equilibrium $((\bar{E}, \bar{q}), (\bar{x}_0, \bar{y}_1))$ with an investment vector $\bar{a}$. We have seen (Proposition 5.1) that firm k's equity $\bar{E}^k$ satisfies

$$\bar{E}^k = \sum_{z \in Z} \mu(z, \bar{a}) \bar{\chi}(z) \sum_{y^k \in Y^k} y^k \overline{Q}^k(y^k | z),$$

where $\bar{\chi}(z)$ is the Z-sdf. Since we have

$$\overline{Q}^k(y^k | z) = P(\{f^k(\bar{a}^k) = y^k\} | \sigma f(\bar{a}) = z),$$

the expression for firm k's equity can then be written as

$$\bar{E}^k = \sum_{z \in Z} \bar{\chi}(z) \sum_{y^k \in Y^k} y^k P\left[\{f^k(\bar{a}^k) = y^k\} \cap \{\sigma f(\bar{a}) = z\}\right].$$

Based on this valuation at equilibrium, there are different ways of defining agents' perception of firm k's equity value when its manager contemplates an alternative investment a^k. A first possibility consists in replacing the expression $\{f^k(\bar{a}^k) = y^k\} \cap \{\sigma f(\bar{a}) = z\}$ by $\{f^k(a^k) = y^k\} \cap \{\sigma f(a^k, \bar{a}^{-k}) = z\}$. This leads to the following objective function

$$\widehat{V}_Z^k(a^k) \equiv -a^k + \sum_{z \in Z} \bar{\chi}(z) \sum_{y^k \in Y^k} y^k P\left[\{f^k(a^k) = y^k\} \cap \{\sigma f(a^k, \bar{a}^{-k}) = z\}\right].$$

Since

$$\{\sigma f(a^k, \bar{a}^{-k}) = z\} = \bigcup_{y \in Y(z)} \{f(a^k, \bar{a}^{-k}) = y\},$$

we find

$$\widehat{V}_Z^k(a^k) = -a^k + \sum_{y \in Y} \bar{\chi}(\sigma y) y^k Q(y, a^k, \bar{a}^{-k}). \tag{6.3}$$

This is in the spirit of Magill and Quinzii (2008) and Magill, Quinzii and Rochet (2011).[12] Under this view, firm k manager takes the stochastic discount factor $\bar{\chi}$ as given but fully considers the impact of new investments on the probability of y and therefore on the aggregate output $z = \sigma y$.

The objective $\widehat{V}_Z^k$ differs from the one we propose since, following the competitive tradition, we assumed that firms do not anticipate the effect of new investments over aggregate variables. Formally, we replaced the expression $\{f^k(\bar{a}^k) = y^k\} \cap \{\sigma f(\bar{a}) = z\}$ by $\{f^k(a^k) = y^k\} \cap \{\sigma f(\bar{a}) = z\}$.

In an environment with non-marginal firms, one may think that firms should maximize the objective function $\widehat{V}_Z^k$. Magill and Quinzii (2009) showed that maximizing this market value function does not always lead to Pareto optimality. This is due to the fact that, when partially taking into account the effect of their actions over aggregate variables, firms end up acting strategically and not competitively.

6.3 The expected social utility

After arguing that market value maximization—computed as in Eq. (6.3)—does not necessarily lead to efficiency, Magill and Quinzii (2009) investigated an alter-

[12] These models are not directly comparable to ours. Magill and Quinzii (2008), for instance, analyze a moral hazard economy in which managers choose unobservable effort levels instead of investments. The framework used in Magill, Quinzii and Rochet (2011) is much closer to ours, but it presents a few differences on the number of goods and structure of preferences.

native objective function for firms. They have shown that if firms maximize an "expected social" objetive function then the resulting allocation is Pareto optimal. We propose to analyse the mechanics behind this result and illustrate the differences with respect to the arguments used in our proof of Theorem 5.1.

Consider a reduced form equilibrium $((\bar{E}, \bar{\rho}), (\bar{x}_0, \bar{z}_1))$ associated with investment $\bar{a}$. In order to identify a sufficient condition for efficiency, we can make a change in investment by modifying date 0 consumption. More formally, for every i, fix $\varepsilon^i \geqslant 0$ such that $\varepsilon \equiv \sum_{i \in I} \varepsilon^i > 0$. We compute the following weighted sum of variations

$$\sum_{i \in I} \frac{1}{\partial u_0^i(\bar{x}_0^i)} \Delta^i(\varepsilon^i)$$

where

$$\Delta^i(\varepsilon^i) \equiv U_Z^i(\bar{x}_0^i - \varepsilon^i, \bar{z}_1^i, a) - U_Z^i(\bar{x}_0^i, \bar{z}_1^i, \bar{a}) \quad \text{and} \quad a = (\bar{a}^k + \varepsilon, \bar{a}^{-k}).$$

Recall that for every $x_0^i \geqslant 0$ we have

$$U_Z^i(x_0^i, \bar{z}_1^i, a) = u_0^i(x_0^i) + \sum_{z' \in Z} u_1^i(\bar{z}_1^i(z'))\mu(z', a). \tag{6.4}$$

When computing the difference $\Delta^i(\varepsilon^i)$—see Eq. (5.7) in the proof of Theorem 5.1—, we have made the following "disintegration":

$$\mu(z', a) = \sum_{y' \in Y(z')} \sum_{z \in Z} \widetilde{Q}(y', a|z)\mu(z, \bar{a}).$$

The term $\mu(z', a)$ is the probability of the event $\{\sigma f(a) = z'\}$, and it can be expressed as a function of the probability $\mu(z, \bar{a})$ of the event $\{\sigma f(\bar{a}) = z\}$ using Bayes' rule. This allows us to compute the difference $\Delta^i(\varepsilon^i)$ factorizing the probabilities and explains why the objective we proposed is comparable with the standard profit maximization criterion.

Magill and Quinzii (2009) followed another route to compute $\Delta^i(\varepsilon^i)$. In accordance with their assumption that each firm k does take into account the effect of its actions over aggregate variables, they made the following simple computation:

$$\Delta^i(\varepsilon^i) = \Delta_0^i(\varepsilon^i) + \sum_{z \in Z} u_1^i(\bar{z}_1^i(z))[\mu(z, a) - \mu(z, \bar{a})],$$

where $\Delta_0^i(\varepsilon^i) = u_0^i(\bar{x}_0^i - \varepsilon^i) - u_0^i(\bar{x}_0^i)$. It follows from concavity of u_0^i that

$$\sum_{i \in I} \frac{1}{\partial u_0^i(\bar{x}_0^i)} \Delta^i(\varepsilon^i) \leqslant -\varepsilon + \sum_{z \in Z} \Phi(z)[\mu(z, \bar{a}^k + \varepsilon, \bar{a}^{-k}) - \mu(z, \bar{a}^k, \bar{a}^{-k})],$$

where

$$\Phi(z) \equiv \sum_{i \in I} \frac{u_1^i(\bar{z}_1^i(z))}{\partial u_0^i(\bar{x}_0^i)}.$$

Given this simple computation, one can exhibit an objective function that leads to Pareto efficiency. Indeed, if firm k maximizes the following "expected social" objective function:

$$\mathrm{MQ}_Z^k(a^k) = -a^k + \sum_{z \in Z} \Phi(z)\mu(z, a^k, \bar{a}^{-k})$$

then the new allocation $((\bar{x}_0^i - \varepsilon, \bar{z}_1^i)_{i \in I}, a)$ does not Pareto dominate $((\bar{x}_0^i, \bar{z}_1^i)_{i \in I}, \bar{a})$.

The point of the above argument is to show how Magill and Quinzii (2009) came up with the "expected social" objective and illustrate the differences with respect to our approach. It turns out that their analysis is particularly important to study decisions of large corporations that are "non-marginal"—in the sense of being aware that their investment decisions affect the probability distribution over the economy's output vector $y = (y^k)_{k \in K}$. The objective function MQ_Z^k plays a crucial role in Magill, Quinzii and Rochet (2011) to provide a theoretical foundation for the theory of stakeholder firms as studied in Allen, Carletti and Marquez (2011). In that perspective, our result shows that the stakeholder theory is not founded by the mere presence of output-contingent contracts. An additional feature (e.g., a non-competitive production sector) is needed to reject profit-maximization as a socially justified objective for firms.

A Appendix: Proof of Proposition 5.1

Consider a stock market equilibrium $((\bar{E}, \bar{q}), (\bar{x}_0, \bar{y}_1))$ given an investment vector $\bar{a}$. For every consumer i, the plan $(\bar{x}_0^i, \bar{y}_1^i)$ is optimal in the budget set $B_Y^i(\bar{E}, \bar{q})$. We let $\bar{z}_1^i = (\bar{z}_1^i(z))_{z \in Z}$ be the conditional expectation of $\bar{y}_1^i$ defined by

$$\forall z \in Z, \quad \bar{z}_1^i(z)\mu(z, \bar{a}) = \sum_{y \in Y(z)} \bar{y}_1^i(y) Q(y, \bar{a}),$$

where we recall that $Y(z) \equiv \{y \in Y \ : \ \sigma y = z\}$. Since markets are Z-complete, for each aggregate outcome z, there exists a portfolio (η^z, θ^z) that replicates the Arrow security paying when aggregate output is z. We let (η^i, θ^i) be the portfolio defined by

$$(\eta^i, \theta^i) = \sum_{z \in Z} \bar{z}_1^i(z)(\eta^z, \theta^z).$$

The portfolio (η^i, θ^i) finances the future consumption represented by $\bar{z}_1^i$ in the sense that

$$\forall y \in Y, \quad \bar{z}_1^i(\sigma y) = R(y) \cdot \theta^i + y \cdot \eta^i.$$

It follows from the market clearing condition for $\bar{y}_1$ that

$$\forall y \in Y, \quad \sum_{i \in I} \bar{z}_1^i(\sigma y) = \sum_{i \in I} \bar{y}_1^i(y).$$

This implies that

$$\forall y \in Y, \quad R(y) \cdot \sum_{i \in I} \theta^i + y \cdot \sum_{i \in I} \eta^i = R(y) \cdot \sum_{i \in I} \bar{\theta}^i + y \cdot \sum_{i \in I} \bar{\eta}^i,$$

where $(\bar{\eta}^i, \bar{\theta}^i)$ is a portfolio financing the equilibrium consumption plan $(\bar{x}^i, \bar{y}_1^i)$. Since equilibrium prices preclude arbitrage opportunities, we obtain

$$\sum_{i \in I} (\bar{q} \cdot \theta^i + \bar{E} \cdot \eta^i) = \sum_{i \in I} (\bar{q} \cdot \bar{\theta}^i + \bar{E} \cdot \bar{\eta}^i).$$

We can now prove that

$$\forall i \in I, \quad \bar{q} \cdot \theta^i + \bar{E} \cdot \eta^i = \bar{q} \cdot \bar{\theta}^i + \bar{E} \cdot \bar{\eta}^i.$$

If this relation did not hold, we should have $\bar{q} \cdot \theta^i + \bar{E} \cdot \eta^i > \bar{q} \cdot \bar{\theta}^i + \bar{E} \cdot \bar{\eta}^i$ for some i. This agent i could then finance the future consumption $\bar{z}_1^i$ together with the present consumption $\bar{x}_0^i + \varepsilon$, for $\varepsilon > 0$ small enough. This leads to a contradiction since u_0^i is increasing and concavity of u_1^i implies

$$\sum_{z \in Z} u_1^i(\bar{z}_1^i(z))\mu(z, \bar{a}) \geqslant \sum_{y \in Y} u_1^i(\bar{y}_1^i(y))Q(y, \bar{a}).$$

This result implies that the consumption plan $(\bar{x}_0^i, \bar{z}_1^i)$ is budget feasible for every agent. Since the Bernoulli functions u_1^i are strictly concave, we must have $\bar{y}_1^i(y) = \bar{z}_1^i(\sigma y)$, for every i and y.

Next, denote by $\bar{\rho}(z)$ the cost of the portfolio (θ^z, η^z), i.e., $\bar{\rho}(z) = \bar{q} \cdot \theta^z + \bar{E} \cdot \eta^z$. Since $(\bar{x}_0^i, \bar{y}_1^i)$ is optimal in $B_Y^i(\bar{E}, \bar{q})$, it follows from the first-order conditions that

$$\bar{q} = \sum_{y \in Y} Q(y, \bar{a}) \frac{\partial u_1^i(\bar{y}_1^i(y))}{\partial u_0^i(\bar{x}_0^i)} R(y) \quad \text{and} \quad \bar{E} = \sum_{y \in Y} Q(y, \bar{a}) \frac{\partial u_1^i(\bar{y}_1^i(y))}{\partial u_0^i(\bar{x}_0^i)} y. \tag{A.1}$$

Given the definition of (θ^z, η^z), we find

$$\bar{\rho}(z) = \sum_{y \in Y(z)} Q(y, \bar{a}) \frac{\partial u_1^i(\bar{y}_1^i(y))}{\partial u_0^i(\bar{x}_0^i)} = \mu(z, \bar{a}) \frac{\partial u_1^i(\bar{z}_1^i(z))}{\partial u_0^i(\bar{x}_0^i)}. \tag{A.2}$$

This implies that agents have homogenous marginal rates of substitution.[13] Since the Bernoulli functions u_1^i are strictly concave, the consumption allocation $(\bar{x}_0, \bar{y}_1)$ is Pareto optimal if and only if $(\bar{x}_0, \bar{z}_1)$ is Pareto optimal. The latter property is satisfied since markets are Z-complete.

Finally, we have

$$\bar{x}_0^i + \sum_{z \in Z} \bar{\rho} \bar{z}_1^i(z) \leqslant e_0^i + \sum_{k \in K} \delta_k^i \left[-\bar{a}^k + \bar{E}^k \right].$$

Combining Eq. (A.1) and (A.2) we have

$$\bar{E} = \sum_{z \in Z} \sum_{y \in Y(z)} \frac{Q(y, \bar{a})}{\mu(z, \bar{a})} \bar{\rho}(z) y = \sum_{z \in Z} \bar{\rho}(z) \sum_{y \in Y} \overline{Q}(y|z) y,$$

where $\overline{Q}(y, z)$ is the condition probability defined by

$$\overline{Q}(y|z) \equiv \begin{cases} Q(y, \bar{a})/\mu(z, \bar{a}), & \text{if } \sigma y = z; \\ 0, & \text{elsewhere.} \end{cases}$$

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[13] This is not a direct or trivial consequence of our completeness hypothesis since we only assume completeness with respect to aggregate outcome and not outcome profiles.

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