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Piecewise Continuous Cumulative Prospect Theory and Behavioral Financial Engineering

Marc Gürtler* and Julia Stolpe**

Abstract:
We extend the continuous Cumulative Prospect Theory (CPT) by considering piecewise continuous distributions with a finite number of jump discontinuities. Such distributions are relevant in practice, for example, within the framework of financial engineering since cash flow distributions of most types of derivatives are only piecewise continuous. In addition, we expand the model with a (piecewise) continuous version of hedonic framing which is, until now, only available in a discrete model setting. We show how to apply the model to a broad class of structured products. Finally, we apply Prospect Theory (PT), CPT, and expected utility theory to a set of different real-life certificates with piecewise continuous and discrete distributions in order to analyze whether there are any significant differences between the theories, and which theory is able to explain the demand behavior of a market participant best. As a result, we recommend the use of the piecewise continuous version of CPT to design products within the framework of behavioral financial engineering.

Keywords: Continuous Cumulative Prospect Theory, Continuous Hedonic Framing, Behavioral Finance, Financial Engineering

JEL classification: G31, G32, G35

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Abstract:

We extend the continuous Cumulative Prospect Theory (CPT) by considering piecewise continuous distributions with a finite number of jump discontinuities. Such distributions are relevant in practice, for example, within the framework of financial engineering since cash flow distributions of most types of derivatives are only piecewise continuous. In addition, we expand the model with a (piecewise) continuous version of hedonic framing which is, until now, only available in a discrete model setting. We show how to apply the model to a broad class of structured products. Finally, we apply Prospect Theory (PT), CPT, and expected utility theory to a set of different real-life certificates with piecewise continuous and discrete distributions in order to analyze whether there are any significant differences between the theories, and which theory is able to explain the demand behavior of a market participant best. As a result, we recommend the use of the piecewise continuous version of CPT to design products within the framework of behavioral financial engineering.

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A central task in the context of financial engineering is the customer-oriented development of financial products. Typical in this context are so-called structured products that are characterized by a combination of individual financial instruments such as stocks, bonds, and derivatives. The most relevant criterion when designing such products is the acceptance of the customers. Against this background, it is less useful to evaluate such products on the basis of a normative theory of rational decision making like the expected utility theory (EUT) of von Neumann and Morgenstern (1944). Rather, it seems preferable to base such an evaluation on descriptive decision theories to consider the actual demand behavior of investors. In this context, the Prospect Theory (PT), suggested by Kahneman and Tversky (1979), and the Cumulative Prospect Theory (CPT), proposed by Tversky and Kahneman (1992), are probably the most successful descriptive theories capturing the preference of decision makers under risk.

Especially from a theoretical perspective, CPT is often considered as the more suitable theory in comparison to PT. Mainly, PT has been criticized for the violation of the first-order stochastic dominance. A further condition satisfied by CPT, in contrast to PT, is continuity, i.e. small changes in a lottery only produce small differences in utility (Rieger and Wang 2008). Furthermore, Fennema and Wakker (1997) confirm that CPT is not merely a mathematical improvement over PT, but also fits better with experimental data. CPT permits the modeling of diminishing sensitivity and gives a better explanation of several empirical findings. However, not all empirical studies confirm the outperformance of CPT in comparison with PT. There is a number of empirical investigations supporting PT in predicting preference patterns (Camerer and Ho 1994, Wu 1994, Humphrey 1995, Birnbaum and McIntosh 1996, Wu and Gonzales 1996, Birnbaum and Navarrete 1998, Luce 1998, Birnbaum und Martin 2003, Birnbaum 2005). Gonzales and Wu (2003) find that both PT and CPT can account for some standard empirical patterns. Wu, Zhang, and Abdellaoui (2005) show that it depends on the design of the gambles which version of the Prospect Theory fits better. Choices of gambles with a certainty effect seem to be consistent with both theories, whereas gambles without involving a certainty effect are consistent with PT only. In sum, from an empirical point of view, there are no unequivocal conclusions whether PT or CPT should be considered as the more favored theory. Against the background of the contentious discussion about the suitability of the single theories, one motivation of this paper is to make a comparative analysis of EUT, PT, and CPT on the basis of structured products to analyze which theory is most suitable within the framework of behavioral financial engineering.

Most of the literature dealing with the subjective evaluation of financial instruments on the basis of PT or CPT focuses on a discrete modeling of the probability distributions (cf. Breuer and Perst 2007 as well as Breuer, Hauten, and Kreuz 2009). However, a more precise description of real decision making requires a generalization of the evaluation approaches to continuous distributions. Therefore, Rieger and Wang (2008) have extended the PT to continuous distributions. Also the extension of CPT to continuous distributions does not cause any problems from a theoretical point of view. However, the cash flow distributions of structured products are usually only piecewise continuous with a finite number of jump discontinuities. Even if the implementation of such distributions is unproblematic on the basis of the continuous PT of Rieger and Wang (2008), the implementation on the basis of the continuous CPT is analytically and numerically hard to handle since necessary derivatives are not available. Against this background, we extend the CPT by considering piecewise continuous distributions. The resulting model is easy to implement and enables us to develop an evaluation formula of the piecewise continuous CPT for a broad class of structured products.

Furthermore, since decision makers are subject to the phenomenon of mental accounting in combination with hedonic framing, there exist approaches that incorporate these phenomena into the discrete version of the CPT (Breuer and Perst 2007). The concept of mental accounting introduced by Thaler (1980) and Tversky and Kahneman (1981) refers to the tendency of decision makers to handle different mental accounts to which they assign specific types of outcomes. Hedonic framing, according to Thaler (1985, 1999), describes the finding that decision makers take the frame – either "integration of certain payments over different mental accounts" or "separation of certain payments" – that leads to the maximum subjective value. Since a corresponding rule in connection with the (piecewise) continuous versions of PT and CPT is still missing, we develop a (piecewise) continuous hedonic framing rule applicable to both theories.

Consequently, we base the comparative analysis on EUT, PT, and CPT for discrete and piecewise continuous distributions with and without the consideration of hedonic framing. The analysis is applied to selected structured products. Precisely, we focus on so-called discount, index, capital guarantee, outperformance, and sprint certificates.

Summarized, the main contributions of the article are as follows: First, we develop an evaluation formula of CPT for a broad class of piecewise continuous cash flow distributions which is easy to implement. Second, we develop a piecewise continuous hedonic framing rule that is applicable to PT and CPT. Third, we apply the different versions of PT, CPT, and EUT on real-life certificates and analyze on the basis of different risk-return profiles, which theory is best in explaining the demand behavior of the investors for this asset class.

The remainder of this paper is structured as follows: Section 1 contains a brief description of PT and CPT for discrete and continuous distributions. Within the framework of Section 2 we develop the CPT model for piecewise continuous distributions. Section 3 deals with the development of a hedonic framing rule in case of PT and CPT for (piecewise) continuous distributions. In Section 4, we present the probability distributions of a broad class of structured products and selected real-life certificates which are necessary for the application of the PT and CPT. Section 5 describes the results of our comparative analysis and Section 6 concludes.

1 Prospect and Cumulative Prospect Theory

1.1 Prospect and Cumulative Prospect Theory for Discrete Distributions

Let $S = \{s_1, \ldots, s_n\}$ be the finite set of possible future states of nature and z_i describes the potential outcome that the occurrence of state s_i, $i = 1, \ldots, n$, entails. A key feature of PT is that decision makers evaluate the possible outcomes $z = (z_1, \ldots, z_n)$ in relation to a reference point x^{ref}. Whereas the traditional EUT is based on the final wealth level z, PT relates to changes in wealth defined as $x_i \coloneqq z_i - x^{ref}$, $i = 1, \ldots, n$. For simplification, it is assumed that $x_i \leq x_j$ for $i, j = 1, \ldots, n$ with $i < j$, and $x_i < 0$, $i = 1, \ldots, k$, and $x_j \geq 0$, $i = k+1, \ldots, n$. A prospect or a lottery is a vector $X = (x_1, p_1; \ldots; x_n, p_n)$, whereby p_i denotes the subjective probability a decision maker assigns to the occurrence of future state s_i, $i = 1, \ldots, n$. Moreover, PT allows for different risk behavior of decision makers in the range of losses ($x_i < 0$) and the range of gains ($x_i > 0$). The tendency of a risk-seeking attitude towards losses and a risk-averse attitude towards gains leads to a value function v that is convex over losses and concave over gains with a "kink" at $x = 0$. The probabilities $p = (p_1, \ldots, p_n)$ are transformed by an S-shaped probability weighting function w. This shape incorporates the empirical finding that decision makers often tend to overweight low and underweight high probabilities. Initially, Kahneman and Tversky (1979) define a PT-value for a lottery with at most two non-zero outcomes. Their formula is generalized to the lottery $X = (x_1, p_1; \ldots; x_n, p_n)$ with n possible outcomes

$$PT(X)=\sum_{i=1}^{n} w(p_i)\cdot v(x_i).^{1} \quad (1)$$

Tversky and Kahneman (1992) proposed the following functional form of the value function

$$v(x)=\begin{cases} x^{\alpha}, & x\geq 0,\\ -\lambda\cdot(-x)^{\beta}, & x<0, \end{cases} \quad (2)$$

with $\alpha \approx \beta \approx 0.88$ describing the degree of diminishing sensitivity and $\lambda \approx 2.25$ measuring the degree of loss aversion. There are several suggestions of the concrete formulation of the probability weighting functions w, e.g. the parametric operationalization of Tversky und Kahneman (1992)

$$w_{\gamma}(p)=\frac{p^{\gamma}}{\left(p^{\gamma}+(1-p)^{\gamma}\right)^{1/\gamma}}, \quad (3)$$

with $\gamma^{+}=0{,}61$ and $\gamma^{-}=0{,}69$ for gains and losses, respectively. In the following, we consider the parametric operationalization of the probability weighting function suggested by Lattimore, Baker, and Witte (1992)

$$w_{\delta,\gamma}(p)=\frac{\delta\cdot p^{\gamma}}{\delta\cdot p^{\gamma}+(1-p)^{\gamma}}=\begin{cases} w^{+}(p):=\dfrac{\delta^{+}\cdot p^{\gamma^{+}}}{\delta^{+}\cdot p^{\gamma^{+}}+(1-p)^{\gamma^{+}}}, & \text{if } x>0,\\ w^{-}(p):=\dfrac{\delta^{-}\cdot p^{\gamma^{-}}}{\delta^{-}\cdot p^{\gamma^{-}}+(1-p)^{\gamma^{-}}}, & \text{if } x<0. \end{cases} \quad (4)$$

The probability weighting function (4) depends on the parameters δ and γ that may be different for gains and losses. δ represents the so-called attractiveness that measures the absolute value of the probability weighting function. The parameter γ describes the discriminability that characterizes to which degree individuals distinguish among different probabilities (for more details see Gonzales and Wu 1999). In the following, we consider a situation under risk in which all probabilities are exogenously given. Therefore, the empirical study of Abdellaoui (2000) provides the parameter estimations $\delta^{+}=0.65$, $\gamma^{+}=0.60$, $\delta^{-}=0.84$ and $\gamma^{-}=0.65$. Since, in most real applications, the probabilities are not given exogenously, decision makers have to estimate them. The uncertainty in connection with the estimation of the probabilities is called ambiguity. The attitude towards ambiguity can be reflected in the shape of the probability weighting function. Depending on the investor's confidence in his or her own probability judgements, the parameters δ and γ have different values. Kilka and Weber (2001) point

[1] The generalization for n outcomes is suggested by Kahneman and Tversky (1979) and has been frequently used by various authors, e.g. Schneider and Lopes (1986), Wakker (1989), Camerer and Ho (1994).

out that the subjectively felt competence with regard to the source of uncertainty being evaluated has a positive influence on discriminability and attractiveness, since an increasing subjectively felt competence leads to a greater confidence in the investor's probability judgements, ceteris paribus. The empirical study of Abdellaoui et al. (2005) provides the typical parameter estimations $\delta^+ = 0.975$, $\gamma^+ = 0.832$, $\delta^- = 1.345$ and $\gamma^- = 0.842$ for decisions under ambiguity. The higher values for discriminability and attractiveness in comparison to decisions under risk lead to the conclusion that ambiguity decision makers systematically overestimate their competence. The values of Abdellaoui (2000) can be interpreted as a decision with low subjectively felt competence level, whereas the values according to Abdellaoui et al. (2005) refer to a decision with high subjectively felt competence level from the investor's point of view.

A fundamental critical point of PT is the violation of first-order stochastic dominance. On this account, Tversky and Kahneman (1992) propose a modified version of the PT, the Cumulative Prospect Theory, that is not marked with this blemish and is defined for more than two outcomes in the first place. For this purpose, the authors introduce decision weights π_i, i = 1, …, n, that are defined for cumulative probabilities in combination with probability weighting functions w^- and w^+ for losses and gains by

$$\pi_i := \begin{cases} \pi_i^- := w^-\left(p_1 + \ldots + p_i\right) - w^-\left(p_1 + \ldots + p_{i-1}\right), & \text{if } i \in \{1, \ldots, k\}, \\ \pi_i^+ := w^+\left(p_i + \ldots + p_n\right) - w^-\left(p_{i+1} + \ldots + p_n\right), & \text{if } i \in \{k+1, \ldots, n\}. \end{cases} \tag{5}$$

On this basis, the CPT-utility for a finite lottery $X = (x_1, p_1; \ldots; x_n, p_n)$ is given as

$$\mathrm{CPT}\left(X\right) = \sum_{i=1}^{n} \pi_i \cdot v\left(x_i\right). \tag{6}$$

1.2 Prospect and Cumulative Prospect Theory for Continuous Distributions

The great weakness of PT is that in case of an infinite number of (continuous distributed) outcomes the PT-formula (1) is not applicable. Therefore, Rieger and Wang (2008) have extended the PT-approach to non-discrete lotteries. Let $\tilde{Z}$ be the continuous outcome random variable and x^{ref} the reference point. The continuous random variable $\tilde{X}$ denotes the continuous relative outcome $\tilde{X} := \tilde{Z} - x^{ref}$. Let $f_{\tilde{X}}$ be the probability density function of $\tilde{X}$. According to Rieger and Wang (2008), the PT-utility for the continuous lottery $X^c := (\tilde{X}, f_{\tilde{X}})$ is expressed as

$$\mathrm{PT}\left(X^{c}\right)=\frac{\int v(x)\cdot f_{\tilde{X}}(x)^{\xi}\,dx}{\int f_{\tilde{X}}(x)^{\xi}\,dx}, \tag{7}$$

with $\xi \in (0,1)$ and

$$\lim_{\varepsilon\to 0}\frac{w(\varepsilon)}{\varepsilon^{\xi}}=C \tag{8}$$

for some finite number $C > 0$ and a probability weighting function w.

The PT-formulation (7) has some features worth mentioning. First, its derivation is based on the discrete PT-formulation of Karmakar (1978)

$$\mathrm{PT}(X)=\frac{\sum_{i=1}^{n} w(p_i)\cdot v(x_i)}{\sum_{i=1}^{n} w(p_i)}, \tag{9}$$

which differs from the well-known PT-valuation formula (1) suggested by Kahneman and Tversky (1979) in the normalization by the sum of weighted probabilities. Rieger and Wang (2008) show that this normalization is necessary to ensure a well-defined expression in the limit. Second, the PT-approach (7) is independent of the probability weighting function w and third, the probability density $f_{\tilde{X}}$ function has the exponent ξ. The second point can be considered as an advantage in the sense that any concrete probability weighting function needs to be determined. Conversely, the probability weighting function helps to display preference patterns. Against this background, the question arises how far this disclaimer of the probability weighting function constitutes a disadvantage. In certain ways, a transformation of the probability takes place by the transformation of the density function $(f_{\tilde{X}})^{\xi}$. To our knowledge, it is still not known if thereby the observed preference patterns are reflected. Additionally, from the third point arises the problem of choosing an arbitrary value $\xi \in (0,1)$ that has to satisfy condition (8). Within the framework of our investigation, we use the probability weighting function $w_{\delta,\gamma}$ according to (4). We set $\xi = \gamma$ and get

$$\lim_{\varepsilon\to 0}\frac{w_{\delta,\gamma}(\varepsilon)}{\varepsilon^{\gamma}}=\lim_{\varepsilon\to 0}\frac{\delta\cdot\varepsilon^{\gamma}}{\varepsilon^{\gamma}\cdot\left(\delta\cdot\varepsilon^{\gamma}+(1-\varepsilon)^{\gamma}\right)}=\delta. \tag{10}$$

Using the parameter estimation in the case of low subjective felt competence, that means the parameter estimation according to Abdellaoui (2000) ($\delta^{+} = 0.65$, $\gamma^{+} = 0.60$, $\delta^{-} = 0.84$ and $\gamma^{-} = 0.65$), and in the case of high subjective felt competence according to Abdellaoui (2005) ($\delta^{+} = 0.975$, $\gamma^{+} = 0.832$), condition (8) is respectively fulfilled. Since the PT-evaluation approach does not incorporate a probability weighting function, it seems to be impossible to depict different subjective felt competence levels as with CPT. However, we use the different determinations of γ for the interpretation of different competence levels. More precisely, we calculate PT-values for continuous lotteries using γ^+ = 0.60 and γ^- = 0.65 and associate calculated PT-values with an investor who subjectively feels a low competence. Conversely, we refer PT-values calculated on the basis of γ^+ = 0.832 to an investor with a high subjectively felt competence level.

Whereas the extension of PT to a continuous distribution is not necessarily easy, in the context of CPT, the transition from a discrete to a continuous probability distribution causes no problems. As in the case of PT, we consider the continuous random variable of relative outcomes $\tilde{X}$. The generalization of CPT (6) for the continuous lottery $X^c = (\tilde{X}, f_{\tilde{X}})$ is given as

$$\mathrm{CPT}\left(X^c\right) = \int_{-\infty}^{0} v(x) \cdot \frac{d}{dx} w^-\left(F_{\tilde{X}}(x)\right) dx - \int_{0}^{\infty} v(x) \cdot \frac{d}{dx} w^+\left(1 - F_{\tilde{X}}(x)\right) dx, \tag{11}$$

where w^- and w^+ are again the probability weighting functions for losses and gains, respectively, and $F_{\tilde{X}}$ denotes the probability distribution function of $\tilde{X}$ (e.g. Rieger and Wang 2006). Consequently, from a theoretical perspective, the generalization of CPT is possible without any problems. However, the same does not apply to the computational feasibility. CPT for continuous distributions is numerically and analytically quite difficult to implement. Rieger and Wang (2008) particularly underline this point when they emphasized that their continuous approach of PT outperforms the CPT-formula (11).

Barberis and Huang (2008) suggest a simplification of equation (11) by

$$\mathrm{CPT}\left(X^c\right) = -\int_{-\infty}^{0} w\left(F_{\tilde{X}}(x)\right) dv(x) + \int_{0}^{\infty} w\left(1 - F_{\tilde{X}}(x)\right) dv(x). \tag{12}$$

For this approach, the authors use the probability weighting function (3) according to Tversky and Kahneman (1992) and assume that the parameter γ (controlling the discriminability) does not differ for gains and losses, i.e. $\gamma^+ = \gamma^-$. On this basis it follows that the probability weighting functions for gains and losses also correspond, $w^+ = w^-$. Since, however, parameter estimations for γ^+ and γ^- provide slightly different values (e.g. Tversky and Kahneman 1992, Abdellaoui 2000), the CPT-formulation (12) can only be considered as an approximation.

However, the numerical problems when implementing the CPT are not considerable if the distribution $F_{\tilde{X}}$ is continuous and $F_{\tilde{X}}(x) \in (0,1)$ for all x. The problem within the framework of the numerical implementation of the CPT arises if the distribution $F_{\tilde{X}}$ is piecewise continuous with a finite number of jump discontinuities and $F_{\tilde{X}}(x) = 0$ or $F_{\tilde{X}}(x) = 1$ for at least an x. The reason is that in the case of jump discontinuities the first derivatives of the function compositions $w^- \circ F_{\tilde{X}}$ and $w^+ \circ (1 - F_{\tilde{X}})$ do not exist in the whole domain and the probability weighting function according to (4) is not differentiable at p = 0 and p = 1. Since these problems do not concern the PT we only show how to handle these problems in the context of the CPT in the next section.

2 Cumulative Prospect Theory for Piecewise Continuous Distributions

We consider a general class of probability distribution functions $F_{\tilde{X}}$ which can be represented for $x \in \mathbb{R}$ according to

$$F_{\tilde{X}}(x) = \sum_{i=0}^{n+1} \alpha_i \cdot H(x - \xi_i) + \sum_{j=1}^{m} \beta_j \cdot F_{\tilde{S}}(\phi_j(x)) \cdot H(x - \psi_j) \qquad (13)$$

where $n, m \in \mathbb{N}$, $\alpha_i, \beta_j \in \mathbb{R}$, $\xi_i, \psi_j \in \mathbb{R} \cup \{-\infty\}$, $\phi_j : \mathbb{R} \to \mathbb{R}$ differentiable functions (i = 0, …, n+1; j = 1,…, m), $F_{\tilde{S}} : \mathbb{R} \to (0,1)$ a differentiable probability distribution, and H the Heaviside function[2]. Furthermore, we assume $\xi_0 < \xi_i < \xi_{n+1}$ for all i = 1, …, n, $\xi_0 \le \psi_j \le \xi_{n+1}$ for all j = 1, …, m, and $F_{\tilde{X}}(x) = 1$ for $x \ge \xi_{n+1}$. This immediately implies $F_{\tilde{X}}(x) = 0$ for $x < \xi_0$. If $\alpha_0 = 0$ and $\alpha_{n+1} = 0$ we set $\xi_0 = -\infty$ and $\xi_{n+1} = +\infty$, respectively. Altogether, we consider a broad class of distribution functions including piecewise continuous functions with a finite number of jump discontinuities.

As mentioned above, we consider the Dirac delta distribution δ because one relevant characteristic of this distribution is

$$\int_{-\infty}^{\infty} \phi(y) \cdot \delta(y - x)\, dy = \phi(x) \qquad (14)$$

[2] The Heaviside function is a non-continuous function whose value is zero for negative arguments and one for positive arguments. The value of the function at zero is ambiguous. Usually, the value is defined as H(0) = 0, H(0) = 0.5, or H(0) = 1. Within the framework of the present paper we set H(0) = 1.

for an integrable function $\phi: \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$.[3] This issue leads to an interesting property of the Heaviside function if we only consider functions ϕ with $\lim_{x\to\infty}(\phi(x)) = 0$:

$$\int_{-\infty}^{\infty} \phi(y) \cdot \frac{d}{dy} H(y-x)\,dy = -\int_{-\infty}^{\infty} \phi'(y) \cdot H(y-x)\,dy = -\int_{x}^{\infty} \phi'(y)dy = \phi(x) = \int_{-\infty}^{\infty} \phi(y) \cdot \delta(y-x)\,dy. \quad (15)$$

Consequently, the first derivative of the Heaviside function H can be identified (under the integral) by the Dirac delta distribution δ.

Since $F_{\tilde{X}}(x) = 0$ for all $x \in (-\infty, \xi_0)$, $F_{\tilde{X}}(x) \in (0,1)$ for all $x \in [\xi_0, \xi_{n+1})$, and $F_{\tilde{X}}(x) = 1$ for all $x \in [\xi_{n+1}, \infty)$ (with $(-\infty,-\infty) := [\infty,\infty) := \emptyset$ and $[-\infty,\xi_{n+1}) := (-\infty, \xi_{n+1})$), we can transform $F_{\tilde{X}}$ according to

$$F_{\tilde{X}}(x) = H(x-\xi_{n+1}) + \hat{F}_{\tilde{X}}(x) \cdot (H(x-\xi_0) - H(x-\xi_{n+1})), \quad (16)$$

where

$$\hat{F}_{\tilde{X}}(x) = \begin{cases} F_{\tilde{X}}(\xi_0) \quad (>0), & x < \xi_0, \\ F_{\tilde{X}}(x), & \xi_0 \le x < \xi_{n+1}, \\ \lim_{\varepsilon\to 0}\left(F_{\tilde{X}}(\xi_{n+1} - |\varepsilon|)\right) \quad (<1), & x \ge \xi_{n+1}. \end{cases} \quad (17)$$

Since $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$, we may represent $w^- \circ F_{\tilde{X}}$ and $w^+ \circ (1 - F_{\tilde{X}})$ according to

$$w^-(F_{\tilde{X}}(x)) = H(x-\xi_{n+1}) + w^-\left(\hat{F}_{\tilde{X}}(x)\right) \cdot (H(x-\xi_0) - H(x-\xi_{n+1})) \quad (18)$$

and

$$w^+(1-F_{\tilde{X}}(x)) = (1 - H(x-\xi_0)) + w^+\left(1-\hat{F}_{\tilde{X}}(x)\right) \cdot (H(x-\xi_0) - H(x-\xi_{n+1})). \quad (19)$$

Against this background, the derivatives of the probability weighting functions (that are only applicable under the integral) can be determined as follows:

$$\begin{aligned} \frac{d}{dx} w^-(F_{\tilde{X}}(x)) = {} & \delta(x-\xi_{n+1}) + w^{-\prime}\left(\hat{F}_{\tilde{X}}(x)\right) \cdot \hat{f}_{\tilde{X}}(x) \cdot (H(x-\xi_0) - H(x-\xi_{n+1})) \\ & + w^-\left(\hat{F}_{\tilde{X}}(x)\right) \cdot (\delta(x-\xi_0) - \delta(x-\xi_{n+1})) \end{aligned} \quad (20)$$

and

$$\begin{aligned} \frac{d}{dx} w^+(1-F_{\tilde{X}}(x)) = {} & -\delta(x-\xi_0) - w^{+\prime}\left(1-\hat{F}_{\tilde{X}}(x)\right) \cdot \hat{f}_{\tilde{X}}(x) \cdot (H(x-\xi_0) - H(x-\xi_{n+1})) \\ & + w^+\left(1-\hat{F}_{\tilde{X}}(x)\right) \cdot (\delta(x-\xi_0) - \delta(x-\xi_{n+1})), \end{aligned} \quad (21)$$

[3] See, e.g., Spall (2003), p. 416.

with $\delta(\infty) = \delta(-\infty) = 0$. For this reason, we may substitute the derivatives $d\,w^{-}\left(F_{\tilde{X}}(x)\right)/dx$ and $d\,w^{+}\left(1-F_{\tilde{X}}(x)\right)/dx$ in CPT-representation (11) by (20) and (21) and get under consideration of the δ-property (14):[4]

$$
\begin{aligned}
&\mathrm{CPT}\left(X^{c}\right)\\
&=\int_{\xi_0^+}^{\xi_{n+1}^-} v(x)\cdot w'\left(\hat{F}_{\tilde{X}}(x)\right)\cdot \hat{f}_{\tilde{X}}(x)\,dx + v(\xi_{n+1})\cdot\left(1-H(\xi_{n+1})\right) + v(\xi_0)\cdot H(\xi_0) \qquad (22)\\
&\quad + v(\xi_0)\cdot w\left(\hat{F}_{\tilde{X}}(\xi_0)\right) - v(\xi_{n+1})\cdot w\left(\hat{F}_{\tilde{X}}(\xi_{n+1})\right),
\end{aligned}
$$

where

$$
w(p) := \begin{cases} w^{-}(p), & \text{if } x \le 0, \\ -w^{+}(1-p), & \text{if } x > 0. \end{cases} \qquad (23)
$$

It remains to determine the integral within the frame of (22). Using the characteristic (15) of the delta distribution leads to the following representation of the density function $\hat{f}_{\tilde{X}}$ which in turn can be applied under the integral for all $x \in (\xi_0, \xi_{n+1})$:[5]

$$
\hat{f}_{\tilde{X}}(x) = \sum_{i=1}^{n} \alpha_i \cdot \delta(x-\xi_i) + \sum_{j=1}^{m} \beta_j \cdot f_{\tilde{S}}\left(\phi_j(x)\right)\cdot \phi_j'(x)\cdot H\left(x-\psi_j\right) + \sum_{j=1}^{m} \beta_j \cdot F_{\tilde{S}}\left(\phi_j(x)\right)\cdot \delta\left(x-\psi_j\right). \qquad (24)
$$

On this basis and under consideration of property (14),

$$
\begin{aligned}
&\int_{\xi_0^+}^{\xi_{n+1}^-} v(x)\cdot w'\left(\hat{F}_{\tilde{X}}(x)\right)\cdot \hat{f}_{\tilde{X}}(x)\,dx\\
&= \sum_{i=1}^{n} \alpha_i \cdot v(\xi_i)\cdot w'\left(\hat{F}_{\tilde{X}}(\xi_i)\right) + \sum_{j=1}^{m} \beta_j \cdot v(\psi_j)\cdot F_{\tilde{S}}\left(\phi_j(\psi_j)\right)\cdot w'\left(\hat{F}_{\tilde{X}}(\psi_j)\right) \qquad (25)\\
&\quad + \sum_{j=1}^{m} \beta_j \cdot \int_{\psi_j^+}^{\xi_{n+1}^-} v(x)\cdot w'\left(\hat{F}_{\tilde{X}}(x)\right)\cdot f_{\tilde{S}}\left(\phi_j(x)\right)\cdot \phi_j'(x)\,dx.
\end{aligned}
$$

Altogether, we get the following CPT-formula for piecewise continuous distributions:

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[4] We define $\int_{a^+}^{b^-} := \lim_{\varepsilon\to 0} \int_{a+|\varepsilon|}^{b-|\varepsilon|}$.

[5] $f_{\tilde{S}}$ denotes the density function corresponding to $F_{\tilde{S}}$.

$$
\begin{aligned}
&\mathrm{CPT}\left(X^c\right)\\
&=\sum_{i=1}^{n}\alpha_i\cdot v(\xi_i)\cdot w'\left(\hat{F}_{\tilde{X}}(\xi_i)\right)+\sum_{j=1}^{m}\beta_j\cdot v(\psi_j)\cdot F_{\tilde{S}}\left(\phi_j(\psi_j)\right)\cdot w'\left(\hat{F}_{\tilde{X}}(\psi_j)\right)\\
&+\sum_{j=1}^{m}\beta_j\cdot\int_{\psi_j^+}^{\xi_{n+1}^-} v(x)\cdot w'\left(\hat{F}_{\tilde{X}}(x)\right)\cdot f_{\tilde{S}}\left(\phi_j(x)\right)\cdot\phi_j'(x)\,dx+v(\xi_{n+1})\cdot(1-H(\xi_{n+1}))+v(\xi_0)\cdot H(\xi_0)\\
&+v(\xi_0)\cdot w\left(\hat{F}_{\tilde{X}}(\xi_0)\right)-v(\xi_{n+1})\cdot w\left(\hat{F}_{\tilde{X}}(\xi_{n+1})\right).
\end{aligned}
\tag{26}
$$

The remaining integral is determinable on the basis of the CPT formula for continuous distributions without further ado. As a result, we have succeeded in developing a CPT formula for piecewise continuous distributions. The application of this formula to a broad class of structured products is shown in Section 4. Previously, we show how to consider mental accounting within the framework of PT and CPT.

3 Mental accounting associated with Prospect Theory and Cumulative Prospect Theory

Within the subjective assessment of outcomes, mental accounting and hedonic framing play an important rule. The concept of mental accounting introduced by Thaler (1980, 1985) refers to the tendency of decision makers to administer different types of mental accounts to which they assign specific outcomes. The outcomes may be mentally aggregated in a single account or may be mentally separated over the different ones. Hedonic framing according to Thaler (1985, 1999) describes the finding that decision makers take the frame that is accompanied by the maximization of subjective assessment. Hence, they combine or separate outcomes in such a way that the kind of mental representation – integration or segregation – provides the subjectively highest possible value. A formal description of the evaluation of two riskless outcomes $\overline{x}$ and $\overline{y}$ according to the principle of hedonic framing is given by Thaler (1999) as

$$\hat{v}\left(\overline{x},\overline{y}\right):=\max\left\{v\left(\overline{x}+\overline{y}\right),v\left(\overline{x}\right)+v\left(\overline{y}\right)\right\}.^{6} \tag{27}$$

The expression $v(\overline{x}+\overline{y})$ describes the integration of the outcomes $\overline{x}$ and $\overline{y}$ and the term $v(\overline{x})+v(\overline{y})$ stands for their segregation. This raises the question of how to implement mental accounting and hedonic framing if at least one outcome is uncertain. In this context, Breuer

[6] Thaler (1999) uses the operator "&" to express the hedonic framing rule, $v(\Delta x \& \Delta y)$:= $\max\{v(\Delta x+\Delta y),v(\Delta x)+v(\Delta y)\}$. Since we think that the description by a two-dimensional function is better suited, we introduce the notation $\hat{v}(\Delta x,\Delta y)$.

and Perst (2007) suggest the following hedonic framing rule, that refers to the assessment of an uncertain discrete lottery $X = (x_1, p_1; \ldots; x_n, p_n)$ in combination with a certain outcome $\overline{y}$:

$$\mathrm{CPT}(X,\overline{y}) := \sum_{i=1}^{n} \pi_i \cdot \hat{v}(x_i,\overline{y}) + \left(1-\sum_{i=1}^{n}\pi_i\right)\cdot \hat{v}(0,\overline{y}). \tag{28}$$

The last summand on the right hand side of (28) is necessary to ensure the result $\mathrm{CPT}(X,\overline{y}) = v(\overline{y})$, if $x_i \to 0$ for all i = 1, …, n. Additionally, it offsets the weighting of the riskless outcome $\overline{y}$ with the decision weight π.

In contrast, we develop a combination rule that reflects mental accounting and hedonic framing in a continuous model environment. For this purpose, we consider a continuous lottery $X^c = (\tilde{X}, f_{\tilde{X}})$ and a certain outcome $\overline{y}$. The case of integration is captured by the random variable $\tilde{X}+\overline{y}$ and the corresponding probability density function is denoted by $f_{\tilde{X}+\overline{y}}$. We suggest the following CPT-evaluation approach against the background of hedonic framing:

$$\begin{aligned}&\mathrm{CPT}(X^c,\overline{y})\\ &:= \mathrm{CPT}\left(\tilde{X}+\overline{y}, f_{\tilde{X}+\overline{y}} \,\middle|\, \underbrace{\hat{v}(\tilde{X},\overline{y}) = v(\tilde{X}+\overline{y})}_{\text{integration field}}\right) + \mathrm{CPT}\left(\tilde{X}, f_{\tilde{X}} \,\middle|\, \underbrace{\hat{v}(\tilde{X},\overline{y}) = v(\tilde{X}) + v(\overline{y})}_{\text{segregation field}}\right) + \mathrm{CPT}(\overline{y}).\end{aligned} \tag{29}$$

In areas where realizations x of the random variable $\tilde{X}$ are such that integration $v(x+\overline{y})$ takes place, we represent integration by the random variable $\tilde{X}+\overline{y}$ and calculate the CPT-value for this aggregation. In areas where the realizations x provide segregation $v(x)+v(\overline{y})$, we fulfill the separate mental accounting in conducting a separate CPT-assessment of the respective accounts. This means, we separately evaluate the certain outcome $\overline{y}$ and the continuous lottery $X^c = (\tilde{X}, f_{\tilde{X}})$ according to CPT as intuitively expected. In this manner, the problem of weighting the certain outcome $\overline{y}$ with a decision weight as presented in Breuer and Perst (2007) does not arise. If $x \to 0$ for all realizations we immediately get $\mathrm{CPT}(X^c,\overline{y}) = \mathrm{CPT}(\overline{y})$.

As already mentioned, within PT as well as CPT gains and losses are evaluated. Thereby, gain and loss are determined by the absolute values of the realizations of the continuous random variable $\tilde{X}$. In the case of hedonic framing, in which two accounts are combined with the intention of an optimal subjective assessment, it is necessary to define what is meant by gain and loss. Within this context, gain and loss do not relate to the absolute values x and $\bar{y}$, but to the subjectively felt values. Hence, the probability weighting function w^- refers to the subjectively felt loss $v(x+\bar{y}) < 0$ or $v(x)+v(\bar{y}) < 0$, respectively, and w^+ refers to the subjectively felt gains $v(x+\bar{y}) \geq 0$ or $v(x) + v(\bar{y}) \geq 0$, respectively. For illustration, we consider the lottery $X^c = (\tilde{X}, f_{\tilde{X}})$ and the certain gain $\bar{y}$. Let this certain gain be described by a random variable $\tilde{Y}$ with probability distribution $F_{\tilde{Y}}$ and density $f_{\tilde{Y}}$. The application of equation (11) in combination with equation (29) leads to the general CPT-presentation

$$
\begin{aligned}
&\mathrm{CPT}\left(X^c,\bar{y}\right)\\
&= \int\limits_{\substack{\{x|\hat{v}(x+\bar{y})\\ =v(x+\bar{y})<0\}}} v(x+\bar{y})\cdot\frac{d}{dx}w^-\left(F_{\tilde{X}+\bar{y}}(x+\bar{y})\right)dx - \int\limits_{\substack{\{x|\hat{v}(x+\bar{y})\\ =v(x+\bar{y})>0\}}} v(x+\bar{y})\cdot\frac{d}{dx}w^+\left(1-F_{\tilde{X}+\bar{y}}(x+\bar{y})\right)dx\\
&\quad+\int\limits_{\substack{\{x|\hat{v}(x+\bar{y})\\ =v(x)+v(\bar{y})<0\}}} v(x)\cdot\frac{d}{dx}w^-\left(F_{\tilde{X}}(x)\right)dx - \int\limits_{\substack{\{x|\hat{v}(x+\bar{y})\\ =v(x)+v(\bar{y})>0\}}} v(x)\cdot\frac{d}{dx}w^+\left(1-F_{\tilde{X}}(x)\right)dx\\
&\quad-\int\limits_{\mathbb{R}_0^+} v(y)\cdot\frac{d}{dy}w^+\left(1-F_{\tilde{Y}}(y)\right)dy.
\end{aligned}
\tag{30}
$$

The CPT-value of the certain outcome $\bar{y}$ is[7]

$$
\mathrm{CPT}(\bar{y}) = -\int\limits_{\mathbb{R}_0^+} v(y)\cdot\frac{d}{dy}w^+\left(1-F_{\tilde{Y}}(y)\right)dy = v(\bar{y}). \tag{31}
$$

This kind of mental accounting against the background of hedonic framing can also be assigned to PT for continuous distributions. The corresponding combination rule is analog

$$
\begin{aligned}
&\mathrm{PT}\left(X^c,\bar{y}\right)\\
&= \mathrm{PT}\left(\tilde{X}+\bar{y}\;\middle|\;\underbrace{\hat{v}(\tilde{X},\bar{y}) = v(\tilde{X}+\bar{y})}_{\text{integration field}}\right) + \mathrm{PT}\left(\tilde{X}\;\middle|\;\underbrace{\hat{v}(\tilde{X},\bar{y}) = v(\tilde{X})+v(\bar{y})}_{\text{segregation field}}\right) + \mathrm{PT}(\bar{y}).
\end{aligned}
\tag{32}
$$

However, in contrast to CPT (31), the last summand of (32) cannot be simplified to $v(\bar{y})$.

4 PT and CPT evaluation of the considered certificates

4.1 Cash flow distribution of a broad class of structured products

As already mentioned, we apply the PT- and CPT-formulas to structured products. Precisely, we consider European-style certificates whose underlying is a stock index. The final reference

[7] The proof is given in Appendix 1.

level of the index is described by the random variable $\tilde{S}_T$, where T stands for the time of maturity. The random variable $\tilde{Z}_T^{cer}$ denotes the payment structure of the certificate (*cer*) and $\tilde{X}_T^{cer} = \tilde{Z}_T^{cer} - x^{ref}$ the corresponding relative outcome. In the following, we consider certificates whose relative outcome corresponds to[8]

$$\tilde{X}_T^{cer} = \tilde{Z}_T^{cer} - x^{ref} = \sum_{i=1}^{n} (a_i + b_i \cdot g_i(\tilde{S}_T)) \cdot \chi_{[x_i, y_i)}(\tilde{S}_T), \qquad (33)$$

in which $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$, g_i a strictly increasing function, $x_i \in \mathbb{R}_0^+$, and $y_i \in (x_i, \infty) \cup \{\infty\}$ ($i = 1, \dots, n$). Furthermore, $x_1 < \dots < x_n$ and $\bigcup_{i=1,\dots,n} [x_i, y_i) = \mathbb{R}_0^+$. On the basis of a given distribution of $\tilde{S}_T$, we are able to determine the probability distribution $F_{\tilde{X}_T^{cer}}$:[9]

$$\begin{aligned} &F_{\tilde{X}_T^{cer}}(x) \\ &= \sum_{i=1}^{n} F_{\tilde{S}_T}(y_i) \cdot H\big(x - (a_i + b_i \cdot g_i(y_i))\big) - \sum_{i=1}^{n} F_{\tilde{S}_T}(x_i) \cdot H\big(x - (a_i + b_i \cdot g_i(x_i))\big) \qquad (34) \\ &\quad + \sum_{j=1}^{n} F_{\tilde{S}_T}\left(g_j^{-1}\left(\frac{x - a_j}{b_j}\right)\right) \cdot H\big(x - (a_j + b_j \cdot g_j(x_j))\big) - \sum_{j=1}^{n} F_{\tilde{S}_T}\left(g_j^{-1}\left(\frac{x - a_j}{b_j}\right)\right) \cdot H\big(x - (a_j + b_j \cdot g_j(y_j))\big). \end{aligned}$$

It is to be noted that $F_{\tilde{S}_T}\left(g_i^{-1}\left((x - a_j)/b_j\right)\right) := 1$ if $b_j = 0$ which implies the respective terms in the last two lines to cancel out. Obviously, distribution (34) is of type (13) if we identify ξ_i of (13) with $a_i + b_i \cdot g_i(x_i)$ and $a_i + b_i \cdot g_i(y_i)$ of (34), respectively. Precisely, $\xi_0 = a_1 + b_1 \cdot g_1(x_1)$. ψ_j can be identified by $a_j + b_j \cdot g_j(x_j)$ and $a_j + b_j \cdot g_j(y_j)$, respectively. Furthermore, α_i corresponds to $-F_{\tilde{S}_T}(x_i)$ and $F_{\tilde{S}_T}(y_i)$, respectively. In addition, $\beta_j = 1$ or $\beta_j = -1$, and the differentiable function $\phi_j(x)$ can be assigned to $g_i^{-1}\left((x - a_j)/b_j\right)$. Altogether, the CPT formula (26) for piecewise continuous distributions is applicable to structured products with relative outcomes according to (33).

In the next section we show that relative outcomes of type (33) are common when considering real-life certificates.

4.2 Cash flow distributions of real-life certificates

Due to the plethora of different designs (e.g. certificates with capital protection, barrier properties) available on the market, our investigation focuses on the specific types of certificates

[8] χ_A stands for the indicator function of set A.

[9] The proof is given in the Appendix 2.

described below: index (*ind*), discount (*dis*), capital guarantee (*gua*), outperformance (*out*) and sprint certificate (*spr*) which are all European-style derivatives with cash flows of type (33). For the sake of simplicity, we make the assumption, that the investor intend to put his whole wealth A_0 into a single product and can purchase any number $m = A_0/CER_0$ of a certificate, where CER_0 is the price of the certificate at valuation date $t = 0$.[10] Furthermore, M denotes the respective subscription ratio. On this basis, the cash flows at maturity are as follows.

Index certificate

The investor participates in the development of the underlying.

$$\tilde{Z}_T^{ind} = \frac{A_0}{IND_0} \cdot M \cdot \tilde{S}_T . \tag{35}$$

Discount certificate

The investor receives the minimum of a cap (CAP) and the current stock index $\tilde{S}_T$.

$$\tilde{Z}_T^{dis} = \frac{A_0}{DIS_0} \cdot M \cdot \min\left(\tilde{S}_T, CAP\right). \tag{36}$$

Capital guarantee certificate

The capital guarantee certificate ensures a capital protection in the amount of CP·NV, independent of the development of the underlying. Here, CP denotes the percentage of the capital protection and NV the nominal value. If the current stock index $\tilde{S}_T$ is not smaller than the initial reference level (B), the investor additionally participates with a participation rate PR on the positive development of the underlying.

$$\tilde{Z}_T^{gua} = \frac{A_0}{GUA_0} \cdot NV \cdot CP + \frac{A_0}{GUA_0} \cdot NV \cdot PR \cdot \frac{\left(\tilde{S}_T - B\right)}{B} \cdot \chi_{[B,\infty)}\left(\tilde{S}_T\right). \tag{37}$$

Outperformance certificate

If the price of the underlying is not greater than the outperformance level (OL), the investor receives a payout according to the price of the underlying. If the price of the underlying is higher than the outperformance level, the certificate provides a higher participation (PR > 1) in the underlying performance.

$$\tilde{Z}_T^{out} = \frac{A_0}{OUT_0} \cdot M \cdot \tilde{S}_T \cdot \chi_{[0,OL)}\left(\tilde{S}_T\right) + \frac{A_0}{OUT_0} \cdot M \cdot \left(OL + PR \cdot \left(\tilde{S}_T - OL\right)\right) \cdot \chi_{[OL,\infty)}\left(\tilde{S}_T\right). \tag{38}$$

Sprint certificate

The sprint certificate functions similarly to the outperformance certificate. If the price of the underlying is below the sprint level (SL), the investor receives a refund corresponding to the

[10] In the following CER will be replaced by the abbreviation of the respective certificate name.

price of the underlying. Within the price range of sprint level and cap, the investor profits disproportionately with a leverage of 2 from the performance of the underlying. However, the investor does not participate in price increases above the cap.

$$\begin{aligned}\tilde{Z}_T^{spr} &= \frac{A_0}{SPR_0}\cdot M\cdot \tilde{S}_T\cdot \chi_{[0,SL)}\left(\tilde{S}_T\right)+\frac{A_0}{SPR_0}\cdot M\cdot\left(2\cdot\tilde{S}_T-SL\right)\cdot\chi_{[SL,CAP)}\left(\tilde{S}_T\right)\\ &+\frac{A_0}{SPR_0}\cdot M\cdot\left(2\cdot CAP-SL\right)\cdot\chi_{[CAP,\infty)}\left(\tilde{S}_T\right).\end{aligned} \tag{39}$$

In order to apply PT and CPT to each product, we take investor's initial wealth A_0 as the reference point x^{ref}. Consequently, $\tilde{X}_T^{cer}=\tilde{Z}_T^{cer}-A_0$ denotes the relative outcome which can easily be determined for each product. For illustration, we consider the discount certificate's relative outcome variable

$$\begin{aligned}\tilde{X}_T^{dis} &= m\cdot M\cdot \min\left(\tilde{S}_T,CAP\right)-A_0\\ &=\left(m\cdot M\cdot\tilde{S}_T-A_0\right)\cdot\chi_{[0,CAP)}\left(\tilde{S}_T\right)+\left(m\cdot M\cdot CAP-A_0\right)\cdot\chi_{[CAP,\infty)}\left(\tilde{S}_T\right).\end{aligned} \tag{40}$$

Thus, the discount certificate leads to an outcome in accordance with (33) on the basis of the parameters

$$\begin{aligned}&a_1=-A_0,\ b_1=m\cdot M,\ x_1=0,\ y_1=CAP,\ g_1(x)=x,\\ &a_2=m\cdot M\cdot K-A_0,\ b_2=0,\ x_2=CAP,\ y_2=\infty,\ g_2(x)=x.\end{aligned}$$

The parameter constellations of all certificates under consideration are presented in Table 1.

Cer	**Coefficients**
Ind	$a_1=-A_0,\ b_1=m\cdot M,\ x_1=0,\ y_1=\infty,\ g_1(x)=x$
Dis	$a_1=-A_0,\ b_1=m\cdot M,\ x_1=0,\ y_1=CAP,\ g_1(x)=x,$ $a_2=m\cdot M\cdot CAP-A_0,\ b_2=0,\ x_2=CAP,\ y_2=\infty,\ g_2(x)=x$
Gua	$a_1=m\cdot NV\cdot(CP-PR),\ b_1=m\cdot NV\cdot PR/B,\ x_1=B,\ y_1=\infty,\ g_1(x)=x,$ $a_2=m\cdot NV\cdot CP-A_0,\ b_2=0,\ x_2=0,\ y_2=B,\ g_2(x)=x$
Out	$a_1=-A_0,\ b_1=m\cdot M,\ x_1=0,\ y_1=OL,\ g_1(x)=x,$ $a_2=m\cdot M\cdot OL\cdot(1-PR)-A_0,\ b_2=m\cdot M\cdot PR,\ x_2=OL,\ y_2=\infty,\ g_2(x)=x$
Spr	$a_1=-A_0,\ b_1=m\cdot M,\ x_1=0,\ y_1=SL,\ g_1(x)=x,$ $a_2=-m\cdot M\cdot SL-A_0,\ b_2=2\cdot m\cdot M,\ x_2=SL,\ y_2=CAP,\ g_2(x)=x,$ $a_3=m\cdot M\cdot(2\cdot CAP-SL)-A_0,\ b_3=0,\ x_3=CAP,\ y_3=\infty,\ g_3(x)=x$

Table 1: Parameter constellations of the certificates according to (33)

Furthermore, it is quite easy to determine the cumulative probability and consequently the probability density function for any kind of presented certificates on the basis of (33). For example, the application of formula (34) to the discount certificate generates the probability distribution

$$F_{\tilde{X}_T^{dis}}(x) = \left(1 - F_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right)\right) \cdot \chi_{[m \cdot M \cdot CAP - A_0, \infty)}(x) + F_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right) \tag{41}$$

which immediately leads to the corresponding probability density function

$$\begin{aligned} f_{\tilde{X}_T^{dis}}(x) = &\left(1 - F_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right)\right) \cdot \delta\left(x - (m \cdot M \cdot CAP - A_0)\right) \\ &+ \frac{1}{m \cdot M} \cdot f_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right) \cdot \left(1 - \chi_{[m \cdot M \cdot CAP - A_0, \infty)}(x)\right). \end{aligned} \tag{42}$$

On the basis of formula (34) under consideration of Table 1, the respective distributions for all considered certificates are easily established (see Table 2).[11]

Cer	Probability distribution function
Ind	$F_{\tilde{X}_T^{ind}}(x) = F_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right)$
Dis	$F_{\tilde{X}_T^{dis}}(x) = \left(1 - F_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right)\right) \cdot \chi_{[m \cdot M \cdot CAP - A_0, \infty)}(x) + F_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right)$
Gua	$F_{\tilde{X}_T^{gua}}(x) = F_{\tilde{S}_T}\left(\frac{B \cdot (x + A_0 - m \cdot NV \cdot CP)}{m \cdot NV \cdot PR} + B\right) \cdot \chi_{[m \cdot NV \cdot CP - A_0, \infty)}(x)$
Out	$F_{\tilde{X}_T^{out}}(x) = F_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right) \cdot \left(1 - \chi_{[m \cdot M \cdot OL - A_0, \infty)}(x)\right) + F_{\tilde{S}_T}\left(\frac{x - m \cdot M \cdot OL \cdot (1 - PR) + A_0}{m \cdot M \cdot PR}\right) \cdot \chi_{[m \cdot M \cdot OL - A_0, \infty)}(x)$
Spr	$F_{\tilde{X}_T^{spr}}(x) = \left(1 - \chi_{[m \cdot M \cdot SL - A_0, \infty)}(x)\right) \cdot F_{\tilde{S}_T}\left(\frac{x + A_0}{m \cdot M}\right) + \chi_{[m \cdot M \cdot SL - A_0, \infty)}(x) \cdot F_{\tilde{S}_T}\left(\frac{x + m \cdot M \cdot SL + A_0}{2 \cdot m \cdot M}\right)$ $- \chi_{[m \cdot M \cdot (2 \cdot CAP - SL) - A_0, \infty)}(x) \cdot F_{\tilde{S}_T}\left(\frac{x + m \cdot M \cdot SL + A_0}{2 \cdot m \cdot M}\right) + \chi_{[m \cdot M \cdot (2 \cdot CAP - SL) - A_0, \infty)}(x)$

Table 2: Probability distribution functions of the different types of certificates

Finally, we need the distribution of the underlying price $\tilde{S}_T$ at maturity. Precisely, we assume $\tilde{S}_T$ to follow a lognormal distribution which is based on the assumption that $(\tilde{S}_t)_t$ follows a geometric Brownian motion (Black and Scholes 1973) and $\ln(\tilde{S}_T)$ is therefore normally distributed,

$$\ln(\tilde{S}_T) \sim \mathcal{N}\left(\ln(S_0) + (\mu_{exp} - \sigma_{exp}^2 / 2) \cdot T, \sigma_{exp}^2 \cdot T\right), \tag{43}$$

where μ_{exp} describes the subjectively estimated expected continuous return of the underlying and σ_{exp} denotes the subjectively estimated volatility from a private investor's point of view. Since we compare the continuous and the discrete variant of PT and CPT we additionally consider the binomial distribution. In this case, we apply a discrete-space approximation according to Cox, Ross, Rubinstein (1979). Following Breuer and Perst (2007), we take the parame-

[11] The corresponding densities can be easily determined analogous to the derivation of (24).

ter specification $u = \exp(\mu_{exp}\Delta t + \sigma_{exp}\sqrt{\Delta t})$, $d = \exp(\mu_{exp}\Delta t - \sigma_{exp}\sqrt{\Delta t})$ and p = 0.5. This leads to the discrete lottery $X^{cer} = (x_1, p_1; \ldots; x_n, p_n)$ with

$$S_i = S_0 u^k d^{i-k},\ x_i = \tilde{X}_T^{cer}(S_i),\ p_i = \binom{i}{k} p^k (1-p)^{i-k},\ i = 1,\ldots,n;\ k = 1,\ldots,i.$$

In the next subsection we explain how to consider mental accounting within the framework of real-life certificates.

4.3 Certificates and mental accounting

Within the framework of the considered certificates mental accounting is only applicable to the capital guarantee certificate since this is the only certificate offering the investor a capital protection in the amount of CP·NV, regardless of the development of the underlying. Additionally, the investor participates in the positive development of the underlying if the final reference level $\tilde{S}_T$ is greater than the initial reference level B. This implies that the payment structure $\tilde{Z}_T^{gua}$ of the capital guarantee certificate can be divided into a certain and an uncertain part, which leads to the possibility of mental accounting. In this case, the investor can assess on the one hand the certain payment structure

$$Z_T^{gua,bond} = m \cdot CP \cdot NV, \tag{44}$$

which he books for example on a kind of "bond account" and on the other hand the uncertain refund given as

$$\tilde{Z}_T^{gua,stock} = m \cdot NV \cdot PR \cdot \left(\frac{\tilde{S}_T - B}{B} \right) \cdot \chi_{[B,\infty)}\left(\tilde{S}_T\right), \tag{45}$$

which he assigns for instance to a "stock account". Consequently, $\tilde{Z}_T^{gua} = Z_T^{gua,bond} + \tilde{Z}_T^{gua,stock}$. However, the question arises which reference points are suitable for the payment structures (44) and (45). Since we took investor's initial wealth as the basis for the reference point of the overall position, it seems to be reasonable to divide the overall reference point into two parts, the share of wealth invested in the "bond position" and the share of wealth invested in the "stock position". Consequently, the reference point of the "bond account" amounts to

$$x^{ref,bond} = \frac{m \cdot CP \cdot NV}{\left(1 + r^{(b)}\right)^T}, \tag{46}$$

where $r^{(b)}$ denotes the term-appropriate cost of debt capital of the certificate issuer. Then, the relative certain outcome is given as

$$X_T^{gua,bond} = m \cdot CP \cdot NV - x^{ref,bond} = m \cdot CP \cdot NV \left(1 - \frac{1}{\left(1+r^{(b)}\right)^T} \right). \quad (47)$$

In addition, we get $x^{ref,stock} = A_0 - x^{ref,bond}$ for the reference point of the uncertain payment. Therefore, the continuous relative outcome variable has the form

$$\begin{aligned} \tilde{X}_T^{gua,stock} &= m \cdot NV \cdot PR \cdot \left(\frac{\tilde{S}_T - B}{B} \right) \cdot \chi_{[B,\infty)}\left(\tilde{S}_T\right) - x^{ref,stock} \\ &= m \cdot NV \cdot PR \cdot \left(\frac{\tilde{S}_T - B}{B} \right) \cdot \chi_{[B,\infty)}\left(\tilde{S}_T\right) - \left(A_0 - \frac{m \cdot CP \cdot NV}{\left(1+r^{(b)}\right)^T} \right). \end{aligned} \quad (48)$$

In order to apply PT and CPT under consideration of mental accounting to the outcomes $X_T^{gua,bond}$ and $\tilde{X}_T^{gua,stock}$, we again need their probability distributions, which are easily derived by the use of (33) and (34). The resulting parameter constellations and probability distributions are presented in Table 3.

Capital guarantee certificate	**Coefficients and probability distribution functions**
Certain payment structure	$a_1 = m \cdot CP \cdot NV\left(1-\left(1+r^{(b)}\right)^{-T}\right)$, $b_1 = 0$, $x_1 = 0$, $y_1 = \infty$, $g_1(x) = x$, $F_{\tilde{X}_T^{gua,bond}}(x) = \chi_{\left[m \cdot CP \cdot NV\left(1-\left(1+r^{(b)}\right)^{-T}\right),\infty\right)}(x)$
Uncertain payment structure	$a_1 = -\left(A_0 - (m \cdot CP \cdot NV)\cdot\left(1+r^{(b)}\right)^{-T}\right)$, $b_1 = 0$, $x_1 = 0$, $y_1 = B$, $g_1(x) = x$, $a_2 = -m \cdot NV \cdot PR - \left(A_0 - (m \cdot CP \cdot NV)\cdot\left(1+r^{(b)}\right)^{-T}\right)$, $b_2 = \frac{m \cdot NV \cdot PR}{B}$, $x_2 = B$, $y_2 = \infty$, $g_2(x) = x$, $F_{\tilde{X}_T^{gua,stock}}(x) = F_{\tilde{S}_T}\left(\frac{B \cdot \left(x + A_0 - (m \cdot NV \cdot CP)\cdot\left(1+r^{(b)}\right)^{-T}\right)}{m \cdot NV \cdot PR} + B \right) \cdot \chi_{\left[-A_0 + (m \cdot CP \cdot NV)\cdot\left(1+r^{(b)}\right)^{-T},\infty\right)}(x)$

Table 3: Coefficients and probability distribution functions of the capital guarantee certificates under consideration of mental accounting

Hence, all parameters and distributions that are necessary to apply PT and CPT under consideration of the mental accounting are determined.[12]

5 Empirical findings

Within our empirical study, we investigate how a fully rational and a boundedly rational investor would assess the above introduced certificates. We determine preference values on the

[12] The concrete assessment of all presented distributions on the basis of the PT- and CPT-formulas is available from the authors upon request.

basis of PT- and CPT-formulas and contrast the results with the preference order according to EUT. For the investigation, we use exemplary 5 real-life certificates that are presented in Table 4.

Certificate	ISIN (DE000…)	Issuer	Issue day/ Exercise day	Time to maturity T as of 08/13/2010 (in years)	Certificate price at 08/13/2010	Further features
Discount certificate	CM8FZW4	Commerz-bank	05/28/2009 06/14/2012	1.8371	€ 25.00	M = 0.01 CAP = 5,700.00
Capital guarantee certificate	DB0SMR4	Deutsche Bank	07/05/2007 06/08/2012	1.8207	€ 97.35	NV = 100.00 € CP = PR = 100 % B = 4,513.18
Index certificate	7093411	Deutsche Bank	09/02/2001 open end	1.8754	€ 27.10	M = 0.01
Sprint certificate	DB2F794	Deutsche Bank	08/12/2010 06/28/2012	1.8754	€ 26.51	M = 0.01 SL = 2,000.00 CAP = 2,700.00
Outperformance certificate	CG8HPC1	Citygroup	03/29/2010 06/15/2012	1.8398	€ 26.08	M = 0.01 OL = 3,250.00 PR = 150.00 %

Table 4: Key features of the considered real-life certificates

Due to the different properties of the respective certificates, it is not possible to find real-life products that have exactly the same issue and exercise day. For the sake of simplicity, we consider a decision situation in which an investor faces the choice to purchase one of the products in the secondary market at 08/13/2010. The underlying of each certificate is the Dow Jones EURO STOXX 50 Index. As of 08/13/2010, the real-life certificates have a nearly identical time to maturity T. Since the focus of our analysis is the comparison of the various preference approaches, the slightly different terms of maturity do not matter in the sense that they affect each evaluation approach equally. Additionally, the investor is assumed to have the possibility of a risk-free investment with maturity T = 1.8754 years. We choose the latter maturity because it is the longest term of the considered certificates and we would like to avoid any disadvantage with regard to the evaluation caused by a lower term. In this connection, the riskless interest rate is interpolated from the German spot rate curve presented by "Deutsche Bundesbank" at 08/13/2010 and results in 0.62 % p.a. Furthermore, the investor's wealth is assumed to amount to A_0 = € 10.000, which he or she completely invests in either one single certificate or the riskless investment.

The subjective evaluation of the capital guarantee certificate in consideration of mental accounting requires the establishment of an adequate cost of capital $r^{(b)}$, which is necessary to calculate the present value of the certain outcome $X_T^{gua,bond}$ according to (46). Since "Deutsche

Bank" is the issuer of the capital guarantee certificate, we calculate the cost of capital on the basis of a zero coupon bond issued by "Deutsche Bank" with ISIN DE000DB7URL7. Issue date is 12/19/2008 and exercise date is 07/20/2012, which nearly corresponds to the exercise date of the capital guarantee certificate. At valuation date 08/13/2012, the zero coupon bond price was 97.00 €. Therefore, a cost of capital $r^{(b)} = 1.58$ % p.a. results.

As mentioned above, we calculate the preference values based on the subjectively estimated expected return μ_{exp} and subjective volatility estimate σ_{exp} from a boundedly rational investor's point of view. According to Breuer and Perst (2007), we allow for the range $-0.05 \leq \mu_{exp} \leq 0.15$ and $0.1 \leq \sigma_{exp} \leq 0.25$. Within the binomial model, we use a step width n = 30, which is easy to implement numerically.

PT for discrete lotteries

Figure 1 shows the numerical results of the discrete evaluation approach according to Karmakar (1978) for different subjectively felt competence levels. On the left-hand side, the 3D pictures display the PT preference values of the considered certificates for different parameter combinations (μ_{exp}, σ_{exp}). On the right-hand side, the shading of the respective area indicates which certificate generates the maximum preference value for a given (μ_{exp}, σ_{exp})-combination. In both cases, low and high competence, the sprint, the discount, the outperformance, and the capital guarantee certificate are the preferred certificates depending on the underlying parameter combinations (μ_{exp}, σ_{exp}). In comparison to these certificates, the riskless investment and the index certificate fail to dominate for any kind of parameter combination. Principally, the sprint certificate is preferred for low- to medium-level expected stock returns. With increasing expected stock return and increasing volatility, the discount certificate gains attractiveness. The cap of the discount certificate is considerably higher than the cap of the sprint certificate. Against this background, a discount certificate seems to be more desirable than the sprint certificate if higher expected returns are under consideration. Since at evaluation date the Dow Jones EURO STOXX 50 value is 2,708.73 and the cap of the sprint certificate is CAP = 2,700.00, it is rather surprising that the sprint certificate is preferred for some positive expected stock returns at all. The preference of the capital guarantee certificate at low and medium expected stock returns comes as no surprise because of the capital protection, guaranteed independently of the development of the underlying. Since, however, the capital guarantee certificate's participation rate is lower than the corresponding rate of the outperformance certificate, the guarantee certificate becomes less attractive at high-level expected stock returns in comparison to the outperformance certificate. Up to the outperformance level, the outperformance certificate exactly reflects the performance of the underlying. Moreover, it supplies a disproportionate participation from the outperformance level. This could explain the subjective desirability of the outperformance certificate for high expected stock returns.

Comparing the results with regard to their different competence levels, it is noticeable that in the case of higher subjectively felt competence the preference area of the sprint and discount certificate grows considerably for increasing expected stock return and volatility. The area of the capital guarantee certificate nearly remains unchanged, however, the area of the outperformance certificate is pushed back. Obviously, a higher subjectively felt competence implies a higher preference towards cap-products. The 3D pictures show that in case of high competence the PT-values are lower than in case of low competence. This is, at the first sight, surprising since a higher subjectively felt competence implies higher probability weights $w_{\delta,\gamma}$ and therefore higher preference values. The reduction is caused by the normalization by the sum of probability weights according to (9). Furthermore, the jagged edge between the areas of the discount and outperformance certificate, which is a result of discrete modeling, is hard to comprehend and seems to contradict the typical behavior of market participants.

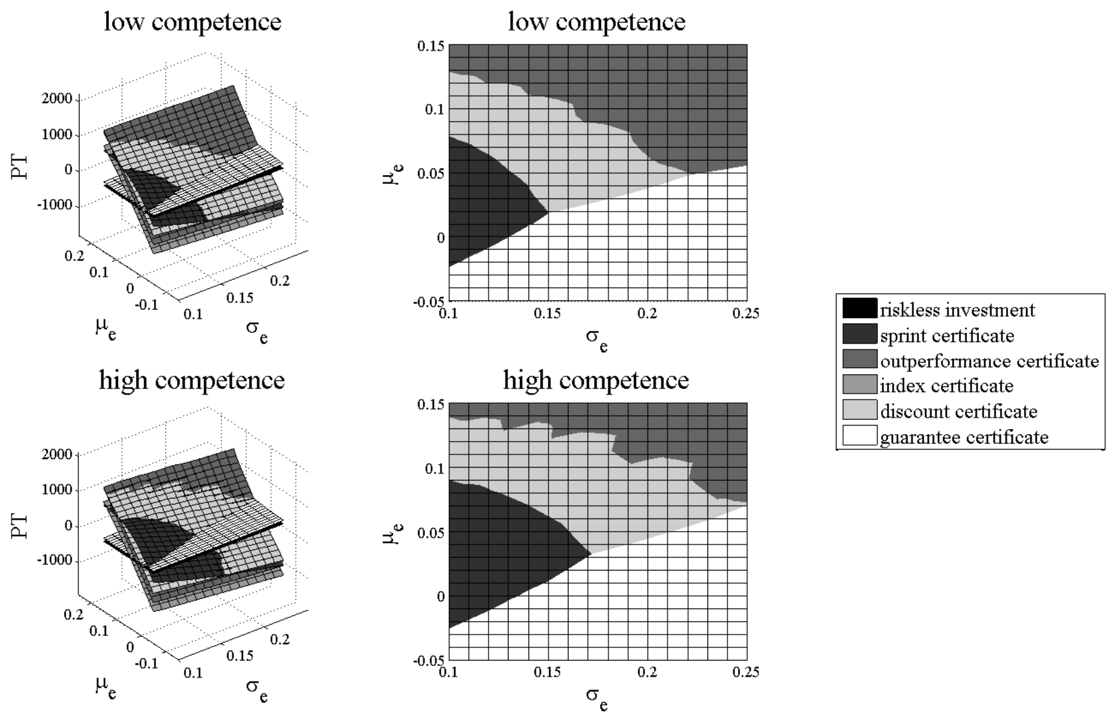

Figure 1: Prospect theory for discrete lotteries, low and high subjectively felt competence

PT for continuous lotteries

In a next step, we investigate the preference values of PT for continuous lotteries. In contrast to the discrete version of PT, according to which sprint, discount, outperformance and guarantee certificate are subjectively more attractive than the riskless investment and the index certificate, the implementation of the continuous approach of PT shows a different preference. In this context, the sprint certificate does not dominate the remaining financial products for some (μ_{exp}, σ_{exp}). This shows, at least for the present products, that the discrete version of PT can explain the demand for more products than the continuous version. The preference values for $\gamma \in \{0.6; 0.65; 0.832\}$ are shown in Figure 2.

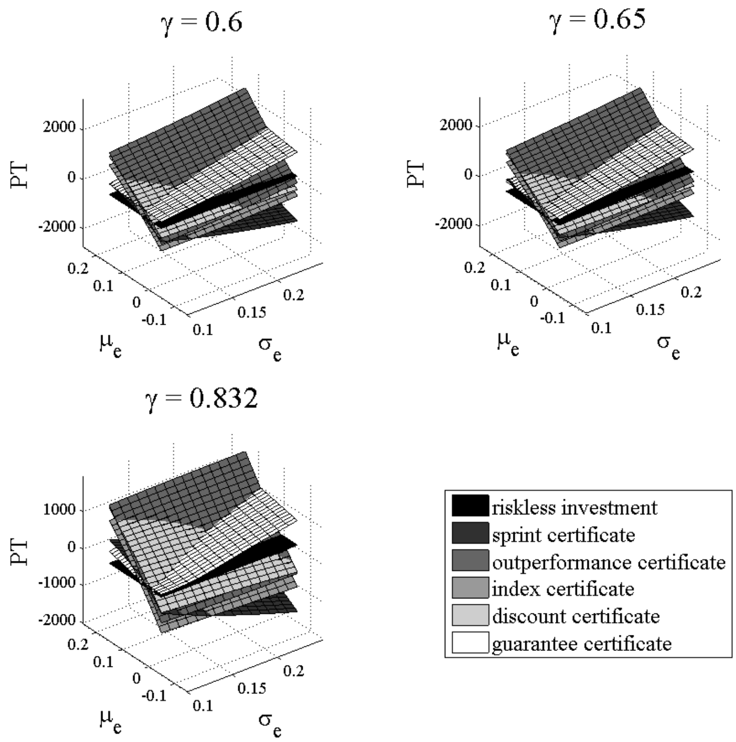

Figure 2: Prospect theory for continuous lotteries for different γ-values

Overall, the 3D pictures of Figure 2 show a different preference pattern in comparison with Figure 1. Obviously, with increasing γ the preference for the discount certificate grows, whereas the preference for the outperformance certificate decreases. Interpreting γ as a subjectively felt competence level, it seems that with increasing competence the desirability of the cap product grows, especially at the expense of the outperformance certificate. Figure 3 shows that the preference for the capital guarantee certificate grows for increasing parameters

μ_{exp} and σ_{exp}, while at the same time the preference diminishes for lower parameters μ_{exp} and σ_{exp} if competence (measured by γ) increases.

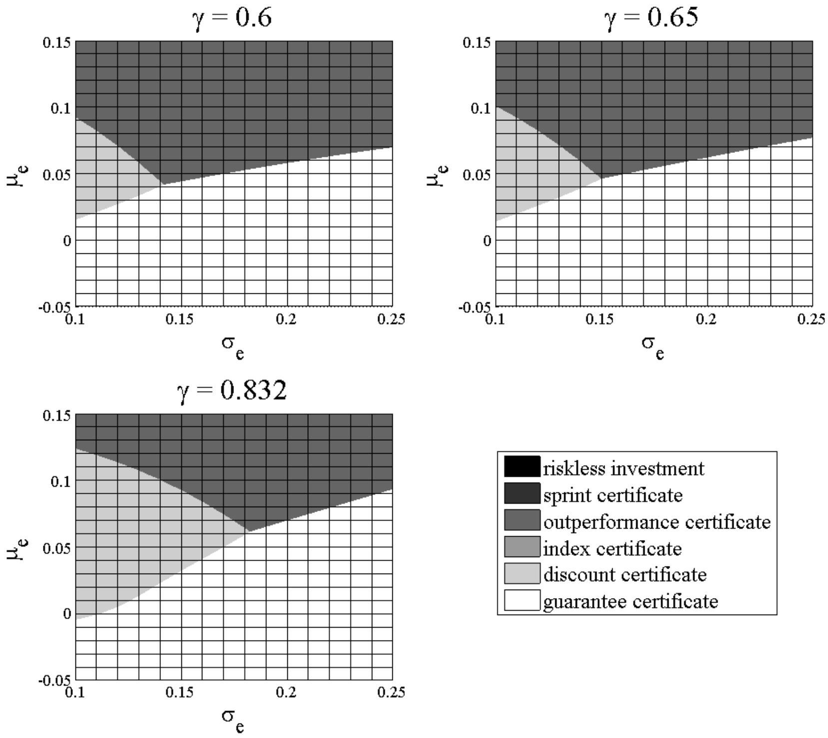

Figure 3: Prospect theory for continuous lotteries for different γ-values

We also apply our suggested approach of modeling mental accounting in combination with hedonic framing to PT. Because of slight variations, we restrict ourselves to the presentation of the results corresponding to the parameters $\gamma = 0.6$ and $\gamma = 0.832$ in Figure 4. For both parameters, the desirability of the capital guarantee certificate grows noticeably, whereas the preference for the discount and outperformance certificate is reduced. However, the consideration of hedonic framing is not able to explain the demand for sprint certificates either.

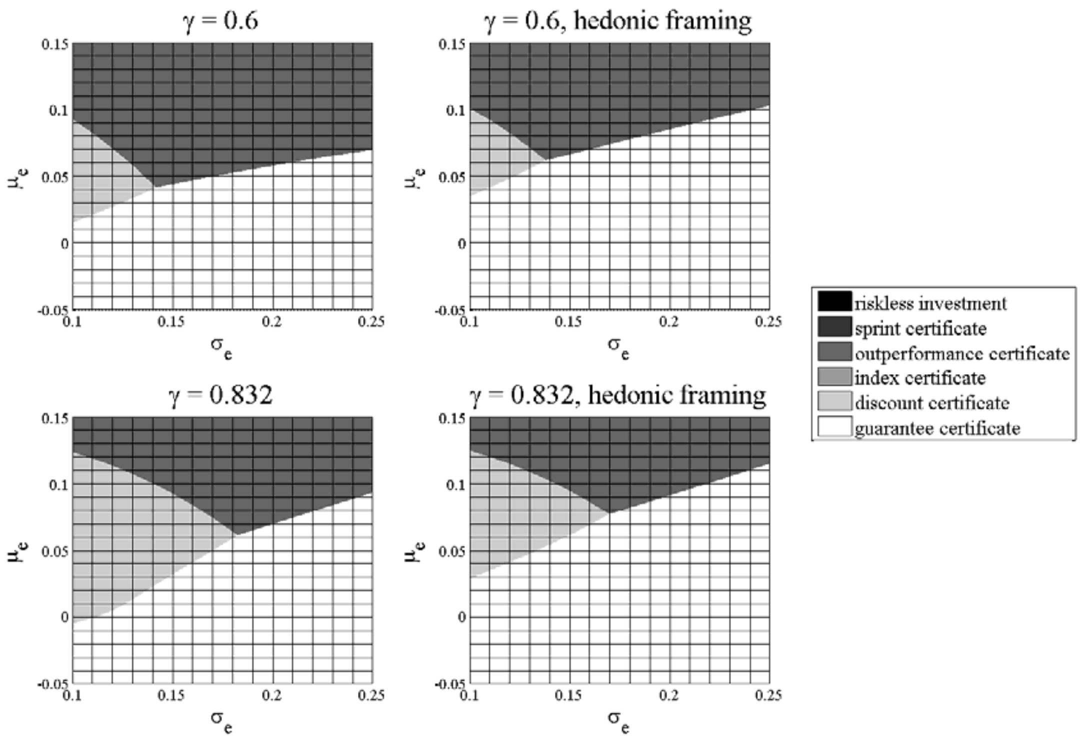

Figure 4: Prospect theory for continuous lotteries, including hedonic framing

CPT for discrete lotteries

Next, we discuss the results of the CPT-evaluation for discrete lotteries in the cases of low and high subjectively felt competence and with the application of the hedonic framing rule according to Breuer and Perst (2007). According to Figure 5, the plots of CPT for different competence levels are similar to those given for PT in Figure 1. Comparing the results of PT and CPT in the case of low competence, Figure 5 shows that for CPT the preference area of the sprint and discount certificate appears slightly compressed, whereas the area of the outperformance certificate increases. The same holds for the presentation of CPT in the case of a high competence level. Furthermore, the application of the hedonic framing rule, or rather the segregation for gains, leads to higher preference values, but the preference order remains unaffected. In addition, similar and weakened in comparison to the discrete PT, there is a jagged edge between the areas of the discount and outperformance certificate, which again stems from discrete modeling and seems to contradict market behavior.

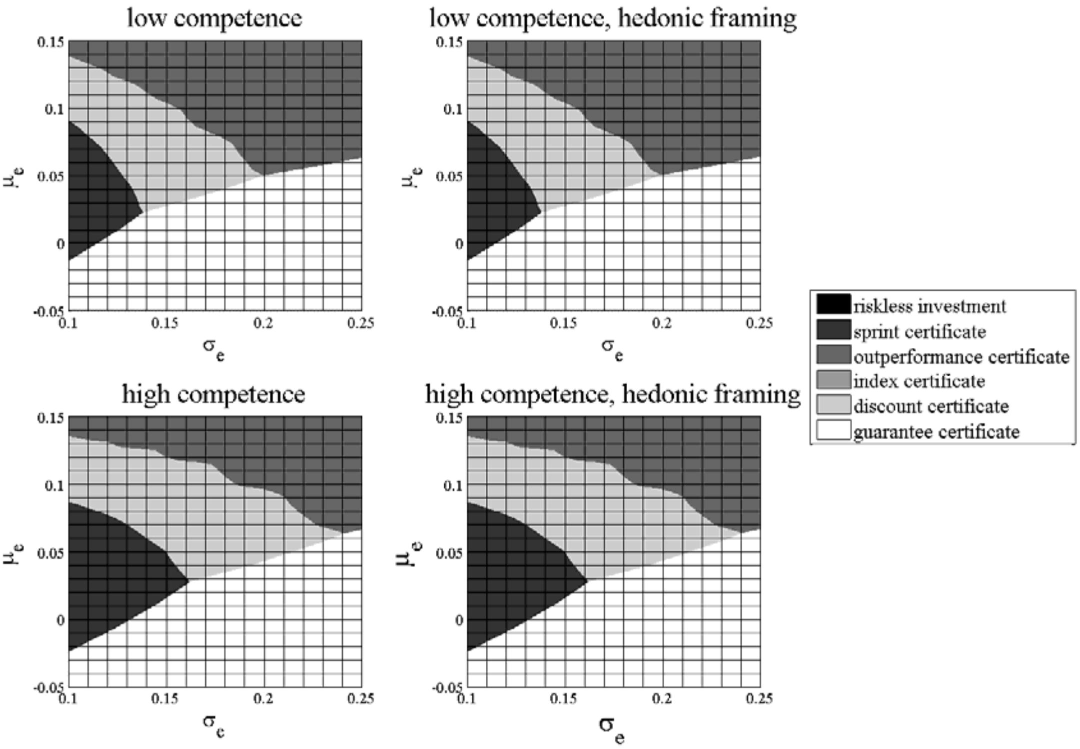

Figure 5: Cumulative Prospect Theory for discrete lotteries, low and high subjective felt competence, including hedonic framing

CPT for continuous lotteries

At first glance, Figure 6 shows that the implementation of the discrete CPT with n = 30 is a good approximation of the continuous CPT. For both cases, low and high subjectively felt competence, the numerical results are quite close together. Thus, preferences in case of discrete lotteries (with n = 30) seem to correspond to the preferences with regard to continuous lotteries. However, Figure 6 and Figure 7 show that in the continuous case and in contrast to the PT application, the cutting planes are totally smooth and therefore more plausible. Additionally, if we take mental accounting in combination with hedonic framing into consideration (see Figure 7), the preference area of the capital guarantee certificate is slightly larger for low as well as high subjectively felt competence. Furthermore, in comparison to PT for continuous distributions, our hedonic framing combination rule has a lower effect on CPT.

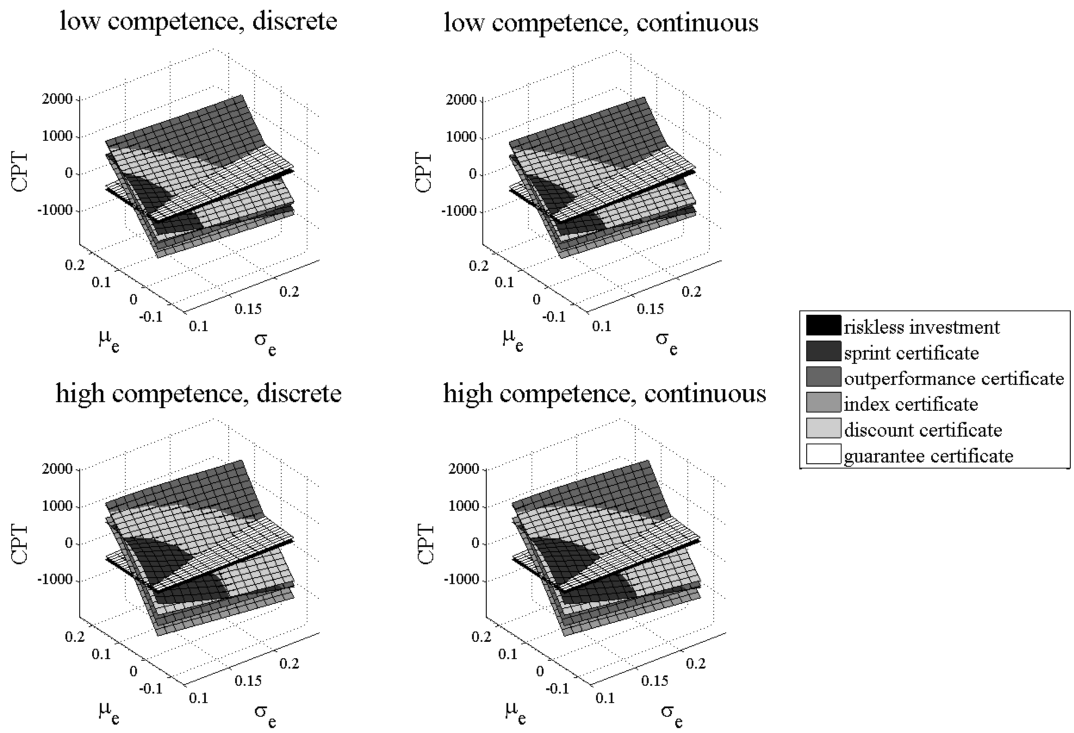

Figure 6: Cumulative Prospect Theory for discrete and continuous lotteries, low and high subjectively felt competence level

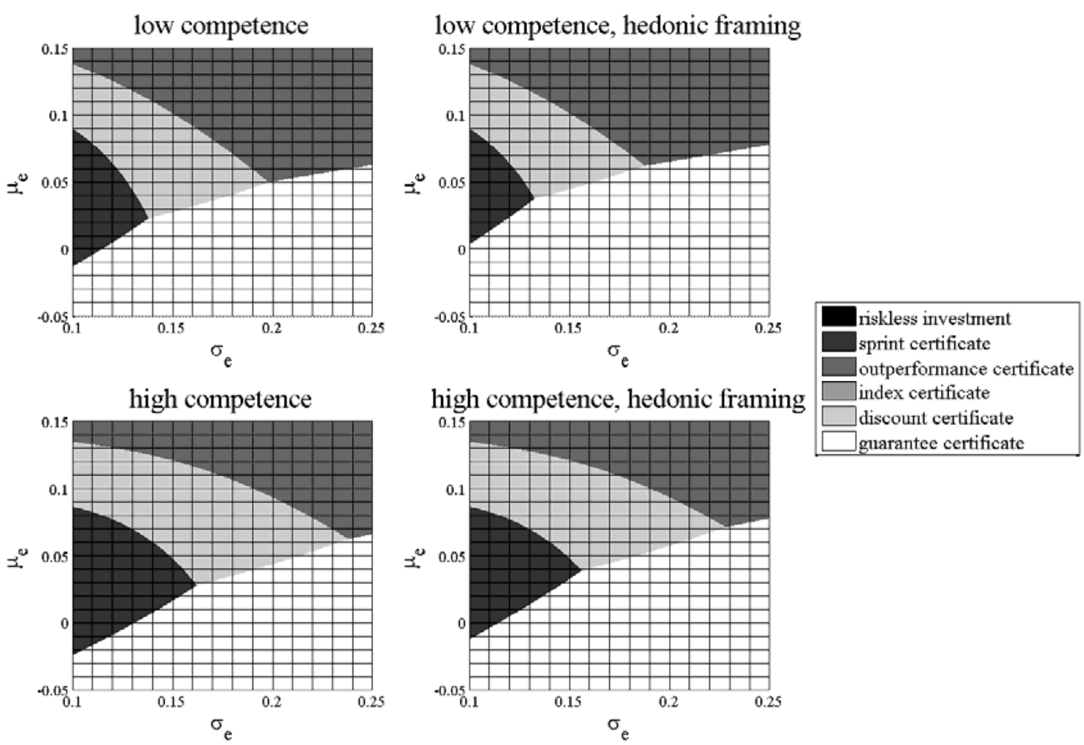

Figure 7: Cumulative Prospect Theory for continuous lotteries, low and high subjectively felt competence, including hedonic framing

EUT for discrete and continuous lotteries

Finally, we want to contrast the subjective evaluations from the point of view of a boundedly rational investor with the evaluation of a fully rational investor. For this purpose, we consider a fully rational individual with a constant risk aversion of one. Since the results of the discrete and the continuous version of the EUT are nearly the same, we restrict ourselves to present the continuous results.

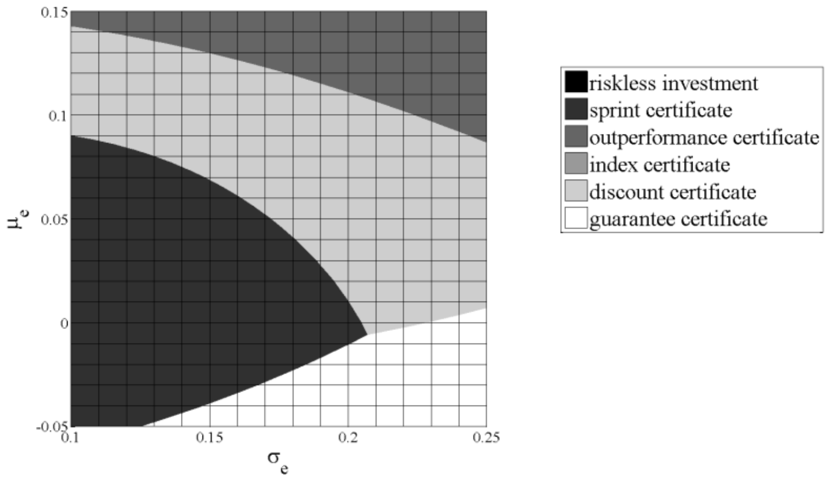

Figure 8: Expected utility theory

Similar to the findings of PT for discrete and CPT for discrete and continuous lotteries, sprint, discount, outperformance, and capital guarantee certificate are the preferred products depending on the corresponding estimated stock return and volatility. For negative expected stock returns and volatilities larger than 12.4%, the capital guarantee certificate has the highest desirability because of the capital protection for any development of the underlying. The outperformance certificate is only of interest in comparison to other products for very high expected returns. The discount certificate's preference is given for any kind of volatility. Noticeable is the relative large preference area of the sprint certificate, plausible for any negative expected stock return, surprising for positive estimated returns. The latter fact appears to be contradictory behavior.

Comparison of the theories

Altogether, except for the continuous version of PT, all theories are able to explain the demand for sprint, discount, outperformance, and capital guarantee certificate. However, application of the discrete versions leads to quite implausible jagged edges between dominance areas, which corresponds to implausible behavior of the market participants. A similar argument relates to EUT because a preference for sprint certificates in the case of positive estimated returns seems to contradict the typical behavior of market participants. Consequently, the application of the continuous version of CPT leads to the most plausible results.

6 Conclusion

We have extended the continuous version of CPT by considering piecewise continuous probability distributions with a finite number of jump discontinuities. We have explained how to apply this theory to a broad class of structured products that comprises inter alia index, discount, capital guarantee, outperformance, and sprint certificates. Additionally, we have developed a hedonic framing rule for mental accounting in the case of piecewise distributions with regard to the combination of an uncertain lottery and a certain gain which we have applied to the capital guarantee certificate. On this basis, we have been able to compare this extended version of CPT with the discrete version of CPT and the respective versions of PT and EUT. In order to analyze the explanatory power of the theories, we have undertaken a study on the basis of real-life certificates of the already mentioned types. We have shown that all theories except for the continuous version of PT justify the demand for four products. However, the discrete versions of PT and CPT as well as EUT lead to partially implausible behavioral patterns. Consequently, if an issuer of certificates intends to design products in the framework of financial engineering, he can base the design on the piecewise continuous version of CPT. In addition, the application of the hedonic framing rule seems not to be necessary, since this rule only has a slight effect on the preference in case of Cumulative Prospect Theory.

Appendix

Appendix 1: (CPT-value of a certain outcome $\bar{y}$)

Let $\bar{y} > 0$ denote a certain gain and let $\tilde{Y}$ describe the corresponding random variable. For all y the value $F_{\tilde{Y}}(y)$ of the probability distribution function obviously corresponds to

$$F_{\tilde{Y}}(y) = P\left(\tilde{Y} \le y\right) = H(y - \bar{y}). \tag{A1}$$

Consequently, this function is contained in the class of distributions according to (13) if we set n = 0, $\alpha_0 = \alpha_1 = 1$, $\beta_1 = 0$, and $\xi_0 = \xi_1 = \bar{y}$. As a result of (25) we immediately get

$$\begin{aligned} CPT(\tilde{Y}, f_{\tilde{Y}}) &= v(\xi_0) \cdot w\left(\hat{F}_{\tilde{X}}(\xi_0)\right) - v(\xi_1) \cdot w\left(\hat{F}_{\tilde{X}}(\xi_1)\right) + v(\xi_1) \cdot (1 - H(\xi_1)) + v(\xi_0) \cdot H(\xi_0) \\ &= v(\bar{y}). \end{aligned} \tag{A2}$$

Appendix 2: (Probability distribution and density function of the general outcome (33))

Using law of total probability, the probability distribution function $F_{\tilde{X}_T^{cer}}$ is given as

$$\begin{aligned} F_{\tilde{X}_T^{cer}}(x) &= P\left(\tilde{X}_T^{cer} \le x\right) \\ &= \sum_{i=1}^{n} P\left(a_i + b_i \cdot g_i\left(\tilde{S}_T\right) \le x \middle| \tilde{S}_T \in M_i\right) \cdot P\left(\tilde{S}_T \in M_i\right) \\ &= \sum_{i=1}^{n} P\left(\tilde{S}_T \le g_i^{-1}\left(\frac{x - a_i}{b_i}\right) \middle| \tilde{S}_T \in M_i\right) \cdot P\left(\tilde{S}_T \in M_i\right) \\ &= \sum_{i=1}^{n} P\left(x_i \le \tilde{S}_T \le \min\left\{y_i, g_i^{-1}\left(\frac{x - a_i}{b_i}\right)\right\}\right) \\ &= \sum_{i=1}^{n} \left(F_{\tilde{S}_T}\left(\min\left\{y_i, g_i^{-1}\left(\frac{x - a_i}{b_i}\right)\right\}\right) - F_{\tilde{S}_T}(x_i)\right) \cdot H\left(x - (a_i + b_i \cdot g_i(x_i))\right). \end{aligned} \tag{A3}$$

In the case $b_i = 0$, we define $g_i^{-1}\left((x - a_i)/b_i\right) := \lim_{b_i \to 0} g_i^{-1}\left((x - a_i)/b_i\right) = \infty$ and consequently, $\min\left\{y_i, g_i^{-1}\left(\frac{x - a_i}{b_i}\right)\right\} = y_i$. Moreover, we have

$$\begin{aligned} &F_{\tilde{S}_T}\left(\min\left\{y_i, g_i^{-1}\left(\frac{x - a_i}{b_i}\right)\right\}\right) \\ &= F_{\tilde{S}_T}\left(g_i^{-1}\left(\frac{x - a_i}{b_i}\right)\right) + \left(F_{\tilde{S}_T}(y_i) - F_{\tilde{S}_T}\left(g_i^{-1}\left(\frac{x - a_i}{b_i}\right)\right)\right) \cdot H\left(x - (a_i + b_i \cdot g_i(y_i))\right). \end{aligned} \tag{A4}$$

This leads to

$$F_{\tilde{X}_T^{cer}}(x) = \sum_{i=1}^{n} F_{\tilde{S}_T}\left(g_i^{-1}\left(\frac{x-a_i}{b_i}\right)\right)\cdot H\left(x-(a_i+b_i\cdot g_i(x_i))\right)$$
$$+\sum_{i=1}^{n} F_{\tilde{S}_T}(y_i)\cdot H\left(x-(a_i+b_i\cdot g_i(y_i))\right)$$
$$-\sum_{i=1}^{n} F_{\tilde{S}_T}\left(g_i^{-1}\left(\frac{x-a_i}{b_i}\right)\right)\cdot H\left(x-(a_i+b_i\cdot g_i(y_i)\right) \qquad \text{(A5)}$$
$$-\sum_{i=1}^{n} F_{\tilde{S}_T}(x_i)\cdot H\left(x-(a_i+b_i\cdot g_i(x_i))\right).$$

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