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by Wolfgang Breuer* and Marc Gürtler**

Abstract. We consider investors with mean-variance-skewness preferences who aim at selecting one out of F different funds and combining it optimally with the riskless asset and direct stock holdings. Direct stock holdings are either exogenously or endogenously determined. In our theoretical section, we derive and discuss several performance measures for the investor's decision problems with a central role of *Kimball*'s (1990) prudence and of several variants of *Sharpe* and *Treynor* measures. In our empirical section, we show that the distinction between exogenous and endogenous stock holding is less important than the issue of skewness preferences. The latter are most relevant for fund rankings, when an investor's skewness preferences are not derived from cubic HARA utility so that the two-fund separation theorem is not valid.

Keywords: investor specific performance measure, performance evaluation, prudence, skewness preferences

JEL classification: G11

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I. Introduction

The main task of performance measures for investment funds is to help investors in identifying the most suitable fund for given preference structures. In general, there are two possible ways to tackle this problem. On the one hand, one can choose a partial-analytical framework, thereby focussing on the decision problem of a given investor for given expectations and neglecting any kind of general capital market considerations. On the other hand, one can analyze capital market price formation processes in order to derive conclusions with respect to the attractiveness of certain funds. For example, the well-known capital asset pricing model (CAPM) as introduced by *Sharpe* (1964) may define such a setting. One may conclude from this equilibrium description that the same performance measure of zero should be assigned to all investment funds, just expressing that the holding of shares of any fund is irrelevant for any capital market participant. Another prominent example of a market-based approach has been developed by Leland (1999) on the basis of power utility functions and lognormal return distributions.

Although such analyses on capital market levels certainly are apt to create interesting general insights, for practical application we prefer the partial-analytical framework focussing on the view of a single investor with given preference structures and expectations who typically acts as a price-taker. If for such an investor the CAPM in its original version or in the modified setting applied by Leland (1999) in fact held, we would learn this from his or her specific expectations. But if this is not true, the CAPM (as any other capital market model) is not of immediate relevance for the investor under consideration.

In what follows, we thus examine an investor with a one-period horizon who faces at time $t = 0$ the problem of selecting just one out of F different funds f in order to combine this investment with the direct holding of a given (reference) portfolio P of equity shares and riskless lending or borrowing until time $t = 1$. As a consequence, for any fund f under consideration we are searching for optimal fractions x_0, x_f, and x_P of the investor's initial wealth optimally invested in the riskless asset, the fund f and the equity portfolio P. After this, resulting preference values for any fund f are used to generate a fund ranking which can be utilized as a recommendation for fund selection.

Certainly, the examination of a situation where only one out of F different funds can be chosen is somewhat restrictive. Nevertheless such a scenario can be interpreted as a classical asset allocation problem with three classes of assets (a fund, direct stock holding and riskless lending or borrowing). As an illustration, this decision problem corresponds to the important case of institutional investors relying only on a single fund manager, a not uncommon practice in many countries. In addition, it is necessary to define different funds as alternative investments if performance measures for single funds shall be derived. Moreover, the analysis of situations with the selection of only one fund at a time may be used as a starting point for the examination of more complex portfolio selection problems in future work. In fact, our derivations remain valid if we reinterpret f = 1, …, F not as single funds but as F different given portfolios of funds. Only the analysis of the determination of the optimal combination of a certain set of funds must then be the object of further research. One recent numerical approach that is devoted to this latter task has been introduced by *Davies/Kat/Lu* (2006). However, because of the complexity of their decision-problem they not even attempt to derive general results that could be interpreted as performance measures. Moreover, owed to computational problems they have to rely on "non-standard ways" of describing investors' preferences. As a consequence, for their approach it does not seem to be possible to derive any connection to expected utility maximizing behavior, as is done for our approach in one of the following sections.

Recently, for simple μ-σ-preferences the decision-problem sketched above has been analyzed for two different settings (see also Figure 1). In the first one which may be called the endogenous case, all three fractions x_0, x_f, and x_P are indeed variable. In *Breuer/Gürtler* (1999, 2000) it has been demonstrated that in such a situation funds can be ranked according to an optimized *Sharpe* measure which coincides with the conventional *Sharpe* ratio[1] of the optimal risky portfolio of fund f and equity portfolio P. Additionally, based on previous findings by *Jobson/Korkie* (1984) it could be shown that for inner solutions the optimized *Sharpe* measure is identical to the *Treynor/Black* appraisal ratio[2] while in the case of short sales restrictions the optimized *Sharpe* measure may lead to border solutions that coincide with the original *Sharpe* ratio, the *Treynor* ratio[3], or *Jensen*'s alpha[4]. As mean-variance preferences are in particular the result of quadratic utility, in what follows we simply speak of the quadratic *Sharpe* measure, *Treynor* measure, *Jensen* measure, and *Treynor/Black* measure. The optimized *Sharpe* measure in situations with short sales restrictions will be named the optimized restricted *Sharpe* measure.

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1 See *Sharpe* (1966).
2 See for the *Treynor/Black* appraisal ratio in particular *Treynor/Black* (1973).
3 See *Treynor* (1965).
4 See *Jensen* (1968).

>>> Insert Figure 1 about here <<<

In the second setting, one may reasonably argue that the fraction x_P of an investor's direct stock holding is exogenously fixed and only x_0 and x_f can be optimized any more, for example, because of former transactions and corresponding transaction costs considerations. This "exogenous case" has intensively been examined by *Scholz/Wilkens* (2003), and in *Breuer/Gürtler* (2005) both approaches have been analyzed with respect to their theoretical and empirical relationships. Quite remarkably, it could be proven that, in general, a fund g is unambiguously preferred to a fund h in the endogenous case as well as in the exogenous case if it exhibits both a higher quadratic *Sharpe* measure and a higher quadratic *Treynor* measure. Nevertheless, besides this finding, *theoretical* relationships between fund rankings in the endogenous case and the exogenous one seem to be quite loose, while *empirical* evidence suggests that at least for simple μ-σ-preferences the distinction between both scenarios is negligible.

>>> Insert Table 1 about here <<<

In this paper, we want to extend the analysis by the explicit consideration of skewness preferences of investors, i.e. preferences regarding the third central moment of uncertain wealth or return, as recent approaches like the ones by *Harvey/Siddique* (2000), *Dittmar* (2002) or *Fletcher/Kihanda* (2005) are in particular stressing the relevance of preferences for higher-order return moments in asset pricing models. As sketched in Table 1, this extension can be done for the endogenous case (cell (3) in Table 1) as well as for the exogenous one (cell (4) in Table 1). Moreover, while *Breuer/Gürtler* (2005) focus on the relationship between cases (1) and (2) of Table 1, we will examine in more depth the relationship between fund rankings for the cases (2) and (4), thus contrasting fund rankings in the exogenous case for mean-variance preferences and mean-variance-skewness preferences. For the endogenous case and with a restriction to the case of cubic utility functions with hyperbolic absolute risk aversion (HARA), such a comparison is presented by *Breuer/Gürtler* (2006). It is shown that funds can be unambiguously ranked according to an optimized "cubic" performance measure which only depends on two arguments: the optimized quadratic *Sharpe* measure of the fund under consideration and a newly introduced performance measure which may be called an optimized cubic *Sharpe* measure. The latter is defined as the quotient of the (third root of the) skewness of the return of the optimal ("preference-independent", i.e. being valid for the whole class of cubic HARA utility functions) combination of a fund f with the reference portfolio P and the corresponding variance of this portfolio return. Thereby, the possibility of preference-independent fund ranking is a consequence of the two-fund separation theorem introduced by *Tobin* (1958) and later on extended by *Hakansson* (1969) and *Cass/Sitglitz* (1970). However, the two-fund separation theorem holds only in the endogenous case with HARA utility with the latter certainly being a relevant restriction of skewness preferences in itself.

Against this background, we start our theoretical exposition in the following Section II with a general discussion of mean-variance-skewness preferences. Certainly, preference parameters are least restricted when we only exclude inefficient solutions from the analysis, i.e. solutions with mean-variance-skewness characteristics that are dominated by other admissible portfolios. However, as is known from simple mean-variance analysis, not every efficient solution may be the outcome of expected utility maximizing behavior. This additional requirement narrows the set of admissible mean-variance-skewness preferences. Moreover, for expected utility maximizing behavior, we are able to show that an investor's optimal portfolio selection is mainly determined by *Kimball*'s prudence, i.e. the negative relation between the third and the second derivative of his or her utility function, as this value governs the relationship between the subjective evaluation of portfolio return skewness (being mainly determined by the third derivative of an investor's utility function) and of portfolio return variance (being mainly determined by the second derivative of an investor's utility function). The range of admissible mean-variance-skewness-preferences under consideration becomes even smaller, when only cubic utility functions of the HARA type are examined.

Based on such a general discussion of mean-variance-skewness preferences, the main theoretical contribution of our paper in Section III aims at the derivation of performance measures for the exogenous case with skewness preferences. In this context, we refrain from restricting ourselves to the analysis of only cubic HARA utility, as the advantage of the HARA property (validity of the two-fund separation theorem) does not hold for the exogenous case. Moreover, in Section IV, we will be able to extend the analysis of the endogenous case to non-HARA skewness preferences as well. We do not know of any other approach attempting to derive general performance measures for arbitrary skewness preferences in the endogenous case or in the exogenous one defined above. We are able to identify several simple submeasures of performance which serve as arguments for our general performance measures and can be interpreted as variants of a fund's (cubic and quadratic) *Sharpe* or *Treynor* measures and thus as a straightforward extension of *Scholz/Wilkens* (2003) and *Breuer/Gürtler* (2005) for the simple mean-variance case. Thereby, for our analysis including skewness preferences, a preference parameter becomes relevant that is directly related to *Kimball*'s prudence in order to define the relative importance of portfolio return skewness and portfolio return variance in performance measurement.

Based on our theoretical derivations, our empirical analysis in Section V addresses the following two issues:

1) Does the empirical finding by *Breuer/Gürtler* (2005) of the irrelevancy of the distinction between the endogenous case and the exogenous one in a mean-variance context carry over to a situation with mean-variance-skewness considerations?

2) Which role does a possible restriction to only cubic HARA utility play for the relevance of performance measures recognizing skewness preferences?

We find that the distinction between the exogenous case and the endogenous one indeed remains to be of only minor importance even if we allow for mean-variance-skewness preferences, while the empirical relevance of skewness preferences seems to be depending on the validity of the two-fund separation. As is well-known, a simple mean-variance approach ceases to be of good approximative quality even in cases with non-quadratic utility when an investor's risk aversion is sufficiently high. Nevertheless, this circumstance can only lead to variations in fund performance in cases without two-fund separation, because otherwise optimal fund rankings are not influenced by variations of an investor's risk aversion. Therefore, it does not seem to be too surprising that skewness preferences affect fund rankings in our empirical example most when mean-variance preferences are not restricted to such parameter constellations that are in line with cubic HARA utility, as derived in Section II of our paper. In any case, this finding sheds additional light on the empirical relevance of preferences for higher-order return moments in performance evaluation for mutual funds.

Section VI tackles the problem of possible ways to practical application of the performance measures developed in this paper. In particular, empirical findings regarding typical values of *Kimball*'s prudence may be an adequate starting point to specify the preference parameter in our performance measures. Section VII concludes. Because of space constraints, all mathematical derivations have been deferred to separate appendices. For the same reasons, several tables (numbered from "Ad 1" to "Ad 7") have been omitted that are not absolutely necessary for the understanding of our exposition. Moreover, Table 2 offers a synopsis of the most relevant symbols utilized in this paper.

>>> Insert Table 2 about here <<<

II. Decision-theoretical background

1. Mean-variance-skewness preferences

The skewness of a wealth distribution can be characterized as its third central moment

(1) $$\gamma_W^3 := E[(\tilde{W} - E(\tilde{W}))^3],$$

with $\tilde{W}$ as the investor's uncertain terminal wealth. As a generalization of the basic mean-variance case, we consider investors who are aiming at the maximization of a μ-σ-γ-preference function Φ_W with

(2) $$\Phi_W(\mu_W, \sigma_w^2, \gamma_w^3) = \mu_W - \kappa_W \cdot \sigma_w^2 + \lambda_W \cdot \gamma_w^3.$$

κ_W and λ_W are positive preference-depending parameters, as risk-averse investors are characterized by negative variance preferences and (typically[5]) by positive skewness preferences. For initial wealth W_0 we can define $\tilde{r} := (\tilde{W} / W_0) - 1$ as the investor's uncertain portfolio return and introduce μ, σ^2, and γ^3 as the relevant moments of the investor's return distribution. Then with given initial wealth W_0, the maximization of the preference function Φ_W is equivalent to the maximization of $\Phi := \Phi_W/W_0 - 1$:

(3) $$\Phi(\mu, \sigma^2, \gamma^3) = \mu - \kappa \cdot \sigma^2 + \lambda \cdot \gamma^3,$$

with $\kappa := W_0 \cdot \kappa_W$ and $\lambda := W_0^2 \cdot \lambda_W$. For $W_0 = 1$, κ and κ_W as well as λ and λ_W are identical. Without loss of generality we therefore will from now on assume $W_0 = 1$. Moreover, (3) can be expressed equivalently as a function of the relevant moments of the investor's excess return $\tilde{u} := \tilde{r} - r_0$ for given riskless interest rate r_0 and with expectation value $\bar{u}$, as we have $\mu = \bar{u} + r_0$, while the second and the third central moment for $\tilde{r}$ and $\tilde{u}$ are identical.

Analogously to μ-σ-dominance and μ-σ-efficiency it is possible to introduce the concept of μ-σ-γ-dominance and μ-σ-γ-efficiency: An alternative 1 is (strictly) dominated by an alternative 2, if we have $\mu_1 \leq \mu_2$, $\sigma_1 \geq \sigma_2$ as well as $\gamma_1 \leq \gamma_2$ with at least one inequality being strict. An alternative is μ-σ-γ-efficient, unless it is (strictly) μ-σ-γ-dominated by at least one alternative.

[5] Among other things, it is well-known that for an expected utility maximizing individual positive skewness preferences are a necessary condition for decreasing absolute risk aversion which in turn seems to be typical for individuals' attitudes towards risk. See, for example, *Arrow* (1971).

Certainly, for preferences according to (3) only μ-σ-γ-efficient alternatives have to be regarded as potential optimal solutions of an investor's decision problem.

In order to solve a portfolio selection problem for given preference function (2) or (3) one has to fix parameters κ and λ. Unfortunately, this straightforward approach does not lead to meaningful general results. We therefore follow another way of derivation, whereby we assume the investor to define a desired (positive) expected overall excess return $\bar{u}^+$ of his or her portfolio which he or she wants to achieve. Since all portfolios under consideration are just characterized by the same desired overall expected rate of return, preference function (3) reduces to

(4) $$\Phi_{\bar{u}^+}(\sigma^2, \gamma^3) := -\omega \cdot \sigma^2 + \gamma^3,$$

with $\omega = \kappa/\lambda > 0$.

As a consequence of this modified approach the determination of preference parameters κ and λ is thus replaced by the specification of $\bar{u}^+$ and ω. Instead of some "absolute" preference levels regarding σ^2 and γ^3, only the relative relevance of "variance aversion" in comparison to "skewness loving" (as expressed by ω) remains relevant. Such an approach seems to be first suggested by *Breuer/Gürtler* (1998). Later on *Berényi* (2002) coined the term "variance equivalent risk measure"[6] for the functional form $-\Phi_{\bar{u}^+}(\sigma^2, \gamma^3)/\omega = \sigma^2 - \gamma^3/\omega$. However, neither *Breuer/Gürtler* (1998) nor *Berényi* (2002) have examined the exogenous case or the endogenous case as defined in this study. Moreover, *Breuer/Gürtler* (1998) present no utiliy-theoretical analysis, while *Berényi* (2002) fails to explicitly consider any portfolio selection problem at all and thus is not able to derive performance measures endogenously.

Apparently, one might wonder about the relationship between optimizing (4) for given portfolio excess return $\bar{u}^+$ and preference parameter ω and the optimization of (3) (or (2)) for given values of κ and λ. This is not a trivial issue. In particular, it should be emphasized that $\bar{u}^+$ is endogenously determined by the investor in question and as such the trade-off between expected (excess) returns and risk properties of return distributions is not neglected at all when applying (4) for means of portfolio optimization. Nevertheless, it remains to be analyzed whether any possible pair ($\bar{u}^+$, ω) is "admissible" in that sense that there is another ("reasonable") pair of preference parameters (κ, λ) that leads to the same optimal portfolio selection.

[6] See also *Berényi* (2003) and *Onorato* (2004).

In order to answer this question we first have to clarify the utility-theoretical background of preference function (2) (or (3)) to some larger extent.

2. Relationships between $\bar{u}^+$ and preference parameters

In the same way, as mean-variance preferences can be derived from the assumption of quadratic utility, it is possible to justify the preference function described by (2) via a cubic *von Neumann-Morgenstern* utility function $U(\tilde{W})$ for uncertain terminal wealth $\tilde{W}$ with

(5) $$U(\tilde{W}) = \hat{a}_3 \cdot \tilde{W}^3 + \hat{a}_2 \cdot \tilde{W}^2 + \hat{a}_1 \cdot \tilde{W} + \hat{a}_0.$$

Using a Taylor expansion around μ_W, expected utility in the case of (5) can be computed as

(6) $$\begin{aligned} E[U(\tilde{W})] &= U(\mu_W) + U'(\mu_W)\cdot(\mu_W - \mu_W) + \frac{1}{2}\cdot U''(\mu_W)\cdot\sigma_W^2 + \frac{1}{6}\cdot U'''(\mu_W)\cdot\gamma_W^3 \\ &= U(\mu_W) + \frac{1}{2}\cdot U''(\mu_W)\cdot\sigma_W^2 + \frac{1}{6}\cdot U'''(\mu_W)\cdot\gamma_W^3. \end{aligned}$$

As in the case of preference function (2) we restrict ourselves to situations with positive skewness preferences which obviously requires $U'''(\mu_W) = 6\cdot\hat{a}_3 > 0 \Leftrightarrow \hat{a}_3 > 0.$ Consequently, the fraction $P := -U'''(\mu_W)/U''(\mu_W)$ becomes positive, too. Actually, *Kimball* (1990) introduced the term "absolute prudence" for this fraction. A positive prudence implies that an investor will increase ceteris paribus his or her riskless lending, when uncertain returns become riskier: The greater the prudence, the more sensitive an investor's reaction by increasing his or her "precautionary saving". Positive skewness preferences thus coincide with a positive prudence and mere mean-variance preferences imply a prudence of zero. Moreover, and rather interestingly, for given value $\bar{u}^+$ of $\bar{u}$ (and thus given μ_W) and with given value W_0 = 1, (6) yields $\omega = \kappa_W/\lambda_W = -3\cdot U''(\mu_W)/U'''(\mu_W) = 3/P$ and hence the preference parameter ω of section II.1 can be interpreted as (three times) the reciprocal value of an investor's prudence P for an excess return realization u with $u = \bar{u}^+$. Moreover, we have $\kappa/\lambda = 3/(W_0\cdot P)$ which also simplifies to 3/P because of our assumption $W_0 = 1$. According to *Kimball* (1990), the product $\mu_W\cdot P$ is called the "relative prudence" for an expected excess return realization $u = \bar{u}^+$. In what follows we simply speak of "prudence" when we mean the absolute one, but will return to the concept of relative prudence in Section VI below.

Because of the cardinality of *von Neumann-Morgenstern* utility functions we can reduce (5) by $\hat{a}_0$ and then divide it by $\hat{a}_3$. Defining $a_2 := \hat{a}_2/\hat{a}_3$ and $a_1 := \hat{a}_1/\hat{a}_3$, (5) can thus be rewritten as $U(\tilde{W}) = \tilde{W}^3 + a_2\cdot\tilde{W}^2 + a_1\cdot\tilde{W}$ so that (6) becomes

(7) $$E[U(\tilde{W})] = \mu_W^3 + a_2 \cdot \mu_W^2 + a_1 \cdot \mu_W + (3 \cdot \mu_W + a_2) \cdot \sigma_W^2 + \gamma_W^3.$$

As long as we restrict ourselves to situations with positive, but diminishing marginal utility (and positive prudence), it is easy to show that the maximization of (7) results in the selection of a μ-σ-γ-efficient alternative.[7] However, not every μ-σ-γ-efficient alternative can be the outcome of the maximization of (7) if we hold on to the requirement of positive, but decreasing marginal utility.[8] In fact, this result is already well-known for simple mean-variance preferences, i.e. the case $\lambda = 0$.[9]

It thus seems reasonable to explicitly allow for the requirement of a positive first and a negative second derivative of the utility function. As a necessary condition for the fulfilment of these properties which is independent[10] of the specific return distribution these signs of the derivatives must be given at least for expected return μ_W, i.e.

(8) $$\begin{aligned} &U'(\mu_W) > 0 \Leftrightarrow 3 \cdot \mu_W^2 + 2 \cdot a_2 \cdot \mu_W + a_1 > 0 \underset{\mu_W > 0}{\Leftrightarrow} a_2 > -\frac{a_1}{2 \cdot \mu_W} - 1.5 \cdot \mu_W, \\ &U''(\mu_W) < 0 \Leftrightarrow a_2 < -3 \cdot \mu_W. \end{aligned}$$

Apparently, (for $\mu_W > 0$) both conditions of (8) can only be simultaneously valid for $a_1 > 3 \cdot \mu_W^2$.

With respect to $U(\tilde{W}) = \tilde{W}^3 + a_2 \cdot \tilde{W}^2 + a_1 \cdot \tilde{W}$, the special case of $a_1 = \frac{a_2^2}{3}$ deserves particular attention, as this leads to a cubic utility function that can be written as

(9) $$U(W) = (W - a)^3 + a^3 = W^3 - 3 \cdot a \cdot W^2 + 3 \cdot a^2 \cdot W,$$

with $a > 0$ and

(10) $$a_2 = -3 \cdot a \Leftrightarrow a = -\frac{a_2}{3}, a_1 = 3 \cdot a^2 \Leftrightarrow a = \sqrt{\frac{a_1}{3}}.$$

Such a cubic utility function exhibits the property of hyperbolic absolute risk aversion mentioned previously, i.e. we have

(11) $$-\frac{U''(W)}{U'(W)} = \frac{1}{\hat{a} + b \cdot W},$$

with risk aversion parameters $\hat{a} = 0.5 \cdot a$ and $b = -0.5$.

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[7] See Appendix 1.

[8] See Appendix 2.

[9] See, for example, *Breuer/Gürtler/Schuhmacher* (2004), p. 171.

[10] It is not difficult to derive stricter restrictions for given domains of uncertain excess returns. However, (8) must be valid in any case and even if we only know the relevant moments of excess returns and not their domains.

As already stated, in order to apply preference function (4), an investor has to determine a pair $(\bar{u}^+,\omega)$ of desired expected overall excess return $\bar{u}^+$ and preference parameter $\omega = \kappa/\lambda$. We are now able to return to the issue of which pairs $(\bar{u}^+,\omega)$ are actually consistent with preference or utility functions (2), (5), and (9). To be more specific, a consistent specification of $(\bar{u}^+,\omega)$ by an investor requires for the case of preference function (2) that there exists at least one corresponding pair of preference parameters κ and λ so that the resulting optimal overall portfolio of the best fund f, reference portfolio P, and the riskless asset leads to an overall expected excess return of $\bar{u}^+$. If such a pair (κ, λ) does not exist, then the resulting ranking for $(\bar{u}^+,\omega)$ lacks any relevance and the pair $(\bar{u}^+,\omega)$ can be called "not admissible". Certainly, (2) imposes the fewest restrictions on admissible pairs $(\bar{u}^+,\omega)$, but even for (2), not all, but only sufficiently great values of expected excess returns $\bar{u}^+$ can be the result of portfolio optimization. Things get even "worse", if we require a cubic *von Neumann-Morgenstern* utility function according to (5), as this implies additional lower or upper bounds for admissible values of ω for given expected excess return. As a consequence of the further restriction of HARA utility, there will be at most two admissible values for ω for any given expected excess return $\bar{u}^+$. These findings are made more precise in

Result 1:

1) In the case of general mean-variance-skewness preferences according to (2) or (3), for any given exogenous value of x_P and given preference parameter ω, it will be possible to justify any desired overall expected excess return $\bar{u}^+$ as preference maximizing when choosing the best fund f, as long as $\bar{u}^+$ is not smaller than the expected excess return of the portfolio that maximizes $\gamma^3 - \omega \cdot \sigma^2$.

2) Define σ^{+2} and γ^{+3} as the variance and the skewness of the return of the investor's overall portfolio for $x_f^+ := (\bar{u}^+ - \hat{x}_P \cdot \bar{u}_P)/\bar{u}_f$, i.e. the necessary share of fund f as part of the investor's overall portfolio in order to attain an overall expected excess return $\bar{u}^+$. Then, in the case of expected utility maximizing behavior with a general cubic utility function according to (5) only preference parameters ω satisfying

$$(12)\quad \begin{aligned} &\text{(a) } \omega > \frac{1.5 \cdot [\sigma^{+2} \cdot \bar{u}_f + (x_f^{+2} \cdot \gamma_f^3 + 2 \cdot x_f^+ \cdot \hat{x}_P \cdot \gamma_{ffP} + \hat{x}_P^2 \cdot \gamma_{fPP})]}{x_f^+ \cdot \sigma_f^2 + \hat{x}_P \cdot \sigma_{fP}}, \text{ if } x_f^+ \cdot \sigma_f^2 + \hat{x}_P \cdot \sigma_{fP} > 0, \\ &\text{(b) } \omega < \frac{1.5 \cdot [\sigma^{+2} \cdot \bar{u}_f + (x_f^{+2} \cdot \gamma_f^3 + 2 \cdot x_f^+ \cdot \hat{x}_P \cdot \gamma_{ffP} + \hat{x}_P^2 \cdot \gamma_{fPP})]}{x_f^+ \cdot \sigma_f^2 + \hat{x}_P \cdot \sigma_{fP}}, \text{ if } x_f^+ \cdot \sigma_f^2 + \hat{x}_P \cdot \sigma_{fP} < 0, \end{aligned}$$

are in line with decreasing positive marginal utility at least with respect to an investor's expected terminal wealth and are consistent with an expected utility maximizing choice of $\bar{u}^+$ regarding the best fund under consideration. Additionally, we need $\omega > 0$ because of our requirement of positive skewness preferences.

3) For cubic HARA utility as described by (9) conditions (12a) and (12b) simplify to

$$\omega = 1.5 \cdot \frac{\partial \sigma^{+2}}{\partial \bar{u}^+} \pm \sqrt{\left(1.5 \cdot \frac{\partial \sigma^{+2}}{\partial \bar{u}^+}\right)^2 - 9 \cdot \sigma^{+2} - 3 \cdot \frac{\partial \gamma^{+3}}{\partial \bar{u}^+}} \tag{13}$$

with the additional requirement of ω being positive. □

Proof: See Appendix 3.

The considerations of this subsection highlight the relationships between the different approaches to justify skewness preferences. We favor the application of the preference function (2) (or (3)), for this objective function encompasses the maximization of expected cubic (HARA) utility as a special case. As in a situation with mean-variance preferences, a utility-theoretic foundation of mean-variance-skewness preferences does not seem to be a sine qua non for the application of (2).[11]

As a last point it should be noted that there is just one drawback of the approach applied in this paper. In what follows we will utilize the preference function (4) with given parameter combination $(\bar{u}^+, \omega)$ to solve F different portfolio selection problems for the exogenous case and for the endogenous case. In each of them, one fund f is optimally combined with the (possibly exogenously given holding of) reference portfolio P and riskless lending or borrowing. Subsequently, funds are ranked according to the corresponding maximum preference values they offer and these preference values – after some algebraic manipulations – are interpreted as performance values. In contrast to the analysis sketched above, we are thus examining not just one, but F different portfolio selection problems with fixed values for $\bar{u}^+$ and ω. Unfortunately, for a given preference function (2) or (3) it is not *sufficient* to be the best fund based on (4) and a given expected return $\bar{u}^+$ for being the best fund at all, because another fund may be better than that fund for another value of $\bar{u}^+$ and it might be that different values for $\bar{u}^+$ describe optimal portfolio selection behavior for different funds. This is a problem typically not discussed in the literature, although there are other approaches that rely on similar stan-

[11] For such an argument in the case of pure mean-variance preferences see *Löffler* (1996).

dardization techniques like the ones by *Graham/Harvey* (1997) and *Modigliani/Modigliani* (1997). In fact, only in the endogenous case with mean-variance preferences it is apparent that there is not any problem, because the fund ranking here is identical for any given desired expected excess return $\bar{u}^+$, a feature which is not generally shared by more complex decision problems.

However, being the best fund for at least one value $\bar{u}^+$ is a *necessary* condition for being the best overall fund for given preferences. Therefore, the investor only has to choose among those funds which are best for at least *one* achievable expected excess return $\bar{u}^+$. Typically, we will expect only a few funds to emerge as candidates and among them an investor should be able to choose without further formal assistance. We will return to this issue in Section VI.

III. The exogenous case

1. Some basic variables

At first glance, the replacement of equation (2) by formula (4) does not seem to be too great an alleviation of the original decision problem. Nevertheless, this approach enables us to derive a measure of performance evaluation that consists of several easily understandable basic elements. In order to do so, we additionally have to introduce the notion of subportfolio Q(f) which consists of the riskless asset as well as of the investor's holding of a fund f and thus describes the variable part of his or her overall portfolio in the exogenous case (see also Figure 1). Correspondingly, R(f) stands for an investor's risky subportfolio consisting of a relative investment $y_f := x_f/(x_f+x_P)$ in a fund f and of $y_P := x_P/(x_f+x_P)$ for direct stock holdings. Furthermore, we define $\bar{u}^+_{Q(f)} := \bar{u}^+ - x_P \cdot \bar{u}_P$ as the contribution of portfolio Q(f) to overall expected excess return $\bar{u}^+$. It should be noted that – in the exogenous case – we have $\bar{u}^+_{Q(f)} = \bar{u}^+_Q(x_P) =: \bar{u}^+_Q = \text{const.}$ for all funds f = 1, ..., F, since $\bar{u}^+$, $x_P = \hat{x}_P$ as well as $\bar{u}_P$ are exogenously given for any fund f.

From now on, we assume all expected excess returns to be nonnegative, as investments with negative expected excess returns are generally not preferable. Moreover, we introduce $\gamma_{Q(f)Q(f)P}$ and $\gamma_{Q(f)PP}$ as symbols for the two co-skewnesses $E[(\tilde{u}_{Q(f)} - \bar{u}_{Q(f)})^2 \cdot (\tilde{u}_P - \bar{u}_P)]$ and $E[(\tilde{u}_{Q(f)} - \bar{u}_{Q(f)}) \cdot (\tilde{u}_P - \bar{u}_P)^2]$. In addition, we need symbols $b_{Q(f)Q(f)P}$ and $b_{Q(f)PP}$ for fractions $\gamma_{Q(f)Q(f)P} / \gamma_P^3$ and $\gamma_{Q(f)PP} / \gamma_P^3$, respectively. Finally, we define $\sigma_{Q(f)P}$ as the covariance between excess returns $\tilde{u}_{Q(f)}$ and $\tilde{u}_P$ and $\beta_{Q(f)P} := \sigma_{Q(f)P} / \sigma_P^2$ as the corresponding regression coefficient. All relevant (co-) moments regarding excess returns $\tilde{u}_f$ and $\tilde{u}_P$ are named analogously and indexed by an “f” or a “P”.

2. The investor-specific cubic performance measure

As already mentioned in Section I, *Scholz/Wilkens* (2003) analyzed the exogenous case for mean-variance preferences. This means that their approach can be interpreted as if examining F portfolio selection problems according to the setting of Figure 1 based on (4) with preference parameter $\lambda = 0$ (i.e. $\omega \to \infty$). From the resulting optimal preference values for each fund f under consideration, they derived a so-called (quadratic) investor-specific performance measure (“qIM”, henceforth), because fund rankings turn out to be depending on investors’ specific preferences, since the two-fund separation theorem does not apply. From the analysis in *Scholz/Wilkens* (2003) and *Breuer/Gürtler* (2005) we know that qIM only depends on the quadratic *Sharpe* measure qSM and the quadratic *Treynor* measure qTM as defined in Table 2. This finding is intuitive appealing, as the first measure applies for the special case $y_P^{(exg)} = 0,$ i.e no direct stock holdings at all, and the latter for the special situation $y_P^{(exg)} \to 1,$ i.e. only marginal fund investments. Up to now, for performance evaluation with mean-variance-skewness preferences, we can only refer to *Breuer/Gürtler* (2006). As has already been mentioned in Section I, they showed that for the endogenous case with cubic HARA utility functions each fund is evaluated on the basis of two basic performance measures. The first one is the optimized quadratic *Sharpe* measure, that is, the value of qSM for the best combination of a fund f and the reference portfolio P. The second one can be interpreted as an optimized cubic *Sharpe* measure. We may define a cubic *Sharpe* measure by replacing the original numerator or denominator of the quadratic *Sharpe* measure $\bar{u}/\sigma$ by γ, i.e. the third root of the respective return skewness. Since this leads to two different versions of a cubic *Sharpe* measure (see also Table 2), we call the one with γ in the numerator the cubic *Sharpe* measure of type 1, and the one with γ in the denominator the cubic *Sharpe* measure of type 2. For the endogenous case with cubic HARA utility *Breuer/Gürtler* (2006) showed the relevance of the cubic *Sharpe* measure of type 1. However, we will shortly see that in the exogenous case the cubic *Sharpe* measure of type 2 becomes relevant. In a similar way, one may distinguish more than just one cubic *Treynor* measure. In the quadratic case there is just one covariance σ_{fP} between $\tilde{u}_f$ and $\tilde{u}_P$. Nevertheless, there are at least two co-skewnesses γ_{ffP} and γ_{fPP} as defined in Table 2 and consequently there are two *Treynor* measures: type 1, defined as $\bar{u}^2/b_{ffP}$ and type 2, defined as $\bar{u}/b_{fPP}$.

In fact, as is revealed by formula (T1) and Result T1 i) of Table 3, a repetition of the analysis of *Scholz/Wilkens* (2003) for $\lambda > 0$, i.e. for an investor with positive skewness preferences, leads to a **c**ubic **i**nvestor-specific performance **m**easure $cIM_f^{(exg)}(\hat{x}_P)$ for the exogenous case that is a function not only of the quadratic *Sharpe* and *Treynor* measure, but also of the cubic *Sharpe* measure of type 2 and of both cubic *Treynor* measures.[12] Other fund specific parameters are not relevant for performance evaluation in the exogenous case. The performance measure (T1), although lengthy, can thus be traced back to only a few fund-dependent determinants. To be more precise, for the typical case of a positive value of $\bar{u}^+_{Q(f)}(\hat{x}_P)$, the performance of a fund f is the better, the greater its quadratic *Sharpe* measure qSM_f. Moreover, in our empirical example of Section V all funds f under consideration as well as the reference portfolio P exhibit negative return skewnesses and co-skewnesses and positive return covariances. For such a situation fund performance is ceteris paribus improving with a higher quadratic *Treynor* measure as well as a lower cubic *Sharpe* performance measure $cSM_f^{(2)}$ of type 2 and becoming better with greater cubic *Treynor* measures. The negative impact of the cubic *Sharpe* measure “2” may appear somewhat surprising, but it is simply caused by the fact that higher values of γ_f, i.e. of the denominator of the cubic *Sharpe* measure 2, lead to higher preference values, while this coincides with a lower cubic *Sharpe* measure 2. In any case, it should be clear that for certain relationships between their respective *Sharpe* measures on the one side and their *Treynor* measures on the other, two funds can be unambiguously ranked regardless of which pair $(\bar{u}_f, \omega)$ is in effect. Result T1 ii) of Table 3 thus tells us under which conditions an investor does not need to bother much about the precise specification of his or her preference parameters. In addition, according to Result T1 ii) it then even plays no role at all, if the exogenous or the endogenous case is considered. Furthermore, the relevance of the fund-dependent submeasures in $cIM_f^{(exg)}(\hat{x}_P)$ may become clearer, if we examine some special cases, as is done in the following subsection.

>>> Insert Table 3 about here <<<

[12] See Appendix 4 for a proof of Result T1.

3. Some special cases

Special cases arise for extreme values of ω and $\hat{x}_P$. Some of them are described in Table 3.[13] Case 2 a) ($\omega \to \infty$) describes a situation with mere mean-variance preferences, while ω = 0 implies a situation with mere mean-skewness preferences. In this context, it should be noted that ω = 0 does not necessarily imply that the investor is not "variance averse" at all. It simply means that the relevance of his or her skewness loving exceeds the relevance of variance considerations by an infinite amount.

Rather interestingly, taking together cases 2 a) and 2 b) gives Result T2 of Table 3 which leads to a second possibility to assess potential fund rankings without the precise specification of ω: For given desired expected excess return $\bar{u}^+$ a fund g is better than a fund h for any preference parameter ω, if its performance measure $cIM_g^{(exg)}(\hat{x}_P)$ is greater than that of fund h, $cIM_h^{(exg)}(\hat{x}_P)$, for both extreme scenarios $\omega \to \infty$ and for ω = 0. To put it another way: For given overall expected excess return, only funds with greater performance measures $cIM_g^{(exg)}(\hat{x}_P)$ for one of these extreme scenarios can be better than a certain fund h even for any other preference parameter ω with $0 < \omega < \infty$. The reason for these findings is that the resulting performance measure for values of ω with $0 < \omega < \infty$ is a linear combination of the performance measures for the two extreme cases $\omega \to \infty$ and ω = 0.

We will use Result T2 of Table 3 in our empirical analysis presented later on, but now turn to special cases described by extreme values of $\hat{x}_P$. In fact, we are more interested in fractions y_f and y_P of fund f and reference portfolio P as parts of the risky subportfolio R(f) than in the fraction $\hat{x}_P$ in itself. Allowing for short sales restrictions we just have to consider situations with $y_f^{(exg)} = 1$ and $y_f^{(exg)} = \varepsilon$ with ε > 0, but small. The first case coincides with $y_P = 0$ and thus requires $\hat{x}_P = 0$. Case 2 c) in Table 3 refers to this situation. According to the last sentence of Result T3, it is even possible to conclude that (for all return skewnesses being of the same sign) a fund g is better than a fund h in the endogenous case in situations with border solutions $y_g^* = y_h^* = 1$, if fund's g cubic *Sharpe* measure "2" is smaller and its quadratic *Sharpe* measure is greater than the corresponding measure of fund h. This is quite remarkable, as according to *Breuer/Gürtler* (2006), in the endogenous case with cubic HARA utility, border solutions with no investment in the reference portfolio P of direct stock holdings at all

[13] The cases 2 a) and 2 b) immediately follow from (T1). See Appendix 5 for the derivation of the special performance measures of situations 2 c) and 2 d).

imply that fund rankings are only (positively) depending on the quadratic *Sharpe* measure qSM_f and the cubic *Sharpe* measure $cSM_f^{(1)}$ of type 1. Only in situations with $\gamma_g > 0$ and $\gamma_h > 0$ it is possible to always derive a greater cubic *Sharpe* measure "1" for a fund g in comparison with a fund h exhibiting both a greater quadratic *Sharpe* measure as well as a smaller cubic *Sharpe* measure "2").[14] Obviously, the performance submeasures according to (T6) thus offer new opportunities for straightforward performance assessments not at hand before.

Now consider the second limiting case described by $y_f^{(exg)*} = \varepsilon$ with $\varepsilon > 0$, but small. For such a situation, portfolio Q(f) just converges to the sole holding of the riskless asset and we thus arrive at a situation with $\bar{u}_Q^+(\hat{x}_P) \to 0$ (i.e. $\hat{x}_P \to \bar{u}^+ / \bar{u}_P$). For this, we get the special performance measure according to case 2 d) of Table 3. In fact, the limiting case $\hat{x}_P \to \bar{u}^+ / \bar{u}_P$ has also been analyzed in *Breuer/Gürtler* (2006) as a possible border solution for the endogenous case with HARA utility and has also led to the derivation of some kind of cubic *Treynor* measure, because for the special case of mean-variance preferences this cubic measure collapses to the (negative inverse of the) quadratic *Treynor* measure. Actually, this cubic *Treynor* measure of *Breuer/Gürtler* (2006) is a special case of the performance measure (T8) of this paper.[15] We thus once again have been able to generalize our findings.

IV. The endogenous case

In the endogenous case, for any given fund f the investor optimizes all three relative portions x_0, x_f, and x_P, simultaneously. Let therefore $x_P^{(f)*}$ stand for the optimal investment in reference portfolio P when combining this portfolio with the riskless asset and fund f, and define optimal fractions $x_0^{(f)*}$ and $x_f^{(f)*}$, analogously. Then, in the endogenous case, each fund f will be evaluated according to the performance measure $cIM_f^{(end)}$ of Table 3. This follows immediately from (T1) of Table 3 if one replaces $\hat{x}_P$ with $x_P^{(f)*}$.

Moreover, in the case of pure mean-variance preferences ($\omega \to \infty$), the best fund according to the optimized quadratic *Sharpe* measure as discussed, for example, in *Breuer/Gürtler* (1999) is always also the best one as well according to (T10) of Table 3 for arbitrary desired overall expected excess return $\bar{u}^+$.[16] It should be noted that such a relationship between (T10) and the

[14] See Appendix 6 for a proof of this statement.
[15] See Appendix 7 for a proof of this statement.
[16] See Appendix 8 for a proof of this statement.

optimized cubic performance measure of *Breuer/Gürtler* (2006) does not generally exist, as the latter performance measure is only based on cubic HARA utility.

With the findings of Sections III and IV, we now turn to the empirical investigation of the relevance of skewness preferences for fund rankings and the importance of the distinction between the endogenous case and the exogenous one.

V. Empirical example

In order to ensure comparability with the results of *Breuer/Gürtler* (2005) we follow their steps of analysis by considering (post tax) return data for 45 mutual funds investing in German equity shares[17] over a period from July 1996 to August 1999 which are calculated on the basis of the development of the respective monthly repurchase prices per share. We assume that all earnings paid out to the investors by a fund f are reinvested in this fund. The riskless interest rate r_0 can be approximated by the expected return of German time deposit running for one month and covering the respective period of time to be observed. We use the DAX 100 as a broadly diversified reference portfolio P. Based on this historical return data, for all 45 funds f and the DAX 100 unbiased estimators for the relevant moments of one-monthly returns are calculated and listed in Table 4.[18]

>>> Insert Table 4 about here <<<

1. Differences in rankings in the exogenous case and the endogenous one

From the analysis in Section III we know that, in the case of short sales restrictions with all skewnesses and co-skewnesses of fund returns being negative, a fund g with a higher quadratic *Sharpe* measure and a higher quadratic *Treynor* measure as well as a lower cubic *Sharpe* measure of type 2 and higher cubic *Treynor* measures than a fund h simultaneously exhibits a greater restricted optimized cubic performance measure and a greater cubic investor specific performance measure $\text{IM}_f^{(exg)}$ (Result T1 of Table 3). While for the case of simple mean-variance preferences, i.e. with the neglection of all cubic submeasures, in *Breuer/Gürtler* (2005) it has been possible to identify 28 of our 45 funds for which the ranking according to their quadratic *Sharpe* measure and their quadratic *Treynor* measure, respectively, was identical, a similarly strong result for mean-variance-skewness preferences cannot be obtained.

[17] In what follows we briefly speak of German funds, though we do not mean their country of origin, but the geographical focus of their investments.
[18] See *Rohatgi* (1976) for the unbiased estimators of the expectation value and the second central moment. Unbiased estimators of other moments can be worked out in just the same manner.

These 28 funds are listed first in Table 4, but only funds # 1 to # 17 can be unambiguously ranked in a situation with skewness preferences as well. The number of each of these first 17 funds in the first column coincides with their ranking position when compared to each other, while the last column of Table 4 is relevant for the first 28 funds and gives their – unambiguous – corresponding ranking place for mean-variance preferences. When taking into account simultaneously all 45 funds, there is no unambiguous fund ranking, i.e. a comparison of the last 17 funds (# 29 to # 45) with funds # 1 to # 28 depends on the specific parameter constellation (ω, $\bar{u}^+$, $\hat{x}_P$) under consideration. According to this result, the recognition of skewness preferences may lead to a greater variety in fund ranking depending on the given fraction $\hat{x}_P$ of the reference portfolio P and the desired overall expected excess return $\bar{u}^+$.

In order to better assess resulting differences in rankings we follow *Breuer/Gürtler* (2005) by calculating Spearman ranking correlation coefficients ρ_{SP} between rankings according to the exogenous cubic investor-specific performance measure (in what follows: "exogenous cubic IM-rankings") for given identical desired overall expected excess returns $\bar{u}^{+(1)} = \bar{u}^{+(2)} = \bar{u}^+$ with $\bar{u}^+ \in \mathsf{U}^+ := \{1.7719\ \%, 1.9\ \%, 2.0\ \%, \ldots, 2.7\ \%, 10\ \%$[19]$\}$ and different values $x_P^{(1)}$ and $x_P^{(2)}$ with $x_P^{(1)}, x_P^{(2)} \in X_P = \{0, 5\ \%, \ldots, 95\ \%, 99.99\ \%\}$. To assure comparability of our results with those of *Breuer/Gürtler* (2005), we thereby restrict ourselves in the same way as *Breuer/Gürtler* (2005) to the analysis of the funds # 29 to # 45 of Table 4. Moreover, we must allow for different intensities of skewness preferences. Thereby, because of space constraints we only consider the two extreme cases $\omega = 0$ (mere mean-skewness preferences, i.e. an infinite prudence) and $\omega = 100{,}000$ (mere mean-variance preferences, i.e. a zero prudence).

For any given expected excess return $\bar{u}^+ \in \mathsf{U}^+$ we compute 21 different fund rankings, as this is the number of exogenous values $x_P^{(1)}$ and $x_P^{(2)}$ taken into account. This leads to an overall sum of $2 \cdot 210 = 420$ different fund rankings for all ten desired expected overall excess returns $\bar{u}^+$ under consideration with 210 of them (for $\omega = 100{,}000$; i.e., a situation with mere mean-variance preferences) already calculated by *Breuer/Gürtler* (2005).

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[19] We add $\bar{u}^+ = 10\ \%$ as an extreme value in order to better assess the stability of our results. Since we refrain from considering situations with short sales of stocks or funds, the minimum accessible value for $\bar{u}^+$ amounts to 1.7719 % because $\bar{u}_Q^+(x_P) = \bar{u}^+ - x_P \cdot \bar{u}_P > 0$ (and thus $x_f > 0$) is only fulfilled for all $0 \leq x_P \leq 1$ if $\bar{u}^+ > \bar{u}_P \approx 1.77189\ \%$.

As has already been pointed out by *Breuer/Gürtler* (2005) for the case of mean-variance preferences and given desired expected overall excess return $\overline{u}^+$, resulting correlation coefficients between two fund rankings do not change much, if differing pairs $(x_P^{(1)}, x_P^{(2)})$ of exogenous direct stock holdings are considered, as long as we have a constant value for $\Delta x_P := \left| x_P^{(1)} - x_P^{(2)} \right|$. In fact, this finding carries over to situations with (even extreme) skewness preferences. For example, for the special case of a desired expected excess return $\overline{u}^+$ = 2.3 % with ω = 0, varying values $x_P^{(1)}$ and $x_P^{(2)}$ with $\left| x_P^{(1)} - x_P^{(2)} \right| = 10\ \%$ imply ranking correlation coefficients from 99.01961 % (for the respective two cubic IM-rankings in the case of $x_P^{(1)}$ = 50 % and $x_P^{(2)}$ = 40 % as well as in the case of $x_P^{(1)}$ = 55 % and $x_P^{(2)}$ = 45 %) to 100 % (e.g., for the respective two cubic IM-rankings in the case of $x_P^{(1)}$ = 20 % and $x_P^{(2)}$ = 10 % as well as in the case of $x_P^{(1)}$ ≈ 100 % and $x_P^{(2)}$ = 90 %) thus leading to a variation of ρ_{SP} of only 0.98039 percentage points.[20] Variations of ρ_{SP} for other expected excess returns $\overline{u}^+$ and given differences Δx_P are of similar scale. Hence, as in *Breuer/Gürtler* (2005), it suffices to take a closer look at average correlation coefficients between exogenous quadratic or cubic IM-rankings, respectively, for different identical values of desired expected returns $\overline{u}^+ \in U^+$ and varying differences $\Delta x_P \in X_P$ between exogenously given holdings of the reference portfolio P.

Once again, the results of *Breuer/Gürtler* (2005) for the case of mean-variance preferences also hold true for situations with skewness preferences: We find out that average correlations between two fund rankings with given difference Δx_P and given identical desired expected return $\overline{u}^+$ are rather high, even if we restrict ourselves to funds which cannot be unambiguously ranked according to the quadratic and cubic submeasures regardless of the intensity of skewness preferences. In fact, for $\overline{u}^+ \in U^+$ and $\Delta x_P \in X_P$, smallest average values of ρ_{SP} amount to 91.17647 % for ω = 100,000 ($\overline{u}^+$ = 1.7719 % and Δx_P ≈ 100 %) and 96.07843 % for ω = 0 ($\overline{u}^+$ = 1.7719 % and Δx_P = 95 % or Δx_P ≈ 100 %).[21] Moreover, average ranking correlation coefficients are slightly decreasing with falling value for $\overline{u}^+$. Finally, average ranking correlation coefficients are smallest for high differences Δx_P which is intuitively appealing. Since $\Delta x_P = 1\ (-\varepsilon)$ implies either $(x_P^{(1)} = 1 - \varepsilon,\ x_P^{(2)} = 0)$ or $(x_P^{(1)} = 0,\ x_P^{(2)} = 1 - \varepsilon)$ as

[20] See also Table Ad 1.
[21] See also Tables Ad 2a and Ad 2b.

well as $\bar{u}^{+} = \bar{u}_{P} + \delta$ (see case 2 d) of Table 3) with ε and δ being positive and small, for Δx_{P} near to 100 % the corresponding (average) ranking correlation coefficient is identical to the correlation coefficient between the rankings according to the special performance measures (T6) and (T8) for $\bar{u}^{+} = \bar{u}_{P} + \delta \approx 1.7719$ %. While (T6) only depends on the quadratic *Sharpe* measure and the cubic type 2 *Sharpe* measure, (T8) is solely determined by the quadratic and the cubic type 2 *Treynor* measure of a fund f. For high enough ω (e.g. ω = 100,000) the relevance of the cubic submeasures vanishes and (T6) and (T8) keep depending only on the quadratic *Sharpe* measure or the quadratic *Treynor* measure as is already well-known from *Breuer/Gürtler* (2005). In contrast, for ω = 0, only the cubic submeasures remain relevant.

In our empirical setting, average correlations are thus increasing in $\bar{u}^{+}$ and decreasing in Δx_{P}. For such a situation, a high correlation between both quadratic measures and between both cubic measures mentioned before implies a high correlation between two fund rankings for Δx_{P} near to 1 and $\bar{u}^{+}$ near to $\bar{u}_{P}$ and necessarily even higher correlations for other parameter values. In our empirical example, all pairs ($\bar{u}^{+}$, Δx_{P}) under consideration thus lead to values for average ranking correlation coefficients not smaller than 91.1764 % (for ω = 100,000) or 96.07843 % (for ω = 0).

The limited independent relevance of the exogenous case even with explicit recognition of positive skewness preferences is also underpinned by ranking correlation coefficients between fund rankings according to the exogenous (quadratic or cubic) IM and the corresponding endogenous ones for different values $\bar{u}^{+}$ and x_P (and either ω = 100,000 or ω = 0). Moreover, two exogenous quadratic or cubic IM-rankings with identical equity holdings as described by x_P, but different values $\bar{u}^{+(1)}$ and $\bar{u}^{+(2)}$ for desired overall expected excess return will generally be very similar, since ranking correlation coefficients between exogenous IM-rankings and the corresponding endogenous ones do not change much for varying expected excess returns $\bar{u}^{+}$. For example, even in the extreme case of $\bar{u}^{+}$ = 1.7719 %, ranking correlation coefficients between the exogenous quadratic (cubic) and the endogenous quadratic (cubic) performance measure only vary between 93.13725 % for x_P = 0 % and 99.50980 % for x_P = 95 % (between 97.54902 % for x_P = 10 %, 15 %, 20 %, 30 %, 35 % and 98.77451 % for x_P = 60 %)

with 93.13725 % (97.54902 %) being the lowest correlation coefficient between the endogenous fund ranking and the exogenous ones for $\overline{u}^+ \in U^+$ and $x_P \in X_P$.[22] Summarizing, we get

Result 2:

At least for our empirical example even in situations with mere mean-skewness preferences there seems to be no need to explicitly distinguish between the exogenous case and the endogenous one. The corresponding result of *Breuer/Gürtler* (2005) for a situation with simple mean-variance preferences is thus confirmed even if skewness preferences are allowed for. □

Certainly, Result 2 *is* of immediate *practical* relevance, as – for purposes of fund rankings – investors need not take care whether their amount of direct stock holdings is endogenous or exogenous. They even need not bother about the scale of their direct stock investments at all.

2. Differences in rankings with and without skewness preferences

However, the explicit recognition of the exogenous case may lead to new insights into the relevance of skewness preferences in performance evaluation. Based on ten German funds, *Breuer/Gürtler* (2006) arrived at a correlation of 95.87 % between the fund ranking according to the restricted optimized quadratic *Sharpe* measure and the restricted optimized cubic performance measure for the endogenous HARA case just suggesting only a limited importance of skewness preferences for fund rankings. However, *Breuer/Gürtler* (2006) showed additionally that welfare losses could be significant when applying a mean-variance approach to approximate cubic HARA utility in order to determine the optimal allocation of initial wealth to riskless lending/borrowing and the holding of risky assets.

The performance measures developed in this paper allow us to go beyond the examination of cubic HARA utility functions. To this end, we compute the rankings of funds for the two cases $\omega = 0$ and $\omega = 100{,}000$ with $x_P = 50$ % and $\overline{u}^+ \in \{1.7719$ %, 2.0 %, 2.2 %, 2.4 %, 2.6 %, 10 %$\}$. Ranking correlation coefficients ρ_{SP} between these two rankings for $\omega = 0$ and $\omega = 100{,}000$ are highest for $\overline{u}^+ = 2.4$ % with 69.8529 % and lowest for $\overline{u}^+ = 10$ % with 64.2157 %.[23] Consequently, the average correlation between these two rankings is as low as 67.6062 %. The same holds true for other values of $x_P \in \{0$ %, 10 %, 20 %, 30 %, 40 %, 50 %, 60 %, 70 %, 80 %, 90 %, 99.99 %$\}$, as average correlation coefficients only vary from 62.5 % for x_P

[22] See also Tables Ad 3a and Ad 3b.
[23] See also Table Ad 4.

= 0 % to 67.8105 % for x_P = 60 %.[24] Obviously, suitable choices of ω may lead to a relevance of skewness considerations for fund rankings that exceeds by far that for the case of cubic HARA utility.

Furthermore, as pointed out in Result T2 of Table 3, from the two extreme rankings ω = 100,000 and ω = 0 with given values x_P and $\bar{u}^+$ possible variations in fund rankings for other values ω can be derived. In Table 5 these ranges have been computed for x_P = 50 % and $\bar{u}^+$ = 1.7719 %. Table 5 shows once again that skewness considerations might lead to a significant variability in fund rankings. Similar results are obtained for other values of $\bar{u}^+ \in$ {1.7719 %, 2.0 %, 2.2 %, 2.4 %, 2.6 %, 10 %}.[25]

>>> Insert Table 5 about here <<<

Based on our findings until now, it should not be too surprising that rankings according to the optimized quadratic *Sharpe* measure approximate only poorly exogenous (and endogenous) cubic IM-rankings in the general case of arbitrary admissible values for ω. To verify this assertion, we calculated correlation coefficients between fund rankings for the exogenous cubic IM (with ω = 0) and the optimized restricted quadratic *Sharpe* measure with different values of $\bar{u}^+$ and of $x_P \in$ {0 %, 10 %, 20 %, 30 %, 40 %, 50 %, 60 %, 70 %, 80 %, 90 %, 99.99 %}. Resulting correlation coefficients only vary between 66.1764 % (e.g., for x_P = 40 % and $\bar{u}^+$ = 2.2 %, 2.3 %, 2.4 %, or x_P = 45 % and $\bar{u}^+$ = 2.3 %, 2.4 %, 2.5 %) and 70.83333 % for, e.g., x_P = 55 % and $\bar{u}^+$ = 2.7 %.[26] The reason for this and the previous findings is that – with non-HARA preferences – we are no longer in a situation where the two-fund separation theorem holds so that variations of risk preferences may influence fund rankings.

With two-fund separation being in effect, variations of an investor's risk aversion are not able to influence the structure of optimal risky portfolios and thus the ranking of funds neither. In fact, for the endogenous case with cubic HARA utility we arrive at a fund ranking according to the restricted optimized cubic performance measure that exhibits a correlation coefficient of 98.2843 % with the performance evaluation for ω = 100,000 (i.e. pure mean-variance preferences), x_P = 50 %, and $\bar{u}^+$ = 2.4 %. For cubic HARA utility in the exogenous case with x_P =

[24] See also Table Ad 5.
[25] See also Table Ad 6.
[26] See also Table Ad 7.

50 % and $\overline{u}^{+}$ = 2.4 % we should get a similarly high correlation. Actually, for the exogenous case to be consistent with cubic HARA utility the choice $\overline{u}^{+}$ = 2.4 % has to be optimal at least for the best fund in question. From (13) we get only one positive solution $\omega \approx 1.468479$ which supports the choice $\overline{u}^{+}$ = 2.4 % for the best fund # 30 and guarantees positive marginal utility for $u = \overline{u}^{+}$. For $\omega \approx 1.468479$ and with $\overline{u}^{+}$ = 2.4 % and x_P = 50 % the resulting fund ranking exhibits a ranking correlation coefficient of 98.7745 % with respect to the fund ranking according to the exogenous quadratic IM. This supports our conjecture of only rather a limited relevance of skewness preferences in the case of cubic HARA utility.

Things change if we turn to arbitrary cubic utility functions, as they do not support the two-fund separation. For the general case of cubic utility we have to take care of conditions (12a) and (12b) with respect to the best fund in question in order to give our fund selection a utility-theoretic foundation. Rather remarkably, (12a) and (12b) lead to a negative lower bound for ω so that we are indeed free to choose ω near to zero without violating the assumption of optimal determination of $\overline{u}^{+}$ with respect to the best fund under consideration as well as positive marginal utility for $u = \overline{u}^{+}$. Moreover, the decrease of preference parameter ω towards 0 actually leads to a ceteris paribus higher prudence (converging towards infinity). This may offer an additional explanation for the emerging significant differences in fund rankings in comparison to mere mean-variance preferences for $\omega \to 0$, as quadratic utility implies a zero prudence. This finding is in line with *Breuer/Gürtler* (2001) who, among others, show that in the case of exponential and power utility functions – without two-fund separation – quadratic utility approximations work quite well in particular for the individual's risk aversion (and thus an individual's prudence) not being too great.

The analysis of the consequences of skewness preferences in this paper hence is of general importance and extends the examination of *Breuer/Gürtler* (2006) of situations with cubic HARA utility (in the endogenous case). Additionally, it becomes clear that cubic HARA utility instead of quadratic utility may become relevant, if one does not look at fund rankings, but at the optimal allocation of initial wealth on the riskless asset and risky assets, since for this decision the two-fund separation theorem does not hold.

Result 3:

As long as the two-fund separation theorem holds, the relevance of skewness preferences for fund rankings compared to rankings based on mere mean-variance preferences will be limited.

The two-fund separation theorem holds exactly for fund rankings in the endogenous case with cubic HARA utility. It also seems to hold approximately sufficiently well for fund rankings in the exogenous case with cubic HARA utility. Skewness preferences thus become most relevant in situations with cubic non-HARA utility or even preference functions recognizing skewness with no utility-theoretic foundation at all. Even for cubic HARA utility skewness preferences must not be neglected when determining the optimal combination of riskless lending/borrowing with the risky subportfolio Q(f), as for this decision the two-fund separation theorem does not hold true, either. □

VI. Issues of practical application

Certainly, the theoretical and empirical findings of the preceding sections have merits of their own. Nevertheless, after having thus thoroughly investigated the relevance of exogenous stock holdings and skewness preferences in performance evaluation, one might wonder how the conclusions of this paper may be put into practical operation. As indicated, it seems to be admissible to give up the distinction between the exogenous case and the endogenous one. It thus remains to analyze the influence of different combinations of $\overline{u}^{+}$ and ω on performance evaluation. Apparently, it is of only little use to present dozens of fund rankings for different pairs $(\overline{u}^{+}, \omega)$. Nevertheless, it would be of no great difficulty to create an interactive online-supply so that investors may input their desired expected excess return and their "prudence" via internet. Obviously, only the specification of the latter parameter might cause some trouble. However, under the simplifying assumption of (relative) prudence being constant, its value might be indirectly derived from the choice of an individual among different return distributions with identical expected returns, but different return variances and skewnesses. As interactive online portfolio management tools are already in effect, it would not be too difficult to enlarge them by going beyond the simple mean-variance analysis. As an alternative, one might restrict oneself to the consideration of only typical values of (relative) prudence. According to *Merrigan/Normandin* (1996) and *Eisenhauer* (2000) such values lie in the range of 1 to 5, corresponding with typical values of our parameter ω between $0.6\cdot\mu_W \approx 0.6$ and $3\cdot\mu_W \approx 3$ for desired expected excess returns $\overline{u}^{+}$ amounting to only some percentage points (and thus μ_W being not very different from 1 for $W_0 = 1$), because we have $\mu_W \cdot P = 3\cdot\mu_W/\omega$ as the relevant relative prudence. Under this prerequisite, an investor would only have to name his or her desired expected excess return u^{+}. Besides this, it is worth mentioning that the empirical finding of parameter values for ω between 0.6 and 3 may be interpreted as a further evidence of the relevance of explicitly taking skewness considerations into account, as the case with mere mean-variance preferences is characterized by a value of infinity for ω.

Moreover, since investors are mainly interested in the best fund out of a given "universe" of funds, in traditional print media, it should be sufficient to only compute which fund ranks on top of the list depending on the parameter specification $(\overline{u}^{+},\omega)$. For our empirical example, there are in fact only two funds out of the funds from # 29 to # 45 of Table 4 which can be the best one for different parameter constellations, as Figure 2 reveals, namely # 30 and # 38. This finding is particularly a consequence of the application of Result T2 of Table 3 which enables us – to some extent – to unambiguously rank funds even if the preference parameter ω is not precisely specified.

It has already been pointed out that being the best fund for at least one parameter constellation is only a *necessary* condition for being selected by a mean-variance-skewness investor. Therefore, in our empirical example, such investors have only to choose between fund # 30 and # 38 of our list. Moreover, if we restrict ourselves to empirically typical values of ω between 0.6 and 3 (and "reasonable" values for $\overline{u}^{+}$), fund # 38 will *always* be ranked before fund # 30. Under these conditions, fund # 38 can unambiguously be identified as the best fund under consideration. Summarizing, the approach presented in this paper may indeed be applicable in real-life fund selection problems.

>>> Insert Figure 2 about here <<<

VII. Conclusion

The main goal of this paper was to develop general performance measures for investors with mean-variance-skewness preferences who aim at selecting one out of F different funds in order to combine this fund optimally with direct stock holdings and the riskless asset. We contributed to the literature by developing such performance measures without the restriction to HARA utility and for situations with (the "exogenous case") and without (the "endogenous case") exogenously given direct stock investments. Resulting performance measures are functions of several fundamental submeasures which can be interpreted as various kinds of quadratic and cubic *Sharpe* and *Treynor* measures. Moreover, performance measures are controlled by two subjective parameters, one of them $(\overline{u}^{+})$ being an investor's desired overall expected excess return and the other (ω) describing his or her intensity of skewness preferences. For the case of *von Neumann-Morgenstern* utility functions the latter parameter can be characterized as (three times) the reciprocal of *Kimball*'s (1990) prudence.

Our empirical example extends the result of *Breuer/Gürtler* (2005) for situations with mean-variance preferences that the distinction between the endogenous case and the exogenous case is hardly of practical relevance. Certainly, *this* result is of practical importance, as it implies that investors need not bother about this issue at all, thus simplifying performance evaluation tremendously. Moreover, with two-fund separation being in effect, mean-variance approaches approximate fund rankings for the case of mean-variance-skewness preferences rather well, as then different degrees of risk aversion (and prudence) cannot influence fund ranking. Since such situations require cubic HARA utility, skewness preferences are most important for fund rankings in situations with an investor's cubic utility function being not of the HARA type or even lacking any utility-theoretical foundation at all. It seems to be interesting to support this result by additional analytical and empirical examinations. For example, additional empirical analyses in particular of hedge funds should be of interest because of their special distributional return properties. This would enable us to analyze whether the recent findings of *Eling/Schuhmacher* (2006, 2007) for hedge funds that the simple quadratic *Sharpe* measure is sufficient for performance evaluation purposes in comparison to the utilization of several other approaches would even hold when applying the performance measures developed in this paper. Prima facie, we would expect fund rankings for mean-variance-skewness preferences not based on cubic HARA utility to possibly deviate considerably from the result of an application of the simple quadratic *Sharpe* measure. However, such considerations have to be reserved for future research.

Although fund rankings in the exogenous case are preference-dependent, the number of possible candidates for being the best fund out of F different ones is typically rather small. As a practical application of the performance measures developed in this paper one may identify the best fund as a function of preference parameter ω and desired expected overall excess return $\overline{u}^+$ and visualize the findings in a $(\overline{u}^+, \omega)-$diagram.

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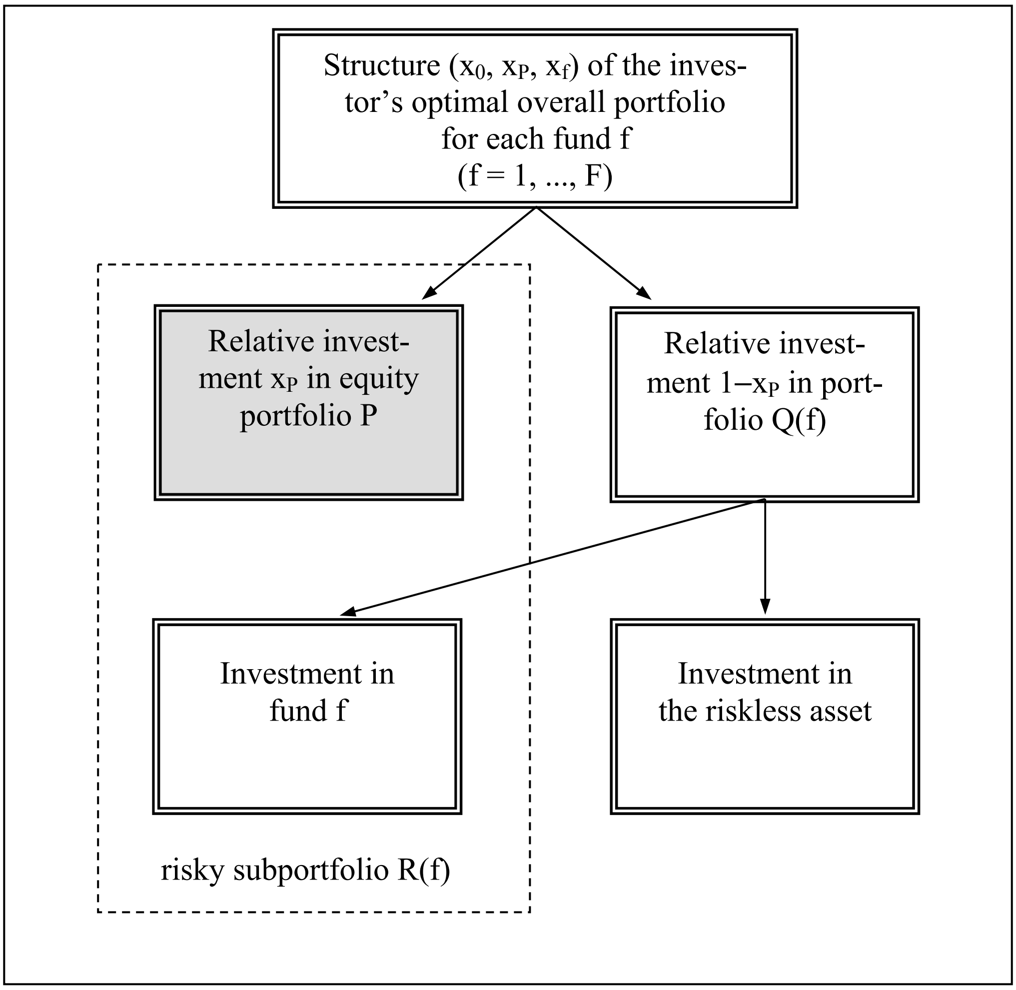
Figure 1

Structure of the Investor's Optimal Overall Portfolio



In the endogenous case, for any fund f under consideration the investor simultaneously optimizes relative shares x_0 (of the riskless asset), x_P (of the equity portfolio P), and x_f (of the fund f). In the exogenous case, only x_f and x_0 (subportfolio Q(f)) can be optimized, since $x_P = x_P^+ = \text{const.}$ (i.e. the "shaded" component in Figure 1 is given).

Figure 2

Pairs (ω, $\bar{u}_f^+$) Leading to an Identical Performance of Funds # 30 and # 38

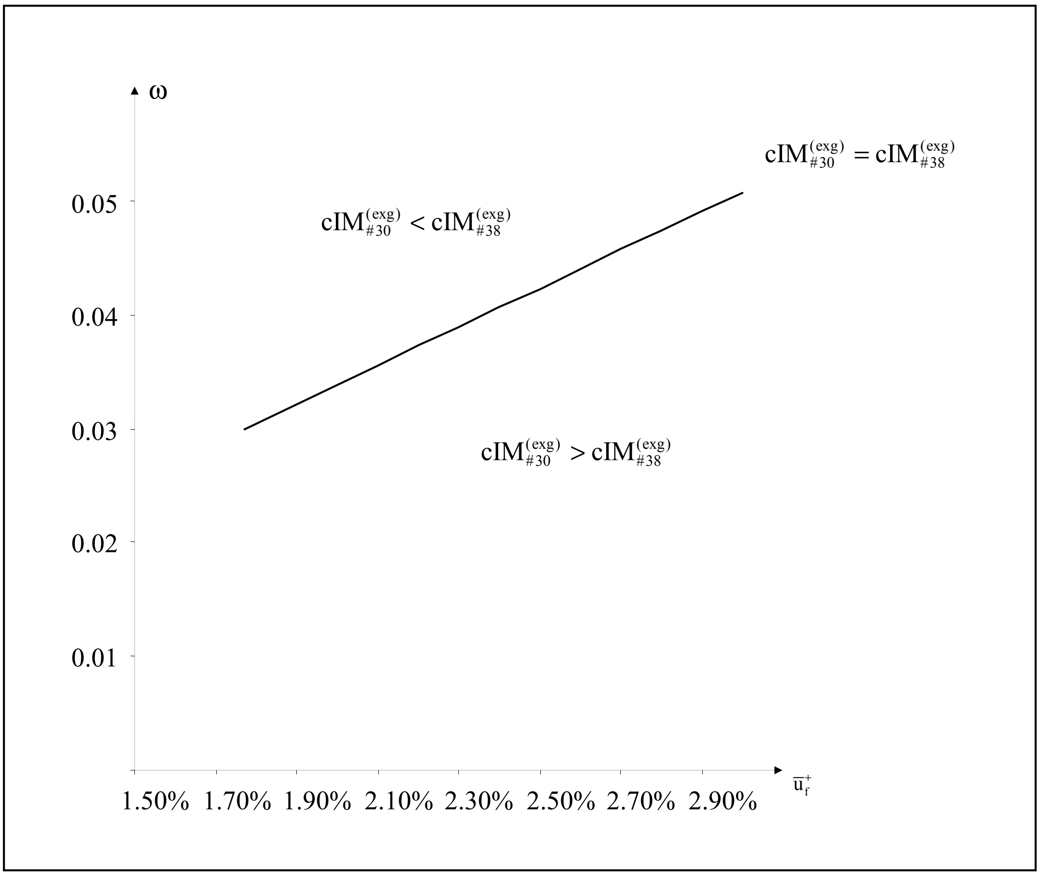


Table 1

Different Scenarios for Performance Evaluation

	Mean-variance preferences	Mean-variance-skewness preferences
Endogenous case	(1)	(3)
Exogenous case	(2)	(4)

Table 2

Synopsis of Relevant Symbols

Assets:

f,g,h: investment funds, F: total number of funds,
P: portfolio of direct stock holdings (serving as the "reference portfolio"), f^*: "best" fund out of all funds f = 1, …, F.

Investor's subportfolios (being part of the investor's total asset holdings):

R(f): risky subportfolio, i.e. (only) investment in fund f and in reference portfolio P,
Q(f): subportfolio which – in the exogenous case – is not already fixed, i.e. (only) investment in fund f and riskless lending or borrowing.

Return characteristics:

r_0 : riskless interest rate,
$\tilde{r}_f$: return of fund f,
$\tilde{r}_P$: return of reference portfolio P,
$\tilde{u}_f$: excess return $\tilde{r}_f - r_0$ of f with expectation value $\bar{u}_f$ and standard deviation σ_f,
$\tilde{u}_P$: excess return $\tilde{r}_P - r_0$ of P with expectation value $\bar{u}_P$ and standard deviation σ_P,
$\tilde{u}_{Q(f)}$: excess return $\tilde{r}_{Q(f)} - r_0$ of Q(f) with expectation value $\bar{u}_{Q(f)}$ and standard deviation $\sigma_{Q(f)}$,
$\tilde{u}_{R(f)}$: excess return $\tilde{r}_{R(f)} - r_0$ of R(f) with expectation value $\bar{u}_{R(f)}$ and standard deviation $\sigma_{R(f)}$,
σ_{fP} : covariance between $\tilde{u}_f$ and $\tilde{u}_P$,
$\beta_{fP} := \sigma_{fP} / \sigma_P^2$ (regression coefficient of a linear regression of $\tilde{u}_f$ with respect to $\tilde{u}_P$),
σ_f^2, σ_P^2 : variance of $\tilde{u}_f$ or $\tilde{u}_P$, respectively,
γ_f^3, γ_P^3 : skewness (i.e. the third central moment) of $\tilde{u}_f$ or $\tilde{u}_P$, respectively ,
γ_{ffP} : co-skewness of type 1 between $\tilde{u}_f$ and $\tilde{u}_P$, i.e. $E[(\tilde{u}_f - \bar{u}_f)^2 \cdot (\tilde{u}_P - \bar{u}_P)]$,
γ_{fPP} : co-skewness of type 2 between $\tilde{u}_f$ and $\tilde{u}_P$, i.e. $E[(\tilde{u}_f - \bar{u}_f) \cdot (\tilde{u}_P - \bar{u}_P)^2]$,
$b_{ffP} := \gamma_{ffP} / \gamma_f^3$, , $b_{fPP} := \gamma_{fPP} / \gamma_f^3$.

Decision variables:

x_0: fraction of monetary wealth risklessly invested ($x_0 < 0$: borrowing of money),
x_P: fraction of monetary wealth invested in reference portfolio P,
x_f: fraction of monetary wealth invested in shares of fund f.

Preference parameters:

Φ: cubic preference function
κ: preference weight on return variance
λ: preference weight on return skewness
ω:= κ/λ.

Specific parameters for the exogenous case:

$\bar{u}^+$: overall expected excess return desired by the investor,
$\hat{x}_P$: percentage of initial wealth already fixed by an investment in the reference portfolio P,
$u_Q^+(x_P)$: contribution of subportfolio Q(f) to an investor's overall achievable expected excess return

Performance measures:

qSM : quadratic *Sharpe* measure of f, i.e. $\bar{u}_f / \sigma_f$,
$cSM_f^{(1)}$: cubic *Sharpe* measure of f (type 1), i.e. γ_f / σ_f ,
$cSM_f^{(2)}$: cubic *Sharpe* measure of f (type 2), i.e. $\bar{u}_f / \gamma_f$,
qTM_f : quadratic *Treynor* measure of f, i.e. $\bar{u}_f / \beta_{fP}$,
$cTM_f^{(1)}$: cubic *Treynor* measure of f (type 1), i.e. $\bar{u}_f^2 / b_{ffP}$,
$cTM_f^{(2)}$: cubic *Treynor* measure of f (type 2), i.e. $\bar{u}_f / b_{fPP}$,
$qIM_f^{(exg)}(\hat{x}_P)$: quadratic investor specific performance measure in the exogenous case,
$cIM_f^{(exg)}(\hat{x}_P)$: cubic investor specific performance measure in the exogenous case,
$cIM_f^{(end)}(x_P^{(f)*})$: cubic investor specific performance measure in the endogenous case,

Optimal values are generally characterized by an asterisk ("*"), "exg" as an index stands for "exogenous", "end" for "endogenous".

Table 3

Performance Measures for Different Decision Situations

Setting	Performance Measure	Consequences
1) Exogenous case (general setting)	(T1) $cIM_f^{(exg)}(\hat{x}_P) := (\bar{u}_Q^+(\hat{x}_P))^3 \cdot \frac{1}{(cSM_f^{(2)})^3} + 3 \cdot \hat{x}_P \cdot \gamma_P^3 \cdot (\bar{u}_Q^+(\hat{x}_P))^2 \cdot \frac{1}{cTM_f^{(1)}} + 3 \cdot \hat{x}_P^2 \cdot \gamma_P^3 \cdot \bar{u}_Q^+(\hat{x}_P) \cdot \frac{1}{cTM_f^{(2)}}$ $+ \omega \cdot \left[(\bar{u}_Q^+(\hat{x}_P))^2 \cdot \frac{-1}{(qSM_f)^2} + 2 \cdot \hat{x}_P \cdot \sigma_P^2 \cdot \bar{u}_Q^+(\hat{x}_P) \cdot \frac{-1}{qTM_f} \right].$ (T2) $qSM_f := \frac{\bar{u}_f}{\sigma_f}$, $qTM_f := \frac{\bar{u}_f}{\beta_{fP}}$, (T3) $cSM_f^{(2)} := \frac{\bar{u}_f}{\gamma_f}$, $cTM_f^{(1)} := \frac{\bar{u}_f^2}{b_{ffP}}$, $cTM_f^{(2)} := \frac{\bar{u}_f}{b_{fPP}}$.	**Result T1:** i) Fund performance is completely determined by the quadratic *Sharpe* measure qSM_f and the quadratic *Treynor* measure qTM_f of fund f as well as by a cubic *Sharpe* $cSM_f^{(2)}$ and two cubic *Treynor* measures: $cTM_f^{(1)}$ and $cTM_f^{(2)}$ (since there are two possible co-skewnesses between returns $\tilde{r}_f$ and $\tilde{r}_P$). ii) For $u_{Q(f)}^+(\hat{x}_P) > 0$ and negative skewnesses and co-skewnesses, a fund g exhibits a better performance measure than a fund h for any desired overall expected excess return $\bar{u}^+$ and any preference parameter ω, if the quadratic submeasures as well as the cubic *Treynor* measures of fund g are both greater than the corresponding measures of fund h, while the cubic *Sharpe* measure of fund g is smaller than that of fund h. In such a situation, the fund g is better than the fund h even in the endogenous case, since $cIM_h^{(exg)}(x_P^{(h)*}) < cIM_g^{(exg)}(x_P^{(h)*}) \le cIM_g^{(exg)}(x_P^{(g)*})$ with "*" denoting optimal, i.e. preference maximizing endogenous, parameter values. □
2) Exogenous case (special settings)		
a) $\omega \to \infty$ (situation with mere mean-variance preferences)	(T4) $qIM_f^{(exg)}(\hat{x}_P) := (\bar{u}_Q^+(\hat{x}_P))^2 \cdot \frac{-1}{(qSM_f)^2} + 2 \cdot \hat{x}_P \cdot \sigma_P^2 \cdot \bar{u}_Q^+(\hat{x}_P) \cdot \frac{-1}{qTM_f}$.	For simple mean-variance-preferences the performance measure of *Scholz/ Wilkens* (2003) and *Breuer/Gürtler* (2005) evolves.
b) $\omega \to 0$ (situation with mere mean-skewness preferences)	(T5) $cIM_f^{(exg,\omega\to 0)}(\hat{x}_P) := (\bar{u}_Q^+(\hat{x}_P))^3 \cdot \frac{1}{(cSM_f^{(2)})^3}$ $+3 \cdot \hat{x}_P \cdot \gamma_P^3 \cdot (\bar{u}_Q^+(\hat{x}_P))^2 \cdot \frac{1}{cTM_f^{(1)}} + 3 \cdot \hat{x}_P^2 \cdot \gamma_P^3 \cdot \bar{u}_Q^+(\hat{x}_P) \cdot \frac{1}{cTM_f^{(2)}}$.	**Result T2:** For given desired expected excess return $\bar{u}^+$ it is just necessary to determine resulting fund rankings for the special situations 2 a) and 2 b). For any other preference value ω at least funds belonging to the intersection of all superior funds for these two rankings are better than a certain fund f. Additionally, at most the union of all these superior funds is better than fund f for any given value ω. □
c) $y_f^{(exg)*} = 1$ (situation with $y_P = 0$ and thus $\hat{x}_P = 0$)	(T6) $cIM_f^{(exg)}(0) = \frac{1}{(cSM_f^{(2)})^3} + \omega^{(mod1)} \cdot \frac{-1}{(qSM_f)^2}$ with (T7) $\omega^{(mod1)} := \kappa/(\lambda \cdot \bar{u}^+)$.	**Result T3:** In situations without direct stock holding, performance evaluation simplifies to a weighted sum of a fund's quadratic *Sharpe* measure and the cubic *Sharpe* measure "2". Apparently, for all return skewnesses being positive or all of them being negative, a fund g with both a greater quadratic *Sharpe* measure and a smaller cubic *Sharpe* measure "2" than a fund g will be better for arbitrary parameters κ, λ, and $\bar{u}^+$. Obviously, in such a situation fund g is better than fund h even in the endogenous case with border solutions $\hat{x}_P := x_P^{(g)*} = x_P^{(h)*} = 0$, when there are no short sales possibilities for risky securities. □
d) $y_f^{(exg)*} = \varepsilon$ with $\varepsilon > 0$, but small (and $\bar{u}_Q^+(\hat{x}_P) > 0$) (Q(f) just converges to the sole holding of the riskless asset and we thus arrive at a situation with $\bar{u}_Q^+(\hat{x}_P) \to 0$ (i.e. $\hat{x}_P \to \bar{u}^+/\bar{u}_P$).)	(T8) $cIM_f^{(exg,\hat{x}_P=\frac{\bar{u}^+}{\bar{u}_P})} := \frac{1}{(cSM_P^{(2)})^3 \cdot cTM_f^{(2)}} + \omega^{(mod2)} \cdot \frac{-1}{(qSM_P)^2 \cdot qTM_f}$, with (T9) $\omega^{(mod2)} := (2 \cdot \kappa)/(3 \cdot \lambda)$.	**Result T4:** The performance of a fund f which is in optimum only marginally added to direct stock holdings can be solely determined by the knowledge of just two fund-dependent parameters with one of them being identical to the quadratic *Treynor* measure and the other one being one of the cubic *Treynor* measures. □
3) Endogenous case (general setting)	(T10) $cIM_f^{(end)} := (\bar{u}_Q^+(x_P^{(f)*}))^3 \cdot \frac{1}{(cSM_f^{(2)})^3} + 3 \cdot x_P^{(f)*} \cdot \gamma_P^3 \cdot (\bar{u}_Q^+(x_P^{(f)*}))^2 \cdot \frac{1}{cTM_f^{(1)}} + 3 \cdot (x_P^{(f)*})^2 \cdot \gamma_P^3 \cdot \bar{u}_Q^+(x_P^{(f)*})$ $\cdot \frac{1}{cTM_f^{(2)}} + (x_P^{(f)*})^3 \cdot \gamma_P^3 + \omega \cdot \left[(\bar{u}_Q^+(x_P^{(f)*}))^2 \cdot \frac{-1}{(qSM_f)^2} + 2 \cdot x_P^{(f)*} \cdot \sigma_P^2 \cdot \bar{u}_Q^+(x_P^{(f)*}) \cdot \frac{-1}{qTM_f} - (x_P^{(f)*})^2 \cdot \sigma_P^2 \right]$, with $x_P^{(f)*}$ being chosen in such a way so as to maximize $cIM_f^{(end)}$.	

Table 4

Unbiased Estimators for the Relevant Moments of German Funds and Reference Portfolio P and Funds Ranking according to Mean-Variance Preferences

No.	name of fund	$\bar{u}_f$	σ_f	σ_{fP}	γ_f	γ_{ffP}	γ_{fPP}	μ-σ-Ranking position
1	Baring German Growth	2.85000 %	7.05836 %	0.33608 %	-3.49081%	-0.02054%	-0.02394%	1
2	Metallbank Aktienfonds DWS	2.07324 %	5.14655 %	0.26836 %	-3.66506%	-0.01298%	-0.01899%	2
3	Plusfonds	2.40324 %	6.83304 %	0.40050 %	-6.14769%	-0.02503%	-0.02475%	4
4	DVG Fonds SELECT INVEST	2.07243 %	6.61112 %	0.40792 %	-6.12317%	-0.02345%	-0.02369%	6
5	SMH Special UBS Fonds 1	1.90811 %	6.60503 %	0.40739 %	-6.01770%	-0.02255%	-0.02320%	9
6	Frankfurter Sparinvest Deka	1.81324 %	6.41583 %	0.39600 %	-5.90287%	-0.02183%	-0.02288%	10
7	DekaFonds	1.91459 %	6.81638 %	0.42138 %	-6.25473%	-0.02444%	-0.02421%	11
8	FT Deutschland Dynamik Fonds	1.79459 %	6.59269 %	0.40786 %	-6.14591%	-0.02328%	-0.02348%	13
9	MK Alfakapital	1.98243 %	7.41669 %	0.45851 %	-6.88045%	-0.02955%	-0.02660%	16
10	MMWI PROGRESS Fonds	1.76081 %	6.71760 %	0.41379 %	-6.26142%	-0.02441%	-0.02414%	18
11	Interselex Equity Germany B	1.72514 %	6.60614 %	0.40989 %	-6.37581%	-0.02520%	-0.02447%	19
12	Parvest Germany C	1.60108 %	6.31697 %	0.39222 %	-6.35149%	-0.02511%	-0.02449%	20
13	DELBRÜCK Aktien UNION-Fonds	1.42919 %	6.25222 %	0.38175 %	-5.98832%	-0.02274%	-0.02352%	24
14	Lux Linea	1.71378 %	7.60317 %	0.46976 %	-7.54059%	-0.03546%	-0.02912%	25
15	Hauck Main I Universal Fonds	1.45865 %	6.58482 %	0.40521 %	-6.54482%	-0.02676%	-0.02532%	26
16	Portfolio Partner Universal G	1.09946 %	6.08717 %	0.32420 %	-4.96447%	-0.01931%	-0.02145%	27
17	Aberdeen Global German Eq	0.46351 %	5.77708 %	0.33096 %	-4.93570%	-0.01549%	-0.01933%	28
18	Incofonds	2.13865 %	6.04074 %	0.34912 %	-5.65640%	-0.02032%	-0.02231%	3
19	ABN AMRO Germany Equity	2.42189 %	7.09676 %	0.42209 %	-6.12601%	-0.02469%	-0.02452%	5
20	DIT Wachstumsfonds	1.88919 %	6.28905 %	0.37674 %	-4.72854%	-0.01478%	-0.01916%	7
21	ADIFONDS	2.16243 %	7.22614 %	0.44304 %	-6.05774%	-0.02361%	-0.02401%	8
22	Concentra	1.85919 %	6.71783 %	0.41575 %	-5.99004%	-0.02245%	-0.02321%	12
23	Thesaurus	1.72811 %	6.36330 %	0.39459 %	-6.24481%	-0.02421%	-0.02401%	14
24	Dexia Eq L Allemagne C	1.67865 %	6.23957 %	0.38700 %	-6.51795%	-0.02649%	-0.02518%	15
25	CB Lux Portfolio Euro Aktien	1.79676 %	6.77890 %	0.42088 %	-6.27426%	-0.02432%	-0.02402%	17
26	EMIF Germany Index plus B	1.57108 %	6.45667 %	0.40139 %	-6.14970%	-0.02346%	-0.02363%	21
27	CS EF (Lux) Germany	1.58297 %	6.66003 %	0.40816 %	-6.99188%	-0.03063%	-0.02721%	22
28	Flex Fonds	1.39730 %	5.98888 %	0.36524 %	-5.49109%	-0.01890%	-0.02132%	23
29	AC Deutschland	1.86378 %	7.09276 %	0.41137 %	-6.49860%	-0.02773%	-0.02600%	-
30	Baer Multistock German Stk A	1.77270 %	5.48620 %	0.32287 %	-4.94105%	-0.01430%	-0.01814%	-
31	BBV Invest Union	1.90946 %	6.30927 %	0.38537 %	-6.27119%	-0.02426%	-0.02397%	-
32	Berlinwerte Weberbank OP	1.57595 %	5.68085 %	0.33807 %	-5.44012%	-0.01901%	-0.02133%	-
33	DIT Fonds für Vermögensbildung	1.32405 %	5.79650 %	0.34777 %	-5.64758%	-0.01978%	-0.02169%	-
34	DWS Deutschland	1.60784 %	6.08441 %	0.36909 %	-6.17714%	-0.02388%	-0.02402%	-
35	Fidelity Fds Germany	1.72892 %	6.24931 %	0.37989 %	-6.62354%	-0.02738%	-0.02554%	-
36	Gerling Deutschland Fonds	1.41054 %	5.19347 %	0.31236 %	-5.59883%	-0.01972%	-0.02188%	-
37	HANSAeffekt	1.73973 %	6.49867 %	0.40096 %	-6.13358%	-0.02351%	-0.02371%	-
38	INVESCO GT German Growth C	1.71649 %	5.67770 %	0.24657 %	-4.60818%	-0.01962%	-0.02241%	-
39	Investa	2.11541 %	6.92485 %	0.42699 %	-6.39576%	-0.02563%	-0.02478%	-
40	Köln Aktienfonds DEKA	1.83865 %	6.54772 %	0.40355 %	-6.32879%	-0.02491%	-0.02436%	-
41	Oppenheim Select	1.69757 %	6.47148 %	0.39475 %	-5.77740%	-0.02145%	-0.02286%	-
42	Ring Aktienfonds DWS	1.86784 %	6.15453 %	0.37430 %	-6.42200%	-0.02570%	-0.02483%	-
43	Trinkaus Capital Fonds INKA	1.71541 %	6.49609 %	0.40013 %	-5.91511%	-0.02193%	-0.02293%	-
44	UniFonds	1.74784 %	6.42735 %	0.39665 %	-6.31308%	-0.02479%	-0.02433%	-
45	Universal Effect Fonds	1.74568 %	6.27421 %	0.38306 %	-6.43703%	-0.02573%	-0.02473%	-
P	DAX 100	1.77189 %	6.24936 %	0.39055 %	-6.19592%	-0.02379%	-0.02379%	

Table 5

Possible Variations in Fund Rankings according to a Variation of ω
(with Given Values $\bar{u}^{+}$ = 1.7719 % and x_P = 50 %)

Fund No.	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Best possible ranking pos.	8	1	4	6	17	13	9	10	8	1	2	5	7	3	7	8	8
Worst possible ranking pos.	15	2	5	9	17	16	14	16	15	3	4	7	16	6	16	15	13

Appendices:

Appendix 1: Proof that the maximization of (7) results in the selection of a μ-σ-γ-efficient alternative if only situations with positive, but diminishing marginal utility (and positive prudence) are considered (footnote 7)

Obviously, it is sufficient to show that $\frac{\partial E[U(\tilde{W})]}{\partial \mu_W} > 0$, $\frac{\partial E[U(\tilde{W})]}{\partial \sigma_W} \leq 0$, and $\frac{\partial E[U(\tilde{W})]}{\partial \gamma_W} \geq 0$.

Since $E[U(\tilde{W})] = U(\mu_W) + (1/2) \cdot U''(\mu_W) \cdot \sigma_W^2 + (1/6) \cdot U'''(\mu_W) \cdot \gamma_W^3$ as well as $U'(\mu_W) > 0$, $U''(\mu_W) < 0$, and $U'''(\mu_W) > 0$, we get:

(A1) $\frac{\partial E[U(\tilde{W})]}{\partial \mu_W} = U'(\mu_W) + \frac{1}{2} \cdot U'''(\mu_W) \cdot \sigma_W^2 > 0$, $\frac{\partial E[U(\tilde{W})]}{\partial \sigma_W} = U''(\mu_W) \cdot \sigma_W \leq 0$, and

$\frac{\partial E[U(\tilde{W})]}{\partial \gamma_W} = \frac{1}{2} \cdot U'''(\mu_W) \cdot \gamma_W^2 \geq 0.$

Appendix 2: Proof that not every μ-σ-γ-efficient alternative can be the outcome of the maximization of cubic expected utility according to (7) if we hold on to the requirement of positive, but decreasing marginal utility (footnote 8)

We consider the availability of only one fund g and the riskless asset with both having identical prices at time t = 0. In addition, there are only two possible states $s^{(1)}$ and $s^{(2)}$ at time t = 1 with equal probability. Fund g leads to uncertain inflows $\tilde{W}_g$ of 2 monetary units in state $s^{(1)}$ and 3 in state $s^{(2)}$ while the riskless asset returns 1 monetary unit regardless of the realized state of nature at time t = 1. Further, the variance minimal portfolio (the riskless asset, i.e. x_0 = 100 %) is obviously μ-σ-γ-efficient, but at the same time we have $\tilde{W}_g > 1$ with certainty and thus $E[U(\tilde{W}_g)] > U(1)$ because of the assumption of positive marginal utility. This immediately implies the postulated statement.

Appendix 3: Proof of Result 1

To show **part 1)** of Result 1, we rewrite the equivalent preference function (3) as

(A2) $$\Phi(\mu, \sigma^2, \gamma^3) = \mu - \lambda \cdot (\omega \cdot \sigma^2 - \gamma^3).$$

For given fund f and given exogenous value of x_P, (A2) can be maximized for varying values of κ and λ so that we have ω = const. Certainly, for $\lambda \to 0$, it will be possible to arrive at optimal portfolios for which the overall expected excess return might be arbitrarily great regardless of the value of ω. Nevertheless, in general it will not be possible to justify very small values of $\bar{u}^+$, as for $\lambda \to \infty$ we will not necessarily arrive at a variance-minimizing behavior. As a consequence, very low values of $\bar{u}^+ \to 0$ can only be justified for preference parameters $\omega \to \infty$. For given ω, the smallest admissible value of $\bar{u}^+$ can be identified by the maximization of $\gamma^3 - \omega \cdot \sigma^2$, as with fixed ω, apparently, there can be no other μ-σ-γ-efficient portfolio of fund f, direct stock holding P and riskless lending or borrowing with a lower value of $\bar{u}^+$.

Additional restrictions have to be allowed for, if we assume – according to **part 2)** of Result 1 – an underlying cubic utility function as described by (5). From (7) we know that $\Phi(\mu_W, \sigma_W, \gamma_W) = U(\mu_W) + (3 \cdot \mu_W + a_2) \cdot \sigma_W^2 + \gamma_W^3$ and thus in such a situation ω is defined as $-(3 \cdot \mu_W + a_2) = -(3 \cdot (1+\mu) + a_2)$ (recall the assumption $W_0 = 1$ according to the main text). Consequently, for given value of $3 \cdot \mu_W^+ + a_2$ (because of $\mu_W^+ = 1 + \mu^+ = 1 + \bar{u}^+ + r_0$) we have to determine the remaining parameter a_1 in such a way so as to support μ^+ as the expected return of the optimal portfolio of the best fund f, the reference portfolio P and riskless lending/borrowing. For varying values of μ and a_2 with ω = const. it must be examined under which conditions there are suitable values for a_1 and whether they fulfil the requirements of (8).

The second requirement of (8) is immediately satisfied, since this condition is equivalent to $\omega > 0$. The first requirement of (8) ($a_1 > \mu_W^+ \cdot (-3 \cdot \mu_W^+ - a_2 - a_2) = \mu_W^+ \cdot (\omega - a_2)$) states a condition for a_1. This condition depends on the pair $(\bar{u}^+, \omega)$, and the parameter constellation of the fund under consideration. With $x_f^+ := (\bar{u}^+ - \hat{x}_P \cdot \bar{u}_P) / \bar{u}_f$ as the necessary share of fund f as part of the investor's overall portfolio in order to attain an overall expected excess return $\bar{u}^+$ and with σ^{+2} and γ^{+3} as the corresponding variance and skewness of the (excess) return of the investor's overall portfolio we are able to transform the condition $a_1 > \mu_W^+ \cdot (\omega - a_2)$. To this end, we look at the first derivative of the preference function (6) that has to fulfil the following necessary condition, since the desired expected excess return $\bar{u}^+$ shall indeed turn out to be optimally chosen for the best fund f under consideration:

$$\text{(A3)}\ U'(\underbrace{1+r_0+\bar{u}^+}_{=:\mu_W^+})+\frac{1}{2}\cdot\left(U'''(\mu_W^+)\cdot\sigma^{+2}+U''(\mu_W^+)\cdot\frac{\partial\sigma^{+2}}{\partial\bar{u}^+}\right)+\frac{1}{6}\cdot U'''(\mu_W^+)\cdot\frac{\partial\gamma^{+3}}{\partial\bar{u}^+}=0$$

$$\Leftrightarrow\ a_1+2\cdot a_2\cdot\mu_W^+ +3\cdot\mu_W^{+2}+3\cdot\sigma^{+2}+\underbrace{(3\cdot\mu_W^+ +a_2)}_{=-\omega}\cdot\frac{\partial\sigma^{+2}}{\partial\bar{u}^+}+\frac{\partial\gamma^{+3}}{\partial\bar{u}^+}=0$$

$$\Leftrightarrow\ a_1=-2\cdot a_2\cdot\mu_W^+ -3\cdot\mu_W^{+2}-3\cdot\sigma^{+2}+\omega\cdot\frac{\partial\sigma^{+2}}{\partial\bar{u}^+}-\frac{\partial\gamma^{+3}}{\partial\bar{u}^+}\overset{!}{>}\mu_W^+\cdot(\omega-a_2)$$

$$\Leftrightarrow\ -\mu_W^+\cdot(\underbrace{a_2+3\cdot\mu_W^+}_{=-\omega})-3\cdot\sigma^{+2}+\omega\cdot\frac{\partial\sigma^{+2}}{\partial\bar{u}^+}-\frac{\partial\gamma^{+3}}{\partial\bar{u}^+}>\mu_W^+\cdot\omega$$

$$\Leftrightarrow\ -3\cdot\sigma^{+2}+\omega\cdot\frac{\partial\sigma^{+2}}{\partial\bar{u}^+}-\frac{\partial\gamma^{+3}}{\partial\bar{u}^+}>0$$

$$\Leftrightarrow\ \omega\cdot\frac{\partial\sigma^{+2}}{\partial\bar{u}^+}>3\cdot\sigma^{+2}+\frac{\partial\gamma^{+3}}{\partial\bar{u}^+}$$

Further, using $\sigma^{+2}=x_f^{+2}\cdot\sigma_f^2+2\cdot x_f^+\cdot\hat{x}_P\cdot\sigma_{fP}+\hat{x}_P^2\cdot\sigma_P^2$ and $\gamma^{+2}=x_f^{+3}\cdot\gamma_f^3+3\cdot x_f^{+2}\cdot\hat{x}_P\cdot\gamma_{ffP}$ $+\ 3\cdot x_f^+\cdot\hat{x}_P^2\cdot\gamma_{fPP}+\hat{x}_P^3\cdot\gamma_P^3$, we have:

$$\text{(A4)}\ \frac{\partial\sigma^{+2}}{\partial\bar{u}^+}=\frac{\partial\sigma^{+2}}{\partial x_f^+}\cdot\frac{\partial x_f^+}{\partial\bar{u}^+}=(2\cdot x_f^+\cdot\sigma_f^2+2\cdot\hat{x}_P\cdot\sigma_{fP})\cdot\frac{1}{\bar{u}_f};$$

$$\frac{\partial\gamma^{+3}}{\partial\bar{u}^+}=\frac{\partial\gamma^{+3}}{\partial x_f^+}\cdot\frac{\partial x_f^+}{\partial\bar{u}^+}=(3\cdot x_f^{+2}\cdot\gamma_f^3+6\cdot x_f^+\cdot\hat{x}_P\cdot\gamma_{ffP}+3\cdot\hat{x}_P^2\cdot\gamma_{fPP})\cdot\frac{1}{\bar{u}_f}.$$

Summarized from (A3) and (A4), we immediately get the condition for admissible values of ω:

$$\text{(A5)}\ \omega\cdot(x_f^+\cdot\sigma_f^2+\hat{x}_P\cdot\sigma_{fP})>1.5\cdot[\sigma^{+2}\cdot\bar{u}_f+(x_f^{+2}\cdot\gamma_f^3+2\cdot x_f^+\cdot\hat{x}_P\cdot\gamma_{ffP}+\hat{x}_P^2\cdot\gamma_{fPP})]$$

This is equivalent to the following inequalities:

$$\text{(A6a)}\ \omega>\frac{1.5\cdot[\sigma^{+2}\cdot\bar{u}_f+(x_f^{+2}\cdot\gamma_f^3+2\cdot x_f^+\cdot\hat{x}_P\cdot\gamma_{ffP}+\hat{x}_P^2\cdot\gamma_{fPP})]}{x_f^+\cdot\sigma_f^2+\hat{x}_P\cdot\sigma_{fP}},\ \text{if}\ x_f^+\cdot\sigma_f^2+\hat{x}_P\cdot\sigma_{fP}>0,$$

$$\text{(A6b)}\ \omega<\frac{1.5\cdot[\sigma^{+2}\cdot\bar{u}_f+(x_f^{+2}\cdot\gamma_f^3+2\cdot x_f^+\cdot\hat{x}_P\cdot\gamma_{ffP}+\hat{x}_P^2\cdot\gamma_{fPP})]}{x_f^+\cdot\sigma_f^2+\hat{x}_P\cdot\sigma_{fP}},\ \text{if}\ x_f^+\cdot\sigma_f^2+\hat{x}_P\cdot\sigma_{fP}<0.$$

(A6a) or (A6b) must be valid for the best fund $f = f^*$ under consideration in order to fulfil a necessary condition for expected utility maximization by selection of this fund. In addition, positive skewness preferences require $\omega > 0$.

Restrictions are even tighter for the case of cubic HARA utility – which is the object of **part 3)** of Result 1 – as the unique parameter a directly determines the optimal overall expected excess return $\bar{u}^+$ and thus ω. As a consequence, there are at most only two values of ω which support the choice of $\bar{u}^+$ for a given fund f.

In the case of cubic HARA utility, we know from (10) that we have $a_2 = -3 \cdot a$ (and thus ω = −3·(μ_W+a)) and $a_1 = 3 \cdot a^2$. With this in mind, the left-hand side of the third line of (A3) becomes

$$\text{(A7)}\quad a_1 = 3 \cdot a^2 = 6 \cdot a \cdot \mu_W^+ - 3 \cdot \mu_W^{+2} - 3 \cdot \sigma^{+2} + \omega \cdot \frac{\partial \sigma^{+2}}{\partial \overline{u}^+} - \frac{\partial \gamma^{+3}}{\partial \overline{u}^+}$$

$$\Leftrightarrow\quad 3 \cdot (\omega/3 + \mu_W^+)^2 = 6 \cdot (\omega/3 + \mu_W^+) \cdot \mu_W^+ - 3 \cdot \mu_W^{+2} - 3 \cdot \sigma^{+2} + \omega \cdot \frac{\partial \sigma^{+2}}{\partial \overline{u}^+} - \frac{\partial \gamma^{+3}}{\partial \overline{u}^+}$$

$$\Leftrightarrow\quad \omega^2 + 6 \cdot \omega \cdot \mu_W^+ + 9 \cdot \mu_W^{+2} = 6 \cdot \omega \cdot \mu_W^+ + 9 \cdot \mu_W^{+2} - 9 \cdot \sigma^{+2} + 3 \cdot \omega \cdot \frac{\partial \sigma^{+2}}{\partial \overline{u}^+} - 3 \cdot \frac{\partial \gamma^{+3}}{\partial \overline{u}^+}$$

$$\Leftrightarrow\quad \omega^2 - 3 \cdot \omega \cdot \frac{\partial \sigma^{+2}}{\partial \overline{u}^+} + 9 \cdot \sigma^{+2} + 3 \cdot \frac{\partial \gamma^{+3}}{\partial \overline{u}^+} = 0$$

$$\Leftrightarrow\quad \omega = 1.5 \cdot \frac{\partial \sigma^{+2}}{\partial \overline{u}^+} \pm \sqrt{\left(1.5 \cdot \frac{\partial \sigma^{+2}}{\partial \overline{u}^+}\right)^2 - 9 \cdot \sigma^{+2} - 3 \cdot \frac{\partial \gamma^{+3}}{\partial \overline{u}^+}}.$$

Moreover, positive skewness preferences require $\omega > 0 \;\Leftrightarrow\; \mu_W^+ := 1 + r_0 + \overline{u}^+ < a$. In addition, the validity of this inequality for the best fund f = f^* according to the performance measure (T1) guarantees the fulfilment of (8) so that the selection of f^* might indeed be in line with the maximization of expected cubic HARA utility with positive, but decreasing marginal utility.[27]

Appendix 4: Proof of Result T1 (footnote 12)

With the preference structure $\Phi_{\overline{u}^+}(\sigma^2, \gamma^3) := -\omega \cdot \sigma^2 + \gamma^3$ according to (4) and

$$\text{(A8)}\qquad \overline{u}^+ = \hat{x}_P \cdot \overline{u}_P + (1 - \hat{x}_P) \cdot \overline{u}_{Q(f)} \Leftrightarrow 1 - \hat{x}_P = \underbrace{(\overline{u}^+ - \hat{x}_P \cdot \overline{u}_P)}_{\overline{u}_Q^+(\hat{x}_P)} / \overline{u}_{Q(f)}$$

an investor prefers fund g to fund h if

[27] The second condition of (8) is obvious. In addition, under consideration of $a_1 = 3 \cdot a^2$ and $a_2 = -3 \cdot a$ the first inequality of (8) is equivalent to $3 \cdot (\mu_W^+ - a)^2 > 0$ which is immediately true for $\mu_W^+ < a$.

(A9) $\gamma^3(\hat{x}_P\cdot\tilde{u}_P+(1-\hat{x}_P)\cdot\tilde{u}_{Q(g)})-\omega\cdot\sigma^2(\hat{x}_P\cdot\tilde{u}_P+(1-\hat{x}_P)\cdot\tilde{u}_{Q(g)})$

$>\gamma^3(\hat{x}_P\cdot\tilde{u}_P+(1-\hat{x}_P)\cdot\tilde{u}_{Q(h)})-\omega\cdot\sigma^2(\hat{x}_P\cdot\tilde{u}_P+(1-\hat{x}_P)\cdot\tilde{u}_{Q(h)})$

$$\Leftrightarrow\ (1-\hat{x}_P)^3\cdot\gamma^3_{Q(g)}+3\cdot\hat{x}_P\cdot(1-\hat{x}_P)^2\cdot\gamma_{Q(g)Q(g)P}+3\cdot\hat{x}_P^2\cdot(1-\hat{x}_P)\cdot\gamma_{Q(g)PP}+\hat{x}_P^3\cdot\gamma_P^3$$
$$-\omega\cdot[(1-\hat{x}_P)^2\cdot\sigma^2_{Q(g)}+2\cdot\hat{x}_P\cdot(1-\hat{x}_P)\cdot\sigma_{Q(g)P}+\hat{x}_P^2\cdot\sigma_P^2]$$
$$>(1-\hat{x}_P)^3\cdot\gamma^3_{Q(h)}+3\cdot\hat{x}_P\cdot(1-\hat{x}_P)^2\cdot\gamma_{Q(h)Q(h)P}+3\cdot\hat{x}_P^2\cdot(1-\hat{x}_P)\cdot\gamma_{Q(h)PP}+\hat{x}_P^3\cdot\gamma_P^3$$
$$-\omega\cdot[(1-\hat{x}_P)^2\cdot\sigma^2_{Q(h)}+2\cdot\hat{x}_P\cdot(1-\hat{x}_P)\cdot\sigma_{Q(h)P}+\hat{x}_P^2\cdot\sigma_P^2]$$

$$\Leftrightarrow\ \left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(g)}}\right)^3\cdot\gamma^3_{Q(g)}+3\cdot\hat{x}_P\cdot\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(g)}}\right)^2\cdot\gamma_{Q(g)Q(g)P}+3\cdot\hat{x}_P^2\cdot\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(g)}}\right)\cdot\gamma_{Q(g)PP}$$
$$-\omega\cdot\left[\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(g)}}\right)^2\cdot\sigma^2_{Q(g)}+2\cdot\hat{x}_P\cdot\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(g)}}\right)\cdot\sigma_{Q(g)P}\right]$$
$$>\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(h)}}\right)^3\cdot\gamma^3_{Q(h)}+3\cdot\hat{x}_P\cdot\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(h)}}\right)^2\cdot\gamma_{Q(h)Q(h)P}+3\cdot\hat{x}_P^2\cdot\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(h)}}\right)\cdot\gamma_{Q(h)PP}$$
$$-\omega\cdot\left[\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(h)}}\right)^2\cdot\sigma^2_{Q(h)}+2\cdot\hat{x}_P\cdot\left(\frac{\overline{u}_Q^+(\hat{x}_P)}{\overline{u}_{Q(h)}}\right)\cdot\sigma_{Q(h)P}\right]$$

$$\Leftrightarrow\ (\overline{u}_Q^+(\hat{x}_P))^3\cdot\frac{1}{\frac{(\overline{u}_{Q(g)})^3}{\gamma^3_{Q(g)}}}+3\cdot\hat{x}_P\cdot\gamma_P^3\cdot(\overline{u}_Q^+(\hat{x}_P))^2\cdot\frac{1}{\frac{(\overline{u}_{Q(g)})^2}{b_{Q^2(g)P}}}+3\cdot\hat{x}_P^2\cdot\gamma_P^3\cdot\overline{u}_Q^+(\hat{x}_P)\cdot\frac{1}{\frac{\overline{u}_{Q(g)}}{b_{Q(g)P^2}}}$$
$$+\omega\cdot\left[(\overline{u}_Q^+(\hat{x}_P))^2\cdot\frac{-1}{\frac{(\overline{u}_{Q(g)})^2}{\sigma^2_{Q(g)}}}+2\cdot\hat{x}_P\cdot\sigma_P^2\cdot\overline{u}_Q^+(\hat{x}_P)\cdot\frac{-1}{\frac{\overline{u}_{Q(g)}}{\beta_{Q(g)P}}}\right]$$
$$>(\overline{u}_Q^+(\hat{x}_P))^3\cdot\frac{1}{\frac{(\overline{u}_{Q(h)})^3}{\gamma^3_{Q(h)}}}+3\cdot\hat{x}_P\cdot\gamma_P^3\cdot(\overline{u}_Q^+(\hat{x}_P))^2\cdot\frac{1}{\frac{(\overline{u}_{Q(h)})^2}{b_{Q^2(h)P}}}+3\cdot\hat{x}_P^2\cdot\gamma_P^3\cdot\overline{u}_Q^+(\hat{x}_P)\cdot\frac{1}{\frac{\overline{u}_{Q(h)}}{b_{Q(h)P^2}}}$$
$$+\omega\cdot\left[(\overline{u}_Q^+(\hat{x}_P))^2\cdot\frac{-1}{\frac{(\overline{u}_{Q(h)})^2}{\sigma^2_{Q(h)}}}+2\cdot\hat{x}_P\cdot\sigma_P^2\cdot\overline{u}_Q^+(\hat{x}_P)\cdot\frac{-1}{\frac{\overline{u}_{Q(h)}}{\beta_{Q(h)P}}}\right]$$

$$\Leftrightarrow\ (\overline{u}_Q^+(\hat{x}_P))^3\cdot\frac{1}{(cSM_g^{(2)})^3}+3\cdot\hat{x}_P\cdot\gamma_P^3\cdot(\overline{u}_Q^+(\hat{x}_P))^2\cdot\frac{1}{cTM_g^{(1)}}+3\cdot\hat{x}_P^2\cdot\gamma_P^3\cdot\overline{u}_Q^+(\hat{x}_P)\cdot\frac{1}{cTM_g^{(2)}}$$
$$+\omega\cdot\left[(\overline{u}_Q^+(\hat{x}_P))^2\cdot\frac{-1}{(qSM_g)^2}+2\cdot\hat{x}_P\cdot\sigma_P^2\cdot\overline{u}_Q^+(\hat{x}_P)\cdot\frac{-1}{qTM_g}\right]$$
$$>(\overline{u}_Q^+(\hat{x}_P))^3\cdot\frac{1}{(cSM_h^{(2)})^3}+3\cdot\hat{x}_P\cdot\gamma_P^3\cdot(\overline{u}_Q^+(\hat{x}_P))^2\cdot\frac{1}{cTM_h^{(1)}}+3\cdot\hat{x}_P^2\cdot\gamma_P^3\cdot\overline{u}_Q^+(\hat{x}_P)\cdot\frac{1}{cTM_h^{(2)}}$$
$$+\omega\cdot\left[(\overline{u}_Q^+(\hat{x}_P))^2\cdot\frac{-1}{(qSM_h)^2}+2\cdot\hat{x}_P\cdot\sigma_P^2\cdot\overline{u}_Q^+(\hat{x}_P)\cdot\frac{-1}{qTM_h}\right].$$

Appendix 5: Proof of (T6) and (T8) according to Table 3 (footnote 13)

The case $y_f^{(exg)*} = 1$ coincides with $y_P = 0$ and thus requires $\hat{x}_P = 0$. For $\hat{x}_P = 0$, (T1) reduces to[28]

(A10)
$$
\begin{aligned}
&cIM_f^{(exg)}(0) = \bar{u}^{+3} \cdot \frac{1}{(cSM_f^{(2)})^3} + \bar{u}^{+2} \cdot \omega \cdot \frac{-1}{(qSM_f)^2} \\
&\Leftrightarrow cIM_f^{(exg)}(0) = \frac{1}{(cSM_f^{(2)})^3} + \omega^{(mod1)} \cdot \frac{-1}{(qSM_f)^2},
\end{aligned}
$$

with $\omega^{(mod1)} := \kappa/(\lambda \cdot \bar{u}^+)$.

Now consider the second limiting case described by $y_f^{(exg)*} = \varepsilon$ with $\varepsilon > 0$, but small. For such a situation, portfolio Q(f) just converges to the sole holding of the riskless asset and we thus arrive at a situation with $u_Q^+(\hat{x}_P) \to 0$ (i.e. $\hat{x}_P \to \bar{u}^+/\bar{u}_P$). For this, we get from (T1) in the typical case of $u_Q^+(\hat{x}_P) > 0$

$$
\text{(A11)} \quad cIM_f^{(exg)}(\hat{x}_P = \tfrac{\bar{u}^+}{\bar{u}_P}) = 3 \cdot \hat{x}_P^2 \cdot \gamma_P^3 \cdot \frac{1}{cTM_f^{(2)}} + 2 \cdot \omega \cdot \hat{x}_P \cdot \sigma_P^2 \cdot \frac{-1}{qTM_f}
$$

$$
\Leftrightarrow \hat{x}_P \cdot cIM_f^{(exg)}(\hat{x}_P = \tfrac{\bar{u}^+}{\bar{u}_P}) = 3 \cdot \left(\frac{\bar{u}^+}{\bar{u}_P}\right)^3 \cdot \gamma_P^3 \cdot \frac{1}{cTM_f^{(2)}} + 2 \cdot \omega \cdot \left(\frac{\bar{u}^+}{\bar{u}_P}\right)^2 \cdot \sigma_P^2 \cdot \frac{-1}{qTM_f}
$$

$$
\Leftrightarrow cIM_f^{(exg,\hat{x}_P = \frac{\bar{u}^+}{\bar{u}_P})} := \frac{1}{(cSM_P^{(2)})^3 \cdot cTM_f^{(2)}} + \omega^{(mod2)} \cdot \frac{-1}{(qSM_P)^2 \cdot qTM_f},
$$

with $\omega^{(mod2)} := (2 \cdot \kappa)/(3 \cdot \lambda)$.

Appendix 6: Proof that only in situations with $\gamma_g > 0$ and $\gamma_h > 0$ it is possible to always derive a greater cubic *Sharpe* measure "1" (defined in Table 2) for a fund g in comparison with a fund h from both a greater quadratic *Sharpe* measure as well as a smaller cubic *Sharpe* measure "2" (according to (T3)) (footnote 14)

For $qSM_g > qSM_h > 0 \Leftrightarrow \frac{\bar{u}_g}{\sigma_g} > \frac{\bar{u}_h}{\sigma_h} > 0$ and $\bar{u}_g = \bar{u}_h$, the case $\gamma_g > 0$ and $\gamma_h > 0$ immediately implies $cSM_g^{(2)} < cSM_h^{(2)} \Leftrightarrow \frac{\gamma_g}{\bar{u}_g} > \frac{\gamma_h}{\bar{u}_h} > 0$ and thus $cSM_g^{(1)} = \frac{\gamma_g}{\sigma_g} = \frac{\bar{u}_g}{\sigma_g} \cdot \frac{\gamma_g}{\bar{u}_g} > \frac{\bar{u}_h}{\sigma_h} \cdot \frac{\gamma_h}{\bar{u}_h} = \frac{\gamma_h}{\sigma_h} =$

[28] The following equivalence means that both performance measures lead to the same fund ranking.

$cSM_h^{(1)}$. The case $\gamma_g < 0$ together with $\gamma_h > 0$ always leads to $cSM_g^{(1)} < cSM_h^{(1)}$ as well as $cSM_g^{(2)} < cSM_h^{(2)}$. Since $\gamma_g > 0$ and $\gamma_h < 0$ corresponds with $cSM_g^{(2)} > cSM_h^{(2)}$, the last case which has to be treated is $\gamma_g < 0$ in connection with $\gamma_h < 0$. Let us look at the situation $\bar{u}_g = \bar{u}_h = 10$ %, $\sigma_g = 10$ %, $\sigma_h = 30$ %, $\gamma_g = -5$ %, and $\gamma_h = -10$ %. This parameter constellation leads to

(A12) $$qSM_g = \frac{0.1}{0.1} = 1 > 1/3 = \frac{0.1}{0.3} = qSM_h,\ cSM_g^{(2)} = \frac{0.1}{-0.05} = -2 < -1 = \frac{0.1}{-0.1} = cSM_h^{(2)}.$$

Nevertheless, we have

(A13) $$cSM_g^{(1)} = \frac{-0.05}{0.1} = -0.5 < -1/3 = \frac{-0.1}{0.3} = cSM_h^{(1)}.$$

Appendix 7: Proof that the cubic *Treynor* measure of *Breuer/Gürtler* (2006) is a special case of the performance measure (T8) of this paper (footnote 15)

With z(0) as a fund-independent constant in the case of cubic HARA utility defined more specifically in *Breuer/Gürtler* (2006), the cubic *Treynor* measure of formula (20) in *Breuer/Gürtler* (2006) can be transformed as follows:

(A14) $$\frac{\gamma_{fPP} + 2\cdot\left(\bar{u}_P - \frac{2}{z(0)}\right)\cdot\sigma_{gP}}{\bar{u}_g} = \frac{\gamma_{fPP}}{\bar{u}_g} + 2\cdot\left(\bar{u}_P - \frac{2}{z(0)}\right)\cdot\frac{\sigma_{fP}}{\bar{u}_g}$$

$$= \frac{1}{\frac{\bar{u}_f}{\gamma_{fPP}}} + 2\cdot\left(\bar{u}_P - \frac{2}{z(0)}\right)\cdot\frac{1}{\frac{\bar{u}_f}{\sigma_{fP}}} = \frac{1}{cTM_f^{(2)}}\cdot\frac{\bar{u}_P^3}{\frac{\bar{u}_P^3}{\gamma_P^3}} + 2\cdot\left(\bar{u}_P - \frac{2}{z(0)}\right)\cdot\frac{1}{qTM_f}\cdot\frac{\bar{u}_P^2}{\frac{\bar{u}_P^2}{\sigma_P^2}}$$

$$= \frac{1}{cTM_f^{(2)}}\cdot\frac{\bar{u}_P^3}{(cSM_P^{(2)})^3} + 2\cdot\left(\bar{u}_P - \frac{2}{z(0)}\right)\cdot\frac{1}{qTM_f}\cdot\frac{\bar{u}_P^2}{(qSM_P)^2}.$$

This measure leads to the same fund ranking as the measure

(A15) $$\frac{1}{(cSM_P^{(2)})^3\cdot cTM_f^{(2)}} + 2\cdot\left(\frac{2}{z(0)\cdot\bar{u}_P} - 1\right)\cdot\frac{-1}{(qSM_P)^2\cdot qTM_f\cdot},$$

which corresponds with (T8) if we identify $\omega^{(\mathrm{mod}\,2)} = (2\cdot\kappa)/(3\cdot\lambda)$ with $2\cdot\left(\frac{2}{z(0)\cdot\bar{u}_P} - 1\right)$.

Appendix 8: Proof that in the case of pure mean-variance preferences, the best fund according to the optimized quadratic *Sharpe* measure as discussed, for example, in *Breuer/Gürtler* (1999) is always also the best one as well according to (T10) of Table 3 for arbitrary desired overall expected excess return $\bar{u}^{+}$ (footnote 16)

According to the optimized quadratic *Sharpe* measure presented in *Breuer/Gürtler* (1999) a fund g will be preferred to a fund h for arbitrary desired overall expected excess return $\bar{u}^{+}$ if and only if

$$\text{(A16)}\quad \frac{\bar{u}^{+}}{(x_g^*)^2\cdot\sigma_g^2+2\cdot x_g^*\cdot x_P^{(g)*}\cdot\sigma_{gP}+(x_P^{(g)*})^2\cdot\sigma_P^2} > \frac{\bar{u}^{+}}{(x_h^*)^2\cdot\sigma_h^2+2\cdot x_h^*\cdot x_P^{(h)*}\cdot\sigma_{hP}+(x_P^{(h)*})^2\cdot\sigma_P^2}$$

$$\Leftrightarrow\ (x_g^*)^2\cdot\sigma_g^2+2\cdot x_g^*\cdot x_P^{(g)*}\cdot\sigma_{gP}+(x_P^{(g)*})^2\cdot\sigma_P^2 < (x_h^*)^2\cdot\sigma_h^2+2\cdot x_h^*\cdot x_P^{(h)*}\cdot\sigma_{hP}+(x_P^{(h)*})^2\cdot\sigma_P^2$$

$$\Leftrightarrow\ \left(\frac{\bar{u}_Q^{+}(x_P^{(g)*})}{\bar{u}_g}\right)^2\cdot\sigma_g^2+2\cdot\frac{\bar{u}_Q^{+}(x_P^{(g)*})}{\bar{u}_g}\cdot x_P^{(g)*}\cdot\sigma_{gP}+(x_P^{(g)*})^2\cdot\sigma_P^2$$

$$<\left(\frac{\bar{u}_Q^{+}(x_P^{(h)*})}{\bar{u}_h}\right)^2\cdot\sigma_h^2+2\cdot\frac{\bar{u}_Q^{+}(x_P^{(h)*})}{\bar{u}_h}\cdot x_P^{(h)*}\cdot\sigma_{hP}+(x_P^{(h)*})^2\cdot\sigma_P^2$$

$$\Leftrightarrow\ (\bar{u}_Q^{+}(x_P^{(g)*}))^2\cdot\frac{-1}{(qSM_g)^2}+2\cdot x_P^{(g)*}\cdot\sigma_P^2\cdot\bar{u}_Q^{+}(x_P^{(g)*})\cdot\frac{-1}{qTM_g}-(x_P^{(g)*})^2\cdot\sigma_P^2$$

$$>(\bar{u}_Q^{+}(x_P^{(h)*}))^2\cdot\frac{-1}{(qSM_h)^2}+2\cdot x_P^{(h)*}\cdot\sigma_P^2\cdot\bar{u}_Q^{+}(x_P^{(h)*})\cdot\frac{-1}{qTM_h}-(x_P^{(h)*})^2\cdot\sigma_P^2.$$

Thus, the implied fund ranking corresponds to the ranking according to (T10) of Table 3 in the quadratic case $\omega\to\infty$.

Addendum:

Example for the validity of the following statement: "Resulting correlation coefficients between two fund rankings do not change much, if differing pairs $(x_P^{(1)}, x_P^{(2)})$ of exogenous direct stock holdings are considered, as long as we have a constant value for $\Delta x_P := \left| x_P^{(1)} - x_P^{(2)} \right|$": All shaded cells present apparently almost identical ranking correlation coefficients between two different respective fund rankings with constant value of Δx_P (fn. 20)

$x_P^{(2)}$ \ $x_P^{(1)}$	0 %	5 %	10 %	15 %	20 %	25 %	30 %	35 %	40 %	45 %	50 %	55 %	60 %	65 %	70 %	75 %	80 %	85 %	90 %	95 %	99.99 %
0 %	100.0000	100.0000	99.75490	99.75490	99.75490	99.75490	99.75490	99.50980	99.01961	99.01961	98.03922	98.03922	98.03922	98.03922	98.03922	98.03922	97.30392	96.56863	96.56863	96.56863	96.56863
5 %	100.0000	100.0000	99.75490	99.75490	99.75490	99.75490	99.75490	99.50980	99.01961	99.01961	98.03922	98.03922	98.03922	98.03922	98.03922	98.03922	97.30392	96.56863	96.56863	96.56863	96.56863
10 %	99.75490	99.75490	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	98.52941	98.52941	98.52941	98.52941	98.52941	98.52941	97.79412	97.30392	97.30392	97.30392	97.30392
15 %	99.75490	99.75490	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	98.52941	98.52941	98.52941	98.52941	98.52941	98.52941	97.79412	97.30392	97.30392	97.30392	97.30392
20 %	99.75490	99.75490	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	98.52941	98.52941	98.52941	98.52941	98.52941	98.52941	97.79412	97.30392	97.30392	97.30392	97.30392
25 %	99.75490	99.75490	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	98.52941	98.52941	98.52941	98.52941	98.52941	98.52941	97.79412	97.30392	97.30392	97.30392	97.30392
30 %	99.75490	99.75490	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	98.52941	98.52941	98.52941	98.52941	98.52941	98.52941	97.79412	97.30392	97.30392	97.30392	97.30392
35 %	99.50980	99.50980	99.75490	99.75490	99.75490	99.75490	99.75490	100.0000	99.75490	99.75490	98.77451	98.77451	98.77451	98.77451	98.77451	98.77451	98.03922	97.54902	97.54902	97.54902	97.54902
40 %	99.01961	99.01961	99.50980	99.50980	99.50980	99.50980	99.50980	99.75490	100.0000	100.0000	99.01961	99.01961	99.01961	99.01961	99.01961	99.01961	98.28431	98.03922	98.03922	98.03922	98.03922
45 %	99.01961	99.01961	99.50980	99.50980	99.50980	99.50980	99.50980	99.75490	100.0000	100.0000	99.01961	99.01961	99.01961	99.01961	99.01961	99.01961	98.28431	98.03922	98.03922	98.03922	98.03922
50 %	98.03922	98.03922	98.52941	98.52941	98.52941	98.52941	98.52941	98.77451	99.01961	99.01961	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	99.50980	99.50980
55 %	98.03922	98.03922	98.52941	98.52941	98.52941	98.52941	98.52941	98.77451	99.01961	99.01961	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	99.50980	99.50980
60 %	98.03922	98.03922	98.52941	98.52941	98.52941	98.52941	98.52941	98.77451	99.01961	99.01961	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	99.50980	99.50980
65 %	98.03922	98.03922	98.52941	98.52941	98.52941	98.52941	98.52941	98.77451	99.01961	99.01961	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	99.50980	99.50980
70 %	98.03922	98.03922	98.52941	98.52941	98.52941	98.52941	98.52941	98.77451	99.01961	99.01961	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	99.50980	99.50980
75 %	98.03922	98.03922	98.52941	98.52941	98.52941	98.52941	98.52941	98.77451	99.01961	99.01961	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000	99.75490	99.50980	99.50980	99.50980	99.50980
80 %	97.30392	97.30392	97.79412	97.79412	97.79412	97.79412	97.79412	98.03922	98.28431	98.28431	99.75490	99.75490	99.75490	99.75490	99.75490	99.75490	100.0000	99.75490	99.75490	99.75490	99.75490
85 %	96.56863	96.56863	97.30392	97.30392	97.30392	97.30392	97.30392	97.54902	98.03922	98.03922	99.50980	99.50980	99.50980	99.50980	99.50980	99.50980	99.75490	100.0000	100.0000	100.0000	100.0000
90 %	96.56863	96.56863	97.30392	97.30392	97.30392	97.30392	97.30392	97.54902	98.03922	98.03922	99.50980	99.50980	99.50980	99.50980	99.50980	99.50980	99.75490	100.0000	100.0000	100.0000	100.0000
95 %	96.56863	96.56863	97.30392	97.30392	97.30392	97.30392	97.30392	97.54902	98.03922	98.03922	99.50980	99.50980	99.50980	99.50980	99.50980	99.50980	99.75490	100.0000	100.0000	100.0000	100.0000
99.99 %	96.56863	96.56863	97.30392	97.30392	97.30392	97.30392	97.30392	97.54902	98.03922	98.03922	99.50980	99.50980	99.50980	99.50980	99.50980	99.50980	99.75490	100.0000	100.0000	100.0000	100.0000

Table Ad 1: Correlation Coefficients between Exogenous Cubic IM-Rankings for Desired Expected Excess Return $\overline{u}^{+}$ = 2.3 %, $\omega = 0$, and Different Values $x_P^{(1)}$ and $x_P^{(2)}$ (in %)

Cells referred to in the main text of the paper are shaded (footnote 21)

Δx_P \ $\bar{u}^+$	**1.7719 %**	**1.90 %**	**2.00 %**	**2.10 %**	**2.20 %**	**2.30 %**	**2.40 %**	**2.50 %**	**2.60 %**	**2.70 %**	**10.00 %**
0 %	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%
5 %	99.75490%	99.77941%	99.79167%	99.79167%	99.79167%	99.80392%	99.80392%	99.80392%	99.81618%	99.84069%	99.91422%
10 %	99.57430%	99.61300%	99.62590%	99.62590%	99.63880%	99.65170%	99.66460%	99.66460%	99.69040%	99.67750%	99.81940%
15 %	99.41449%	99.45534%	99.45534%	99.46895%	99.48257%	99.50980%	99.52342%	99.52342%	99.55065%	99.53704%	99.72767%
20 %	99.27912%	99.32238%	99.30796%	99.32238%	99.35121%	99.36563%	99.38005%	99.38005%	99.40888%	99.40888%	99.61073%
25 %	99.06556%	99.17279%	99.15748%	99.18811%	99.23407%	99.24939%	99.26471%	99.28002%	99.28002%	99.28002%	99.52512%
30 %	98.85621%	98.93791%	98.93791%	98.97059%	99.03595%	99.10131%	99.13399%	99.16667%	99.15033%	99.15033%	99.42810%
35 %	98.63445%	98.72199%	98.72199%	98.75700%	98.84454%	98.84454%	98.89706%	98.96709%	99.00210%	99.00210%	99.33473%
40 %	98.41629%	98.51056%	98.49170%	98.52941%	98.62368%	98.62368%	98.68024%	98.77451%	98.73680%	98.75566%	99.24585%
45 %	98.20261%	98.34559%	98.28431%	98.32516%	98.36601%	98.36601%	98.44771%	98.54984%	98.50899%	98.52941%	99.14216%
50 %	97.95009%	98.15062%	98.08378%	98.12834%	98.19519%	98.15062%	98.23975%	98.28431%	98.23975%	98.26203%	99.01961%
55 %	97.59804%	97.89216%	97.81863%	97.89216%	97.99020%	97.94118%	98.03922%	98.06373%	98.01471%	98.03922%	98.92157%
60 %	97.22222%	97.54902%	97.49455%	97.57625%	97.68519%	97.68519%	97.82135%	97.90305%	97.84858%	97.87582%	98.82898%
65 %	96.84436%	97.12010%	97.08946%	97.18137%	97.42647%	97.42647%	97.57966%	97.70221%	97.61029%	97.67157%	98.71324%
70 %	96.42857%	96.74370%	96.63866%	96.77871%	97.05882%	97.05882%	97.26891%	97.40896%	97.33894%	97.37395%	98.56443%
75 %	96.03758%	96.40523%	96.28268%	96.40523%	96.56863%	96.60948%	96.85458%	97.05882%	96.97712%	97.01797%	98.40686%
80 %	95.58824%	95.98039%	95.83333%	95.98039%	96.12745%	96.22549%	96.27451%	96.51961%	96.42157%	96.51961%	98.23529%
85 %	94.97549%	95.46569%	95.28186%	95.34314%	95.52696%	95.58824%	95.71078%	95.71078%	95.77206%	95.89461%	98.10049%
90 %	94.03595%	94.60784%	94.52614%	94.60784%	94.68954%	94.93464%	95.09804%	95.09804%	95.26144%	95.01634%	97.95752%
95 %	92.64706%	93.50490%	93.99510%	93.99510%	94.11765%	94.24020%	94.48529%	94.48529%	94.73039%	94.36275%	97.67157%
99.99 %	91.17647%	91.91176%	92.89216%	92.89216%	92.89216%	93.13725%	93.13725%	93.13725%	93.62745%	93.62745%	97.30392%

Table Ad 2a: Average Correlation Coefficients between IM-Rankings for Varying Identical Values of Desired Expected Excess Return $\bar{u}^+$ and Identical Differences $\Delta x_P = |x_P^{(1)} - x_P^{(2)}|$ between Exogenous Investments in Reference Portfolio P ($\omega = 100{,}000$)

Cells referred to in the main text of the paper are shaded (footnote 21)

Δx_P \ $\bar{u}^+$	1.7719 %	1.90 %	2.00 %	2.10 %	2.20 %	2.30 %	2.40 %	2.50 %	2.60 %	2.70 %	10.00 %
0 %	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%	100.00000%
5 %	99.87745%	99.87745%	99.88971%	99.88971%	99.88971%	99.88971%	99.88971%	99.88971%	99.88971%	99.88971%	99.98775%
10 %	99.74200%	99.75490%	99.76780%	99.76780%	99.76780%	99.76780%	99.76780%	99.76780%	99.76780%	99.78070%	99.97420%
15 %	99.60512%	99.63235%	99.64597%	99.64597%	99.64597%	99.64597%	99.64597%	99.65959%	99.65959%	99.67320%	99.95915%
20 %	99.46655%	99.49539%	99.50980%	99.50980%	99.50980%	99.50980%	99.52422%	99.53864%	99.53864%	99.56747%	99.94233%
25 %	99.26471%	99.34130%	99.35662%	99.35662%	99.37194%	99.37194%	99.38725%	99.41789%	99.41789%	99.44853%	99.92341%
30 %	99.05229%	99.11765%	99.13399%	99.15033%	99.21569%	99.23203%	99.24837%	99.28105%	99.28105%	99.31373%	99.90196%
35 %	98.84454%	98.87955%	98.91457%	98.93207%	99.01961%	99.01961%	99.03711%	99.08964%	99.08964%	99.15966%	99.87745%
40 %	98.58597%	98.64253%	98.71795%	98.73680%	98.79336%	98.79336%	98.81222%	98.86878%	98.86878%	98.92534%	99.86802%
45 %	98.38644%	98.46814%	98.46814%	98.50899%	98.57026%	98.57026%	98.59069%	98.61111%	98.61111%	98.67239%	99.85703%
50 %	98.17291%	98.26203%	98.26203%	98.30660%	98.35116%	98.28431%	98.32888%	98.35116%	98.35116%	98.44029%	99.84403%
55 %	97.91667%	98.01471%	98.01471%	98.06373%	98.11275%	98.03922%	98.08824%	98.16176%	98.16176%	98.28431%	99.82843%
60 %	97.57625%	97.71242%	97.71242%	97.76688%	97.84858%	97.90305%	97.95752%	98.03922%	98.03922%	98.14815%	99.80937%
65 %	97.24265%	97.30392%	97.51838%	97.57966%	97.67157%	97.73284%	97.79412%	97.94730%	97.94730%	98.06985%	99.78554%
70 %	97.09384%	97.09384%	97.23389%	97.33894%	97.51401%	97.58403%	97.65406%	97.82913%	97.82913%	98.00420%	99.75490%
75 %	96.89542%	96.97712%	97.05882%	97.18137%	97.30392%	97.42647%	97.50817%	97.71242%	97.71242%	97.91667%	99.75490%
80 %	96.71569%	96.91176%	97.00980%	97.00980%	97.15686%	97.15686%	97.30392%	97.54902%	97.54902%	97.79412%	99.75490%
85 %	96.56863%	96.81373%	96.93627%	96.93627%	96.93627%	96.93627%	97.12010%	97.30392%	97.30392%	97.61029%	99.75490%
90 %	96.32353%	96.65033%	96.81373%	96.81373%	96.81373%	96.81373%	96.81373%	97.05882%	97.05882%	97.30392%	99.75490%
95 %	96.07843%	96.32353%	96.56863%	96.56863%	96.56863%	96.56863%	96.56863%	96.56863%	96.56863%	96.93627%	99.75490%
99.99 %	96.07843%	96.07843%	96.56863%	96.56863%	96.56863%	96.56863%	96.56863%	96.56863%	96.56863%	96.56863%	99.75490%

Table Ad 2b: Average Correlation Coefficients between IM-Rankings for Varying Identical Values of Desired Expected Excess Return $\bar{u}^+$ and Identical Differences $\Delta x_P = |x_P^{(1)} - x_P^{(2)}|$ between Exogenous Investments in Reference Portfolio P ($\omega = 0$)

Cells referred to in the main text of the paper are shaded (footnote 22)

x_P \ $\bar{u}^+$	1.7719 %	1.90 %	2.00 %	2.10 %	2.20 %	2.30 %	2.40 %	2.50 %	2.60 %	2.70 %	10.00 %
0 %	93.13725%	93.13725%	93.13725%	93.13725%	93.13725%	93.13725%	93.13725%	93.13725%	93.13725%	93.13725%	93.13725%
5 %	94.60784%	94.60784%	94.60784%	94.60784%	94.60784%	94.60784%	94.60784%	94.60784%	94.60784%	94.36275%	93.87255%
10 %	96.32353%	96.32353%	94.85294%	94.85294%	94.85294%	94.85294%	94.85294%	94.85294%	94.85294%	94.85294%	94.36275%
15 %	97.54902%	97.54902%	96.81373%	96.81373%	96.81373%	96.32353%	96.32353%	96.32353%	96.32353%	96.32353%	94.36275%
20 %	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	96.81373%	96.81373%	94.60784%
25 %	97.30392%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	94.60784%
30 %	97.30392%	97.30392%	97.30392%	97.30392%	97.30392%	97.54902%	97.54902%	97.54902%	97.54902%	97.54902%	94.85294%
35 %	98.28431%	98.28431%	97.30392%	97.30392%	97.30392%	97.30392%	97.30392%	97.30392%	97.54902%	97.54902%	94.85294%
40 %	98.52941%	98.52941%	98.28431%	98.28431%	97.30392%	97.30392%	97.30392%	97.30392%	97.30392%	97.30392%	94.85294%
45 %	98.52941%	98.52941%	98.52941%	98.52941%	98.52941%	98.28431%	98.28431%	97.30392%	97.30392%	97.30392%	94.85294%
50 %	98.28431%	98.28431%	98.52941%	98.52941%	98.52941%	98.52941%	98.52941%	98.28431%	98.28431%	98.28431%	94.85294%
55 %	98.28431%	98.28431%	98.28431%	98.28431%	98.52941%	98.52941%	98.52941%	98.52941%	98.52941%	98.52941%	96.32353%
60 %	98.77451%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.52941%	98.52941%	98.52941%	98.52941%	96.32353%
65 %	98.77451%	98.77451%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.52941%	98.52941%	98.52941%	96.32353%
70 %	98.77451%	98.77451%	98.77451%	98.77451%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.52941%	96.81373%
75 %	99.01961%	98.77451%	98.77451%	98.77451%	98.77451%	98.77451%	98.28431%	98.28431%	98.28431%	98.28431%	96.81373%
80 %	99.26471%	99.01961%	99.01961%	98.77451%	98.77451%	98.77451%	98.77451%	98.28431%	98.28431%	98.28431%	97.54902%
85 %	99.26471%	99.26471%	99.01961%	99.01961%	98.77451%	98.77451%	98.77451%	98.77451%	98.77451%	98.28431%	97.54902%
90 %	99.26471%	99.26471%	99.26471%	99.01961%	99.01961%	98.77451%	98.77451%	98.77451%	98.77451%	98.77451%	97.54902%
95 %	99.50980%	99.26471%	99.26471%	99.26471%	99.01961%	99.01961%	98.77451%	98.77451%	98.77451%	98.77451%	97.54902%
99.99 %	98.77451%	99.50980%	99.26471%	99.26471%	99.26471%	99.01961%	99.01961%	99.01961%	98.77451%	98.77451%	97.54902%

Table Ad 3a: Correlation Coefficients between Fund Rankings according to the Exogenous Quadratic IM and the (Restricted) Endogenous Quadratic IM for Different Values of Desired Expected Excess Return $\bar{u}^+$ and of Exogenous Investment x_P in Reference Portfolio P (ω = 100,000)

Cells referred to in the main text of the paper are shaded (footnote 22)

$\bar{u}^+$ / x_P	1.7719 %	1.90 %	2.00 %	2.10 %	2.20 %	2.30 %	2.40 %	2.50 %	2.60 %	2.70 %	10.00 %
0 %	97.79412%	99.26471%	97.79412%	99.75490%	99.75490%	99.75490%	99.75490%	99.75490%	99.75490%	99.75490%	97.79412%
5 %	97.79412%	99.26471%	97.79412%	99.75490%	99.75490%	99.75490%	99.75490%	99.75490%	99.75490%	99.75490%	97.79412%
10 %	97.54902%	99.01961%	97.54902%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	97.79412%
15 %	97.54902%	99.01961%	97.54902%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	97.79412%
20 %	97.54902%	99.01961%	97.54902%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	97.79412%
25 %	98.03922%	99.01961%	97.54902%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	97.79412%
30 %	97.54902%	99.50980%	98.03922%	99.75490%	99.75490%	99.50980%	99.50980%	99.50980%	99.50980%	99.50980%	97.79412%
35 %	97.54902%	99.01961%	97.54902%	99.26471%	99.75490%	99.75490%	99.75490%	99.75490%	99.75490%	99.50980%	97.54902%
40 %	98.52941%	98.03922%	97.54902%	99.26471%	99.26471%	99.26471%	99.26471%	99.75490%	99.75490%	99.75490%	97.54902%
45 %	98.52941%	98.03922%	98.52941%	98.28431%	98.52941%	99.26471%	99.26471%	99.26471%	99.26471%	99.26471%	97.54902%
50 %	98.52941%	98.03922%	98.52941%	98.28431%	98.28431%	98.28431%	98.52941%	99.26471%	99.26471%	99.26471%	97.54902%
55 %	98.52941%	98.03922%	98.52941%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.52941%	97.54902%
60 %	98.77451%	98.03922%	98.52941%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	97.54902%
65 %	98.03922%	97.30392%	98.52941%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	97.54902%
70 %	98.03922%	96.56863%	98.77451%	97.54902%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	97.54902%
75 %	98.03922%	96.56863%	98.03922%	97.54902%	97.54902%	98.28431%	98.28431%	98.28431%	98.28431%	98.28431%	97.54902%
80 %	98.03922%	96.56863%	98.03922%	96.81373%	97.54902%	97.54902%	97.54902%	98.28431%	98.28431%	98.28431%	97.54902%
85 %	98.03922%	96.56863%	98.03922%	96.81373%	96.81373%	96.81373%	97.54902%	97.54902%	97.54902%	98.28431%	97.54902%
90 %	98.28431%	96.56863%	98.03922%	96.81373%	96.81373%	96.81373%	96.81373%	97.54902%	97.54902%	97.54902%	97.54902%
95 %	98.28431%	96.56863%	98.03922%	96.81373%	96.81373%	96.81373%	96.81373%	96.81373%	96.81373%	97.54902%	97.54902%
99.99 %	98.28431%	96.81373%	98.03922%	96.81373%	96.81373%	96.81373%	96.81373%	96.81373%	96.81373%	96.81373%	97.54902%

Table Ad 3b: Correlation Coefficients between Fund Rankings according to the Exogenous Cubic IM and the (Restricted) Endogenous Cubic IM for Different Values of Desired Expected Excess Return $\bar{u}^+$ and of Exogenous Investment x_P in Reference Portfolio P ($\omega = 0$)

Cells referred to in the main text of the paper are shaded (footnote 23)

$\bar{u}^+$	1.7719 %		2.00 %		2.20 %		2.40 %		2.60 %		10.00 %	
No. \ ω	0	100,000	0	100,000	0	100,000	0	100,000	0	100,000	0	100,000
29	11	12	11	12	11	12	11	12	11	12	10	15
30	1	2	1	2	1	2	1	2	1	2	1	1
31	4	5	4	5	4	5	4	5	4	5	4	5
32	9	6	9	6	9	6	9	6	9	7	9	8
33	17	17	17	17	17	17	17	17	17	17	17	17
34	15	14	15	14	15	14	15	14	15	14	15	13
35	14	9	14	9	14	9	14	9	14	9	14	9
36	16	10	16	10	16	10	16	10	16	10	16	10
37	10	13	10	13	10	13	10	13	10	13	11	12
38	3	1	3	1	3	1	3	1	3	1	2	2
39	2	4	2	3	2	3	2	3	2	3	3	3
40	5	7	5	7	5	7	6	7	7	6	7	6
41	7	16	7	16	7	16	7	16	5	16	5	16
42	6	3	6	4	6	4	5	4	6	4	6	4
43	8	15	8	15	8	15	8	15	8	15	8	14
44	12	11	12	11	12	11	12	11	12	11	12	11
45	13	8	13	8	13	8	13	8	13	8	13	7
ρ_{SP}	68.1373%		69.1176%		69.1176%		69.8529%		65.1961%		64.2157%	

Table Ad 4: Ranking of Funds for the Two Cases ω = 0 As Well As ω = 100,000 with x_P = 50 % and $\bar{u}^+ \in$ {1.7719 %, 2.0 %, 2.2 %, 2.4 %, 2.6 %, 10 %} and Corresponding Ranking Correlation Coefficients ρ_{SP}

Cells referred to in the main text of the paper are shaded (footnote 24)

x_P	0 %	10 %	20 %	30 %	40 %	50 %	60 %	70 %	80 %	90 %	99.99 %
Av. Corr. Coeff.	62.5000%	64.1748%	65.5637%	65.5229%	65.8905%	67.6062%	67.8105%	67.5654%	67.4837%	66.9526%	66.3399%

Table Ad 5: Average Correlation Coefficients between the Two Rankings according to $\omega = 0$ and $\omega = 100{,}000$ for Varying Values of x_P

Generalization of Table 5 of the main text to situations with varying values for desired expected excess return (footnote 25)

$\bar{u}^{+}$	1.7719 %		2.00 %		2.20 %		2.40 %		2.60 %		10.00 %	
No.	**Best possible ranking pos.**	**Worst possible ranking pos.**	**Best possible ranking pos.**	**Worst possible ranking pos.**	**Best possible ranking pos.**	**Worst possible ranking pos.**	**Best possible ranking pos.**	**Worst possible ranking pos.**	**Best possible ranking pos.**	**Worst possible ranking pos.**	**Best possible ranking pos.**	**Worst possible ranking pos.**
29	8	15	8	15	8	15	8	15	8	15	9	16
30	1	2	1	2	1	2	1	2	1	2	1	1
31	4	5	4	5	4	5	4	5	4	5	4	5
32	6	9	6	9	6	9	6	9	7	9	7	10
33	17	17	17	17	17	17	17	17	17	17	17	17
34	13	16	13	16	13	16	13	16	13	16	12	16
35	9	14	9	14	9	14	9	14	9	14	9	14
36	10	16	10	16	10	16	10	16	10	16	10	16
37	8	15	8	15	8	15	8	15	8	15	8	15
38	1	3	1	3	1	3	1	3	1	3	2	2
39	2	4	2	3	2	3	2	3	2	3	3	3
40	5	7	5	7	5	7	6	7	6	7	6	7
41	7	16	7	16	7	16	7	16	5	16	5	16
42	3	6	4	6	4	6	4	5	4	6	4	6
43	7	16	7	16	7	16	7	16	7	16	7	15
44	8	15	8	15	8	15	8	15	8	15	8	15
45	8	13	8	13	8	13	8	13	8	13	7	13

Table Ad 6: Possible Variations in Fund Rankings according to a Variation of ω

Cells referred to in the main text of the paper are shaded (footnote 26)

x_P \ $\bar{u}^+$	1.7719 %	1.90 %	2.00 %	2.10 %	2.20 %	2.30 %	2.40 %	2.50 %	2.60 %	2.70 %	10.00 %
0 %	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%
5 %	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%	67.15686%
10 %	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.15686%
15 %	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.15686%
20 %	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.15686%
25 %	66.42157%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.15686%
30 %	66.17647%	66.42157%	66.42157%	66.42157%	66.42157%	67.40196%	67.40196%	67.40196%	67.40196%	67.40196%	67.15686%
35 %	66.17647%	66.17647%	66.17647%	66.17647%	66.42157%	66.42157%	66.42157%	66.42157%	66.42157%	67.40196%	67.40196%
40 %	70.09804%	70.09804%	66.17647%	66.17647%	66.17647%	66.17647%	66.17647%	66.42157%	66.42157%	66.42157%	67.40196%
45 %	70.09804%	70.09804%	70.09804%	70.09804%	70.83333%	66.17647%	66.17647%	66.17647%	66.17647%	66.17647%	67.40196%
50 %	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.83333%	66.17647%	66.17647%	66.17647%	67.40196%
55 %	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.83333%	67.40196%
60 %	69.85294%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	67.40196%
65 %	69.11765%	69.85294%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	67.40196%
70 %	69.11765%	69.11765%	69.85294%	69.85294%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	67.40196%
75 %	69.11765%	69.11765%	69.11765%	69.85294%	69.85294%	70.09804%	70.09804%	70.09804%	70.09804%	70.09804%	67.40196%
80 %	69.11765%	69.11765%	69.11765%	69.11765%	69.85294%	69.85294%	69.85294%	70.09804%	70.09804%	70.09804%	67.40196%
85 %	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.85294%	69.85294%	69.85294%	70.09804%	67.40196%
90 %	68.62745%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.85294%	69.85294%	69.85294%	67.40196%
95 %	68.62745%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.85294%	67.40196%
99.99 %	68.62745%	68.62745%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	69.11765%	67.40196%

Table Ad 7: Correlation Coefficients between Fund Rankings According to the Exogenous Cubic IM and the Restricted Optimized Quadratic *Sharpe* Measure for Different Values of Desired Expected Excess Return $\bar{u}^+$ and of Exogenous Investment x_P in Reference Portfolio P ($\omega = 0$)