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Coherent Banking Capital and Optimal Credit Portfolio Structure

by Wolfgang Breuer*and Marc Gürtler**

Abstract. "Coherent" measures of a bank's whole risk capital imply a structure of a bank's optimal credit portfolio that is independent of its deposits and the expected deposit rate, of expected bankruptcy costs and of expected costs of regulatory capital.

Keywords: Basel II, Regulatory Capital, Coherent Risk Capital, Separation

JEL classification: G21, G28

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1. Introduction

In almost every developed country there are compulsory banking capital requirements that aim at sustaining the stability of the financial sector. According to the new capital adequacy framework (Basel II) finally adopted by the Basel Committee in June 2004 a bank's required regulatory capital (RC) is based on so-called economic capital describing that capital amount "that a firm believes is necessary to absorb potential losses associated with each of the included risks" (Basel Committee on Banking Supervision, 2003, p. 1). But since banks are legally obliged to make provisions against expected losses, the capital requirement RC only aims at the problem of unexpected losses (see Kupiec, 2003). We thus may define a bank's required whole risk capital (WRC) as the sum of its economic or regulatory capital and of its expected losses, because the latter leads to provisions a bank can only make if it is furnished with enough capital.

Mostly, the WRC is calculated as the Value-at-Risk (VaR), although the Value-at-Risk exhibits some well-known unfavorable features. In particular, according to Artzner et al. (1997), as a consequence of the missing subadditivity of the VaR, two banks that merge may ceteris paribus face higher capital requirements than the sum of their individual requirements before the merger. Certainly, this finding is counterintuitive and may induce rather adverse incentive effects for banks regarding their decisions on organizational structures.

Therefore, with $\tilde{L}_1$ and $\tilde{L}_2$ as two uncertain losses from two different credits 1 and 2, Artzner et al. (1997, 1999) and Frey and McNeil (2002) postulated four axioms that should be met by a reasonable ("coherent") measure of the whole risk capital WRC which – after subtraction of overall expected losses – yields the required regulatory capital:

(Ax1) $\text{WRC}(\tilde{L}_1 + \tilde{L}_2) \leq \text{WRC}(\tilde{L}_1) + \text{WRC}(\tilde{L}_2)$ (subadditivity),

(Ax2) $\tilde{L}_1 \leq \tilde{L}_2 \;\Rightarrow\; \text{WRC}(\tilde{L}_1) \leq \text{WRC}(\tilde{L}_2)$ (monotonicity), (Ax3) $\text{WRC}(\lambda \cdot \tilde{L}_1) = \lambda \cdot \text{WRC}(\tilde{L}_1)$ for all $\lambda \geq 0$ (positive homogeneity),

(Ax4) $\text{WRC}(\tilde{L}_1 + L) = \text{WRC}(\tilde{L}_1) + L$ for all certain losses L (translation-invariance).

While (Ax1) has already been explained, without (Ax2) ceteris paribus higher losses with probability one would possibly result in lower requirements of WRC. (Ax3) just implies that the summation of an arbitrary number of perfectly correlated credits should lead to a corresponding increase in the whole risk capital, and according to (Ax4) certain losses have to be accounted for completely by additional capital requirements regardless of a bank's other loss risks. Additionally, (Ax4) implies the natural identity $\text{WRC}(\tilde{L} - \text{WRC}(\tilde{L})) = 0$: The supply of the adequate amount $\text{WRC}(\tilde{L})$ of capital according to a loss distribution $\tilde{L}$ reduces additional requirements to zero.

Obviously, axioms (Ax1) to (Ax4) sound quite reasonable. Moreover, as we want to show in the next section, as long as the whole risk capital satisfies these axioms, the optimal structure of a bank's credits may be independent of certain influence factors like the costs of regulatory capital and a bank's expected bankruptcy costs, thus simplifying a bank's portfolio decision to a great extent.

2. The setting

To be more precise, consider a risk-neutral bank in a one-period-framework with uncertain cash flows $\tilde{c}_1$ at time 1. Define B_b as the amount of money lent to borrower b = 1, …, $\hat{b}$ at a contractual interest rate of $r_b^{(B)}$ and B = $(B_1, ..., B_{\hat{b}})$ as the vector of all credits granted by the bank at time t = 0 to borrowers. Thus, $B_b \cdot (1 + r_b^{(B)})$ characterizes the amount of money borrower b has to return at t = 1. Certainly, some borrowers will not be able to repay the whole amount $B_b \cdot (1 + r_b^{(B)})$. We use $\tilde{L}(B)$ to describe the bank's uncertain overall loss from such problems.

Moreover, the bank may lend or borrow on the interbanking money market at a given interest rate $r^{(f)}$ for lending and borrowing with a total investment of A (A > 0: lending, A < 0: borrowing). We assume this lending and borrowing to be due just a short time before credits and deposits are so that we can consider lending and borrowing on the interbanking market to be risk-free. It seems plausible to assume that a bank's deposit rates are lower than $r^{(f)}$. Each bank will therefore choose the maximum possible amount $D^{(max)}$ of deposits, as choices $D < D^{(max)}$ would forgo riskless gains otherwise possible by investing additional deposits $D^{(max)}-D$ at $r^{(f)}$ on the interbanking market. We denote with $E(\tilde{r}_d^{(D^{(max)})})$ the corresponding expected (average) deposit rate to be paid by the bank under consideration.

Under the plausible premise of binding banking capital regulation, the bank's required whole risk capital consists of provisions for expected losses $E(\tilde{L}(B))$ and the bank's (additional) regulatory capital $RC(B)$. While we assume that the bank may acquire capital to account for expected losses at a given interest rate $\tilde{r}^{(WRC_1)}$, additional capital requirements $RC(B)$ may lead to increasing (marginal) costs of capital $\tilde{r}^{(WRC_2)} =: \tilde{r}^{(RC)}$. We thus have $\tilde{r}^{(RC)}(RC(B))$ with $\partial\tilde{r}^{(RC)}/\partial RC \geq 0$ while $\tilde{r}^{(WRC_1)} = \text{const.}$ Finally, we have to account for the bank's budget constraint $B_1 + ... + B_{\hat{b}} + A = D^{(max)} + E(\tilde{L}(B)) + RC(B).$ Now define

$$
\begin{aligned}
&E(\tilde{c}_1^-(B)) \\
&:= A \cdot (1 + r^{(f)}) + \sum_{b=1}^{\hat{b}} B_b \cdot (1 + r_b^{(B)}) - E(\tilde{L}(B)) \cdot (2 + E(\tilde{r}^{(WRC_1)})) - D^{(max)} \cdot (1 + E(\tilde{r}_d^{(D^{(max)})})) \qquad (1) \\
&= \sum_{b=1}^{\hat{b}} B_b \cdot [r_b^{(B)} - r^{(f)}] - E(\tilde{L}(B)) \cdot (1 + E(\tilde{r}^{(WRC_1)}) - r^{(f)}) - D^{(max)} \cdot (E(\tilde{r}_d^{(D^{(max)})}) - r^{(f)})
\end{aligned}
$$

as the bank's expected cash flow before subtracting additional expected costs of capital holding RC(B). A "perfect" bank regulation might accomplish that the probability of default and thus expected bankruptcy costs are (almost) independent of a bank's behavior by requiring a certain regulatory capital RC. However, for real-life situations it sounds more plausible to assume that situations with ceteris paribus higher requirements of RC coincide with higher banks' default probabilities and thus higher expected bankruptcy costs C(RC) for the bank under consideration: $\partial C(RC)/\partial RC \geq 0$. Summing up, we have

$$E(\tilde{c}_1(B)) = E(\tilde{c}_1^-(B)) - RC(B)\cdot[1+E(\tilde{r}^{(RC)}(RC(B)))] - C(RC(B)) \;\rightarrow\; \max_B.! \qquad (2)$$

as the bank's objective function. Ceteris paribus, the bank will prefer alternatives with higher values for $E(\tilde{c}_1^-)$ and lower values for RC.

3. Implications

The axioms (Ax1) to (Ax4) for a bank's whole risk capital imply certain properties for its corresponding regulatory capital that serves as a defense against unexpected losses:

Lemma. *Let (Ax1) to (Ax4) be true, then the regulatory capital* $RC = WRC - E(\tilde{L})$ *satisfies (Ax1), (Ax3) and the property*

(P4) $RC(\tilde{L}_1 + L) = RC(\tilde{L}_1)$ *for all certain losses L.*

Proof. *See the Appendix.*

RC thus possesses all relevant features that are known from the standard deviation as the most prominent measure of risk. Consider now a bank that is aiming at the maximization of the objective function (2) and that is facing a legally defined RC that corresponds to WRC satisfying (Ax1) to (Ax4). Let $B^{(sum)}$ be defined as $\sum_{b=1}^{\hat{b}} B_b$, i.e. as the overall outstanding risky credits so that we may define $y_b = B_b/B^{(sum)}$ as the fractions of uncertain credits in the bank's overall risky credit subportfolio. Then, the Lemma enables us to derive the following

Proposition. *1) Assume that an inner solution for the optimization of $B^{(sum)}$ does exist. Then optimal fractions* $y_b^* = B_b^* / B^{(sum)*}$ *do not depend on* $E(\tilde{r}^{(RC)}(RC))$, *$D^{(max)}$, and $r^{(D)}$. Moreover, they are the same for all monotone increasing functions C(RC). Changes of the regulatory bank capital from RC(B) to f(RC(B)) with ∂f(RC(B))/∂RC(B) ≥ 0 do not affect the optimal credit portfolio structure, either. 2) Sufficient conditions for the existence of an inner solution for $B^{(sum)}$ to the bank's optimization problem are expected bankruptcy costs C and/or expected costs* $E(\tilde{r}^{(RC)})$ *of regulatory capital being strictly convex in $B^{(sum)}$ given optimal credit portfolio structures.*

Proof. *See the Appendix.*

According to our Proposition, as a consequence of coherent whole risk capital, the *structure* of a bank's optimal credit portfolio (if existing) does not depend on the general level of regulatory capital requirements, i.e. any tightening of capital requirements RC to f(RC) with ∂f/∂RC ≥ 0 will not affect the structure of a bank's risky subportfolio, but only its *amount*. Moreover, a bank has not to bother about the precise functional relationship between regulatory capital and expected bankruptcy costs as long as these relationships can be described by arbitrary monotone increasing functions. Further, it does not matter, either, how the expected costs of regulatory capital rise with a bank's additional demand for RC. Finally, the structure of the credit portfolio does not depend on the amount of deposits that is chosen by the bank nor on the corresponding deposit rate.

The rationale for our Proposition is that, for given credit portfolio structure, variations of the amount of a bank's riskless lending or borrowing imply combinations of $E(\tilde{c}_1^-)$ and RC that all lie on a straight line with a fixed starting point ($E(\tilde{c}_1^-) = D^{(max)} \cdot (r^{(f)} - E(\tilde{r}_d^{(D^{(max)})}))$; RC = 0) being independent of the credit portfolio structure. All achievable lines thus differ only by their slopes and can therefore be unambiguously compared as long as a bank prefers ceteris paribus higher values for $E(\tilde{c}_1^-)$ and lower values for RC. The situation is analogously to that underlying the famous two-fund separation theorem of Tobin (1958) for mean-variance investors.

4. Discussion

The separation result of the last section simplifies the bank's decision problem tremendously and may serve as a further argument for coherent risk capital definitions. Certainly, our results are the consequences of some simplifying assumptions. In particular, we assume credits to be perfectly divisible and the bank to act as a pure price-taker at least on the credit market. However, since any single credit contributes only to a minor degree to a bank's overall portfolio, the assumption of divisibility might be a satisfactory approximation of the real-life situation. Moreover, intense interbanking competition will prevent banks from acting in another way than as price-takers.

More importantly, we assume a possibility for riskless lending and borrowing on the interbanking market, although we may take positive default probabilities of a bank into account. We therefore must propose that lending and borrowing on the interbanking market is due, before credits and deposits are payable. Since interbanking money market transactions are indeed often very short-term oriented, our setting seems to be reasonable. In addition, according to part 2) of the Proposition, even for $C(RC) \equiv 0$, i.e. (in particular) for situations without any default problems for regulated banks so that interbanking lending and borrowing are "naturally" riskless, an inner solution for the bank's optimization problem may exist and thus the separation result remains in effect. Moreover, part 1) of the Proposition will also hold true for situations with no riskless borrowing opportunity at all, as long as there is a possibility for riskless lending (by buying government notes and bonds) and the bank management is sure to make use of this option. Summarizing, we deem it interesting to further analyze the portfolio selection consequences of coherent risk capital requirements in more detail.

Appendix

Proof of the Lemma:

(Ax1):

$$
\begin{aligned}
& \mathrm{WRC}(\tilde{L}_1+\tilde{L}_2) \le \mathrm{WRC}(\tilde{L}_1)+\mathrm{WRC}(\tilde{L}_2) \\
\Leftrightarrow\ & \mathrm{RC}(\tilde{L}_1+\tilde{L}_2)+\mathrm{E}(\tilde{L}_1+\tilde{L}_2) \le \mathrm{RC}(\tilde{L}_1)+\mathrm{E}(\tilde{L}_1)+\mathrm{RC}(\tilde{L}_2)+\mathrm{E}(\tilde{L}_2)\,. \\
\Leftrightarrow\ & \mathrm{RC}(\tilde{L}_1+\tilde{L}_2) \le \mathrm{RC}(\tilde{L}_1)+\mathrm{RC}(\tilde{L}_2).
\end{aligned}
$$

(Ax3):

$$
\begin{aligned}
& \mathrm{WRC}(\lambda\cdot\tilde{L}_1) = \lambda\cdot\mathrm{WRC}(\tilde{L}_1) \\
\Leftrightarrow\ & \mathrm{RC}(\lambda\cdot\tilde{L}_1)+\mathrm{E}(\lambda\cdot\tilde{L}_1) = \lambda\cdot\mathrm{RC}(\tilde{L}_1)+\lambda\cdot\mathrm{E}(\tilde{L}_1) \\
\Leftrightarrow\ & \mathrm{RC}(\lambda\cdot\tilde{L}_1) = \lambda\cdot\mathrm{RC}(\tilde{L}_1).
\end{aligned}
$$

(P4):

$$
\begin{aligned}
& \mathrm{WRC}(\tilde{L}_1+L) = \mathrm{WRC}(\tilde{L}_1)+L \\
\Leftrightarrow\ & \mathrm{RC}(\tilde{L}_1+L)+\mathrm{E}(\tilde{L}_1+L) = \mathrm{RC}(\tilde{L}_1)+\mathrm{E}(\tilde{L}_1)+L \\
\Leftrightarrow\ & \mathrm{RC}(\tilde{L}_1+L) = \mathrm{RC}(\tilde{L}_1).
\end{aligned}
$$

Proof of the Proposition:

1)

Define $\tilde{\ell}_b := \tilde{L}_b/[B_b\cdot(1+r_b^{(B)})]$ as the uncertain “loss rate” of credit b. Then we have:

$$
\tilde{L}(B)=\sum_{b=1}^{\hat{b}}\tilde{L}_b=\sum_{b=1}^{\hat{b}}\tilde{\ell}_b\cdot B_b\cdot(1+r_b^{(B)})=B^{(sum)}\cdot\sum_{b=1}^{\hat{b}}y_b\cdot\tilde{\ell}_b\cdot(1+r_b^{(B)}). \tag{A.1}
$$

With this we get:

$$
\begin{aligned}
\mathrm{E}(\tilde{c}_1^-(B)) &= \sum_{b=1}^{\hat{b}}B_b\cdot(r_b^{(B)}-r^{(f)})-\mathrm{E}(\tilde{L}(B))\cdot(1+\mathrm{E}(\tilde{r}^{(WRC_1)})-r^{(f)})-D^{(max)}\cdot(\mathrm{E}(\tilde{r}_d^{(D^{(max)})})-r^{(f)}) \\
&= \sum_{b=1}^{\hat{b}}B_b\cdot[r_b^{(B)}-r^{(f)}-\mathrm{E}(\tilde{\ell}_b)\cdot(1+r_b^{(B)})\cdot(1+\mathrm{E}(\tilde{r}^{(WRC_1)})-r^{(f)})]-D^{(max)}\cdot(\mathrm{E}(\tilde{r}_d^{(D^{(max)})})-r^{(f)}).
\end{aligned} \tag{A.2}
$$

For given deposits, a higher value of $E(\tilde{c}_1^-(B))$ corresponds with a higher value of

$$\mu(B) := \sum_{b=1}^{\hat{b}} B_b \cdot [r_b^{(B)} - r^{(f)} - E(\tilde{\ell}_b) \cdot (1 + r_b^{(B)}) \cdot (1 + E(\tilde{r}^{(WRC_1)}) - r^{(f)})]. \quad (A.3)$$

For a given expectation value $\mu := \mu(B)$, the credit portfolios with a minimum of regulatory capital can be determined as follows:

$$RC[B_1^{(\mu)}, ..., B_{\hat{b}}^{(\mu)}] \ \rightarrow \ \min_{B_1^{(\mu)}, ..., B_{\hat{b}}^{(\mu)}} .! \quad (A.4)$$

$$\text{s.t. } \mu = \sum_{b=1}^{\hat{b}} B_b^{(\mu)} \cdot [r_b^{(B)} - r^{(f)} - E(\tilde{\ell}_b) \cdot (1 + r_b^{(B)}) \cdot (1 + E(\tilde{r}^{(WRC_1)}) - r^{(f)})]. \quad (A.5)$$

With λ_μ as a Lagrangian multiplier, the necessary conditions are

$$\frac{\partial RC}{\partial B_b^{(\mu)}} = \lambda_\mu \cdot [r_b^{(B)} - r^{(f)} - E(\tilde{\ell}_b) \cdot (1 + r_b^{(B)}) \cdot (1 + E(\tilde{r}^{(WRC_1)}) - r^{(f)})] \quad (b = 1, ..., \hat{b}). \quad (A.6)$$

Since (from axiom (Ax3)) $RC[c \cdot B_1^{(\mu)}, ..., c \cdot B_{\hat{b}}^{(\mu)}] = c \cdot RC[B_1^{(\mu)}, ..., B_{\hat{b}}^{(\mu)}]$ for all c ≥ 0, RC is homogeneous of degree one so that we get from Euler's Homogeneous Function Theorem

$$RC\left[B_1^{(\mu)}, ..., B_{\hat{b}}^{(\mu)}\right] = \sum_{b=1}^{\hat{b}} B_b^{(\mu)} \cdot \frac{\partial RC}{\partial B_b^{(\mu)}}. \quad (A.7)$$

Under consideration of (A.5) and (A.6), the result (A.7) implies

$$RC\left[B_1^{(\mu)}, ..., B_{\hat{b}}^{(\mu)}\right] = \lambda_\mu \cdot \mu \quad (A.8)$$

Since

$$\mu = B^{(sum)} \cdot \sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot [r_b^{(B)} - r^{(f)} - E(\tilde{\ell}_b) \cdot (1 + r_b^{(B)}) \cdot (1 + E(\tilde{r}^{(WRC_1)}) - r^{(f)})], \quad (A.9)$$

a variation of μ can be reached by a variation of $B^{(sum)}$ which in turn does not affect

$$\lambda_\mu = \frac{RC\left(B^{(sum)} \cdot \sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot \tilde{\ell}_b\right)}{B^{(sum)} \cdot \sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot [r_b^{(B)} - r^{(f)} - E(\tilde{\ell}_b) \cdot (1+r_b^{(B)}) \cdot (1+E(\tilde{r}^{(WRC_1)}) - r^{(f)})]} \underset{\substack{\text{Axiom} \\ (\text{Ax3})}}{=} \frac{RC\left(\sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot \tilde{\ell}_b\right)}{\sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot [r_b^{(B)} - r^{(f)} - E(\tilde{\ell}_b) \cdot (1+r_b^{(B)}) \cdot (1+E(\tilde{r}^{(WRC_1)}) - r^{(f)})]}. \tag{A.10}$$

As a consequence, for given credit portfolio structure ($y_1, \ldots, y_{\hat{b}}$) variations of $B^{(sum)}$ imply (μ, RC)-combinations that are all located on the same straight line. The optimal credit portfolio structure thus does not depend on the given values of μ and $B^{(sum)}$. Since the set of risky credit portfolios is convex in the μ–RC space due to the Lemma, the second-order conditions for a minimum regarding the problem (A.4) are satisfied.

A variation of D and $r^{(D)}$ does not influence the fact that the bank (following optimization problem (2)) will prefer alternatives with higher values for $\mu(B)$ and lower values for RC. Thus, the proof above is independent of the concrete specification of D and $r^{(D)}$. The same argument can be used to justify the last two sentences of part 1) of the Proposition.

2)

The objective function

$$\begin{aligned} E(\tilde{c}_1(B)) &= E(\tilde{c}_1^-(B)) - RC(B) \cdot [1 + E(\tilde{r}^{(RC)}(RC(B)))] - C(RC(B)) \\ &= B^{(sum)} \cdot \sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot [r_b^{(B)} - r^{(f)} - E(\tilde{\ell}_b) \cdot (1+r_b^{(B)}) \cdot (1+E(\tilde{r}^{(WRC_1)}) - r^{(f)})] - D^{(max)} \cdot (E(\tilde{r}_d^{(D^{(max)})}) - r^{(f)}) \\ &\quad - B^{(sum)} \cdot RC\left(\sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot \tilde{\ell}_b\right) \cdot \left[1 + E\left(\tilde{r}^{(RC)}\left(B^{(sum)} \cdot RC\left(\sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot \tilde{\ell}_b\right)\right)\right)\right] \\ &\quad - C\left(B^{(sum)} \cdot RC\left(\sum_{b=1}^{\hat{b}} y_b^{(\mu)} \cdot \tilde{\ell}_b\right)\right) \end{aligned} \tag{A.11}$$

of the maximization problem (2) implies an inner solution for $B^{(sum)}$, if C and/or $E(\tilde{r}^{(RC)})$ are strictly convex functions.

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