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Performance Evaluation, Portfolio Selection, and HARA Utility

by Wolfgang Breuer[†] and Marc Gürtler[‡]

Abstract. Our main goal is the generalization of the approach of Jobson and Korkie(1984) for funds performance evaluation. Therefore, we consider the portfolio selection problem of an investor who faces short sales restrictions when choosing among F different investment funds and assume the investor's utility function to be of the HARA type. We develop a performance measure and discuss its relationships to Treynor(1965), Sharpe(1966), Jensen(1968), Prakash and Bear(1986), and Grinblatt and Titman(1989). Particular attention is given to the special case of cubic utility implying skewness preferences. Our findings are illustrated by an empirical example.

Keywords: HARA utility , performance evaluation, portfolio selection, skewness

JEL-Classification: G11

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1. Introduction

Investors use performance measures for choosing among alternative funds engagements. In general, there are two possible ways to tackle the problem of the development of new performance measures. On the one hand, one can choose a partial-analytical framework, thereby focusing on the decision problem of a given investor for given expectations and neglecting any kind of general capital market considerations. On the other hand, one can analyze capital market price formation processes in order to derive conclusions with respect to the attractiveness of certain funds. For example, the well-known capital asset pricing model (CAPM) as introduced by Sharpe(1964), Lintner(1965), and Mossin(1966) may define such a setting. Recent approaches like the ones by Harvey and Siddique(2000), Dittmar(2002) or Fletcher and Kihanda(2005) are in particular stressing the relevance of preferences for higher-order return moments like skewness and kurtosis in asset pricing models. However, in general one may conclude from such equilibrium descriptions that the same performance measure of zero should be assigned to all investment funds, just expressing that – compared to direct stock holdings – the holding of shares of any fund is irrelevant for any capital market participant.

Although such analyses on capital market levels certainly are apt to create interesting general insights, for practical application we prefer a partial-analytical framework as the one sketched in Bodie et al(2005), pp. 870-874, for the case of simple mean-variance preferences. This means that we focus on the view of a single investor with given preference structures and expectations who typically acts as a price-taker. If for such an investor the CAPM in fact should hold, we would learn this from the investor’s specific expectations. But if this is not true, the CAPM (as any other capital market model) is not of immediate relevance for the investor under consideration. Nevertheless there are attempts to derive performance measures even on the basis of a capital market equilibrium approach as the one presented by the CAPM. Apparently, to keep consistency with capital market requirements only marginal investments in funds with out-of-equilibrium returns can be taken into account. Even in this regard a partial-analytical framework may prove useful as it helps to better understand the relevance of the performance measures derived from equilibrium consideration. We therefore will return to this aspect later on.

To be more specific with respect to our own approach, we use the framework by Jobson and Korkie(1984) as our starting point. Jobson and Korkie(1984) – among other things – considered an investor at time 0 with mean-variance preferences who chooses among F different (alternative) funds which can each be combined with a fixed reference portfolio P until time 1. Moreover, the investor is able to borrow or lend any amount at a riskless interest rate R. This last assumption leads to the validity of the well-known two-funds separation theorem first established by Tobin(1958). As a consequence of Tobin's separation result, any investor with mean-variance preferences should select that fund which offers the highest attainable Sharpe ratio of a combination of a fund f with the reference portfolio P. Moreover, Jobson and Korkie(1984) verified that the resulting funds ranking according to this optimized Sharpe ratio is equivalent to a ranking according to the square of the so-called appraisal ratio of Treynor and Black(1973). Henceforth, we simply speak of the Treynor/Black measure. In the same way the Sharpe ratio will often be called by us the Sharpe measure.

The approach by Jobson and Korkie(1984) therefore presents a portfolio-theoretical foundation of the well-known Treynor/Black measure. Unfortunately there are at least two shortcomings connected with their analysis. First, they did not allow for possible short-sales restrictions. As a consequence, the fund f exhibiting the highest square of the Treynor/Black measure might lead to the highest possible Sharpe measure of accessible portfolios only by being sold short. In general funds cannot be sold short and there is therefore a need for considering this additional restriction when developing measures for performance evaluation. This characterizes the first aim of our analysis.

Moreover, as already sketched, there are several studies which underpin the necessity for allowing for more general preferences than the simple mean-variance case. In particular, Levy and Markowitz(1979), Kroll et al(1984), Hlawitschka(1994) as well as Breuer and Gürtler(2001) indicate that for high risk aversion and/or the use of options the approximative quality of mean-variance preferences may be poor. Moreover, the relevance of higher-order return moments in portfolio selection models has been emphasized, for example, by Chunhachinda et al(1997) and Patton(2004). The analysis by Jobson and Korkie(1984) therefore should be broadened to allow for more general preferences. Indeed, Hakansson(1969) and Cass and Stiglitz(1970) were able to extend the original two-funds separation theorem to the whole class of HARA utility functions, i.e. utility functions with hyperbolic absolute risk aversion. This generalization of preferences for the original portfolio selection problem considered by Jobson and Korkie(1984) is the second main goal of our paper.

To fulfil these two goals, section 2. formally describes the general portfolio selection problem under consideration. As already mentioned, we consider a one-period problem with an investor just identifying one out of F different funds and optimally combining this one with a direct stock investment and riskless lending or borrowing. Certainly, the examination of a situation where only one out of F different funds can be chosen is somewhat restrictive. Nevertheless such a scenario can be interpreted as a classical asset allocation problem with three classes of assets (a fund, direct stock holding and riskless lending or borrowing). As an illustration, this decision problem corresponds to the important case of institutional investors relying only on a single fund manager, a not uncommon practice, for example, in the U.K. In addition, it is necessary to define different funds as alternative investments if performance measures for single funds shall be derived. Moreover, the analysis of situations with the selection of only one fund at a time may be used as a starting point for the examination of more complex portfolio selection problems in future work. In fact, our derivations remain valid if we reinterpret f = 1, …, F not as single funds but as F different given portfolios of funds. Only the analysis of the determination of the optimal combination of a certain set of funds must then be the object of further research.

From the basic presentations at the beginning of section 2. we derive generalized versions of the classical performance measures suggested by Sharpe(1966), Treynor(1965), and Jensen(1968) for the case of HARA preferences with the last one belonging to the class of period weighting measures introduced by Grinblatt and Titman(1989). We show that all these measures are only border solutions if we exclude short sales of fund shares and equity portfolios. For inner optima of the portfolio selection problem we get a performance measure which can best be viewed as the generalized Sharpe measure not of a fund itself but of its optimal combination with the reference portfolio P. Moreover, when we refer to the optimized performance measure we mean one which can be endogenously derived from the investor's portfolio selection problem when taking care of border solutions. Section 2. offers a complete description of adequate measures of funds ranking in the case of HARA utility functions for the portfolio selection problem under consideration.

In section 3. we consider the special case of cubic HARA utility functions and determine the special functional form of the Sharpe measure for optimal combinations of an arbitrary fund under consideration and the reference portfolio. Prakash and Bear(1986) were one of the first to apply the Three-Moment CAPM by Kraus and Litzenberger(1976) in order to derive a modified Treynor ratio recognizing skewness preferences We therefore additionally highlight the relationship between their results and the ones of this paper. As stated above, our partial-analytical framework will help to understand better the relevance of performance measures derived from pure capital market equilibrium considerations.

In section 4. we analyze empirically the relevance of funds rankings on the basis of the generalized performance evaluation established in sections 2. and 3. for funds investing in either German, British, or French shares. We restrict ourselves to the case of quadratic and cubic (HARA) utility. Since there are considerable differences in funds rankings according to the several performance measures mentioned above our findings suggest that there is in fact a need for an optimized performance measure explicitly recognizing skewness preferences. Section 5. concludes.

As is necessarily the case for any theoretical paper our results are based on several proofs that, because of space constraints, are in several separate appendices which are available from the authors on request. Moreover, it should be mentioned that we assume in our propositions and lemmas that the order of differentiation and integration may be exchanged and that all expected values under consideration exist.

2. Performance measurement in the general case of HARA utility

We consider an expected utility maximizing investor with an initial endowment W_0 at time 0 who can invest in exactly one of F different funds f = 1, …, F as well as in a portfolio P of shares. Moreover, riskless lending or borrowing at a constant interest rate R is possible. For a given fund f let x be the fraction of W_0 which is invested securely for one period from time 0 until time 1 with x < 0 denoting riskless lending The amount $y \cdot (1-x) \cdot W_0$ will be given to the portfolio manager of fund f. The remaining amount $(1-y) \cdot (1-x) \cdot W_0$ will be invested in the portfolio P of equity shares directly available on the capital market. Let R_f and R_P be the uncertain rates of return of investment fund f and portfolio P and r_f as well as r_P stand for the corresponding excess returns $R_f - R$ and $R_P - R$, respectively. We assume that $E[r_P] > 0$ and $E[r_f] > 0$ for all funds f.

Then the investor's excess return on the risk component of the portfolio is

$$y \cdot r_f + (1-y) \cdot r_P = r_q \text{, say,} \tag{1}$$

and total wealth at time 1 is

$$W_1 = W_0 \cdot (x \cdot (1+R) + (1-x) \cdot (1+R+r_q)) = W_0 \cdot (1+R+(1-x) \cdot r_q)\,. \tag{2}$$

The investor faces three decision problems. First the investor has to select an investment fund f and then has to determine the optimal values of x and y in order to maximize the expectation value of the utility function.

Funds rankings should not be preference-dependent; that is they should only depend on objective market data. Since we want to rank funds for an investor with the decision problem described above we only allow for utility functions which all assure the same kind of funds evaluation. From Hakansson(1969) and Cass and Stiglitz(1970) it is known that for certain classes of utility functions with hyperbolic absolute risk aversion (HARA) the optimal value for y as well as the assessment of different investment funds does not depend on the specification of the investor's preferences. More precisely, a HARA utility function $U_{(a,b)}$ is described by an absolute risk aversion

$$-U''_{(a,b)}(W_1) / U'_{(a,b)}(W_1) = 1/(a + b \cdot W_1)\,. \tag{3}$$

HARA utility functions with an identical parameter b belong to the same class. All members of such a class lead to the same optimal value for y independent of the preference parameter a. As pointed out, e.g. Dybvig and Ross(2003), pp. 629-631, this property of HARA utility function is the reason for their central relevance in portfolio theory.

The reciprocal of absolute risk aversion, i.e.

$$-U'_{(a,b)}(W_1)/U''_{(a,b)}(W_1),$$

is called an individual's risk tolerance. In the case of HARA utility we get a linear function $a+b\cdot W_1$. As shown by Borch(1960), because of this linearity, pareto-efficient risk sharing designs among individuals are linear in terminal wealth for a given class of HARA utility functions and thus very easy to establish. This circumstance just highlights another important feature of HARA preferences. Moreover, linear risk tolerances can be interpreted as a first order Taylor series approximation of arbitrary risk tolerances thus emphasizing the high practical relevance of HARA utility. Thereby, we must have $a+b\cdot W_1>0$ for all possible terminal wealth levels. According to (3), this condition is necessary for simultaneously guaranteeing $U'(W_1)>0$ and $U''(W_1)<0$ for all W_1, that is positive and decreasing marginal utility and therefore risk-averse behavior. In Breuer and Gürtler(2005) the requirement $a+b\cdot W_1>0$ is analyzed in more detail. As parameters a and b can assume arbitrary values as long as we always have $a+b\cdot W_1>0$, HARA utility functions describe situations with falling (b < 0), rising (b > 0) and constant (b = 0) risk tolerance. Thereby falling (rising) risk tolerance is equivalent to rising (falling) absolute risk aversion.

As already pointed out by Hakansson(1969) and Stiglitz(1970), classes of HARA utility functions comprise exponential, logarithmic, and power utility functions and can be completely described in the following way:

$$U_{(a,b)}(W_1) = -\exp\left(-\frac{1}{a}\cdot W_1\right), \quad b = 0;$$

$$U_{(a,b)}(W_1) = \ln(a + W_1), \quad b = 1; \tag{4}$$

$$U_{(a,b)}(W_1) = \frac{1}{b-1}\cdot(a + b\cdot W_1)^{1-\frac{1}{b}}, \text{ otherwise } (b \in \Re \setminus \{0, 1\}).$$

Using (3) and (4) as well as the abbreviation

$$z = ((1-x)\cdot W_0)/(a + b\cdot W_0\cdot(1+R)) \tag{5}$$

in Appendix A it is shown that

$$U_{(a,b)}(W_1) = \exp\left(-\frac{W_0\cdot(1+R)}{a}\right)\cdot U_{(1,b)}(z\cdot(y\cdot r_f + (1-y)\cdot r_P)), \quad b = 0;$$

$$U_{(a,b)}(W_1) = \ln(a + W_0\cdot(1+R)) + U_{(1,b)}(z\cdot(y\cdot r_f + (1-y)\cdot r_P)), \quad b = 1; \tag{6}$$

$$U_{(a,b)}(W_1) = (a + b\cdot W_0\cdot(1+R))^{1-\frac{1}{b}}\cdot U_{(1,b)}(z\cdot(y\cdot r_f + (1-y)\cdot r_P)), \text{ otherwise } (b \in \Re \setminus \{0, 1\}).$$

Since positive marginal utility in the case of an exclusive riskless investment requires a positive value of $a + b\cdot W_0\cdot(1+R)$, the maximization of $E[U_{(a,b)}(W_1)]$ can be replaced by the maximization of

$$E[U_{(1,b)}(z\cdot(y\cdot r_f + (1-y)\cdot r_P))].$$

For any given fund f, the above expectation is maximized with respect to y and z and does not depend on the investor's initial endowment W_0 or preference parameter a any more. It is obvious that this result immediately implies the well-known two-funds separation theorem mentioned earlier. In what follows these optimal values of y and z (as well as other corresponding optimal values) are indicated by an asterisk.

From all funds under consideration the investor will select that one with the highest attainable value of

$$E[U_{(1,b)}(z\cdot(y\cdot r_f+(1-y)\cdot r_P))].$$

In the case of quadratic utility this leads us to the realization of the maximum Sharpe ratio by the optimal combination of a fund f and the portfolio P of equity shares as has already been described by Jobson and Korkie(1984). Brennan and Solanki(1981) derived a similar result for situations with lognormally distributed security returns and risk neutral market valuations.

We therefore introduce the following definition.

Definition 1. *Consider an investor with HARA utility facing the portfolio selection problem described at the beginning of section 2. Also define U as $U_{(1,b)}$. The quantity*

$$E[U(z\cdot(y\cdot r_f+(1-y)\cdot r_P))]=GSM(y,z),\ say, \tag{7}$$

is defined as the generalized Sharpe measure for the portfolio structure (y,z).

This definition leads to the following proposition.

Proposition 1. *An investor with a HARA utility function facing the portfolio selection problem defined above will rank funds according to the maximum values of $GSM(y,z)$ for each fund.*

Proof. *See derivation above.*

Certainly, y should be restricted to avoid situations where a fund or the reference portfolio P of direct stock holding is sold short by the investor. To analyze the characterization of possible border solutions with respect to y we have to introduce the following new lemma.

Lemma. *Let f be a fund with* $E[r_f] \neq E[r_P]$ *and define* $z(y)$ *as the optimal value of z for given fund f and share y of the fund f in the risky portfolio . The following statements obtain.*

(i) *z(y) has a unique root* $\hat{y} = E[r_P]/(E[r_P] - E[r_f])$.

(ii) *The sign of z(y) equals the sign of* $y \cdot E[r_f] + (1-y) \cdot E[r_P]$ *for all y.*

(iii) *The sign of the derivative* $\partial z(y)/\partial y\big|_{y=\hat{y}}$ *equals the sign of* $E[r_f] - E[r_P]$*. Thus, if* $E[r_f] \underset{<}{>} E[r_P]$*, it follows:* $\hat{y} \underset{>1}{<0}$, $z(y) \underset{>}{<} 0$ *for all* $y < \hat{y}$ *and* $z(y) \underset{<}{>} 0$ *for all* $y > \hat{y}$.

(iv) $GSM(y, z(y)) = E[U(z(y) \cdot (y \cdot r_f + (1-y) \cdot r_P))]$ *has a unique (unrestricted) maximum at a value* y^* *and a unique (unrestricted) minimum at* $\hat{y}$.

(v) $z(y^*) \cdot (1-y^*) < 0 \;\Rightarrow\; \underset{y \in [0,1]}{\arg\max}[GSM(y, z(y))] = 1$,

$z(y^*) \cdot y^* < 0 \;\Rightarrow\; \underset{y \in [0,1]}{\arg\max}[GSM(y, z(y))] = 0$.

Proof. *See Appendix B.*

Let optimal values for y and z (as well as other variables) in the case of x ∈ $\Re$ and y ∈ $[\underline{y}, \overline{y}]$ ($\underline{y}, \overline{y} \in \Re$) be characterized by two asterisks ** while a single asterisk denotes optimal solutions for x, y ∈ $\Re$. According to part (v) of the Lemma we get a border solution $y^{**} = 1$ in the case of a restriction y ≤ 1 if the investor would – without this restriction – prefer to sell the equity portfolio short. To see this it is necessary to recognize that we have $z(y^*) \underset{(>)}{<} 0 \;\Rightarrow\; x(y^*) \underset{(<)}{>} 1$ and that we get $y^* > 1$ or $1 - y^* > 1$ for $z(y^*) < 0$ from part (ii) of the Lemma. This implies that situations with $z(y^*) > 0$ and $1 - y^* < 0$ as well as situations with

$z(y^*) < 0$ and $1 - y^* > 0$ are characterized by the optimality of short sales of (only) equity portfolio P if possible. Consequently, the introduction of short sales restrictions in such a situation then thus leads to a border solution $y^{**} = 1$. Exactly this relationship is described by the first line of part (v) of the Lemma. As shown by Sharpe(1966), for quadratic utility the setting y = 1 implies the ranking of funds according their simple Sharpe measure. We therefore may speak of the generalized Sharpe measure (for HARA utility) of a fund f in the following sense.

Definition 2. *The special case*

$$GSM(1, z(1)) = E[U(z(1) \cdot r_f)] \tag{8}$$

is called the generalized Sharpe measure of a fund f.

The second line of part (v) of the Lemma characterizes a situation where it is best for the investor to invest nothing in fund f. However, there may be situations where the investor wants to realize a (marginal) minimum positive engagement in funds so that the domain of y is then [ε, 1] with ε > 0. Under such circumstances an unrestricted optimal positive investment in f leads to $y_f^{**} = \varepsilon$. In this case it is possible to derive a simpler performance measure than the one presented in Proposition 1. To do so, we just have to look at the investor's utility depending on y for optimal choice z(y):

$$U(z(y) \cdot (y \cdot r_f + (1 - y) \cdot r_P)) = U(r(y)), \text{ say.} \tag{9}$$

For $y = \varepsilon \to 0$ the investor's expected utility converges to the same value $E[z(0) \cdot r_P]$ for any fund under consideration. Thereby, z(0) corresponds with that value z in the case y = 0 that leads to the maximum expected value of the utility function

$$E[U_{(1,b)}(z \cdot (y \cdot r_f + (1 - y) \cdot r_P))].$$

Since no dependency on the elected fund is recognizable, z(0) is equal for all funds. In order to compare funds f with $y = \varepsilon \to 0$ we thus have to derive $E[U(r(y))]$ with respect to y at y = 0. We get

$$\left.\frac{\partial E[U(r(y))]}{\partial y}\right|_{y=0} = E[U'(z(0)\cdot r_P)\cdot((\partial z(y)/\partial y)_{y=0}\cdot r_P + z(0)\cdot(r_f - r_P))] \\ = E[U'(z(0)\cdot r_P)\cdot r_f]\cdot z(0), \tag{10}$$

since $E[U'(z(0)\cdot r_P)\cdot r_P] = 0$ according to the first-order necessary condition for z(0).

As a result of part (ii) of the Lemma the sign of z(0) is positive. According to Appendix C, a fund g with $y = y_g$ and $z(y) = z_g(y)$ is thus better than a fund h for $y_g = y_h = \varepsilon \to 0$ with $y = y_h$ and $z(y) = z_h(y)$ if the following relationship holds:

$$E[U'(z_g(0)\cdot r_P)\cdot r_g] \; > \; E[U'(z_h(0)\cdot r_P)\cdot r_h] \\ \Leftrightarrow \; E[r_g] - \frac{cov[U'(z_g(0)\cdot r_P), r_g]}{cov[U'(z_g(0)\cdot r_P), r_P]}\cdot E[r_P] \; > \; E[r_h] - \frac{cov[U'(z_h(0)\cdot r_P), r_h]}{cov[U'(z_h(0)\cdot r_P), r_P]}\cdot E[r_P]. \tag{11}$$

In the case of quadratic utility function (11) becomes the original Jensen measure. (11) therefore suggests the following definition.

Definition 3. *The quantity*

$$E[r_f] - \beta_{fP}\cdot E[r_P] = GJM\text{, say,} \tag{12}$$

with β_{fP} *defined as*

$$\frac{cov[U'(z(0)\cdot r_P), r_f]}{cov[U'(z(0)\cdot r_P), r_P]}$$

is called a generalized Jensen measure for HARA utility.

The generalized Jensen measure is already known in the literature and may be interpreted as the marginal expected utility from adding a small fraction of a fund f to a reference portfolio

P as has been pointed out by e.g. Grinblatt and Titman(1989), p. 407, for the special case of quadratic utility, and in general by Leland(1999), pp. 28, 33. It is interesting to note that this measure should only be used to rank funds f which are inferior in such a sense that the investor would prefer to sell them short. All other funds are better than those inferior ones and will be ranked separately according to the performance measure of Proposition 1 (possibly allowing for border solutions $y^{**} = 1$ in the case of unrestricted optimal values $y^{*} > 1$). Such funds are called superior ones.

Since inferior funds are optimally sold short, they are characterized by a negative Jensen measure, while GJM > 0 holds for all funds f in which the investor prefers a positive holding. Furthermore, a negative sign of the reversed Jensen measure $E[r_P] - \beta_{Pf} \cdot E[r_f]$ shows us that we have a border solution $y^{**} = 1$ for the investors would like to sell portfolio P short. Summing up, the generalized Jensen measure and its reversed formulation make it possible to easily check for any fund f whether there will be a border solution with $y^{**} = \varepsilon > 0$ or $y^{**} = 1$ or not.

Finally, the performance measure according to Definition 3 belongs to the class of period weighting measures introduced by Grinblatt and Titman(1989) if we interpret period weights as marginal utilities. This follows immediately from the analysis in Grinblatt and Titman(1989), p. 407. The performance measure according to Proposition 1 does not belong to the class of period weighting measures as shown in Appendix D. As a by-product we get the following general assessment of the usefulness of period weighting measures for fund performance issues in our model. As in Grinblatt and Titman(1989), period weighting measures can be effectively used to identify superior funds since these funds are unambiguously characterized by a positive sign of their Jensen measures. In contrast to Grinblatt and Titman(1989) this does not rely on multivariate normal return distributions. Moreover, in order to determine a complete funds ranking, period weighting measures can only be applied to rank inferior funds, but not the superior ones. For our context, period weighting measures thus lack general applicability. Summarizing, we have the following proposition.

Proposition 2. *Consider an investor with HARA utility and facing a portfolio selection problem as described at the beginning of section 2. with short sales restrictions* $y \in [\varepsilon, 1]$, $\varepsilon > 0$, *but small. Funds which should optimally be sold short are inferior compared to all other funds and can be identified by their negative generalized Jensen measure. They should be ranked separately behind the other funds according to the generalized Jensen measure. Any other fund has to be characterized by the generalized Sharpe measure of the optimal combination of this fund f and equity portfolio P. Funds which lead to optimal short sales of the equity portfolio P are characterized by a negative reversed generalized Jensen measure. For them performance evaluation reduces to the generalized Sharpe measure according to Definition 2. In the case of quadratic utility functions all generalized performance measures can be simplified to their counterparts based on a quadratic utility function.*

Proof. *See derivation above.*

In an identical manner to its equivalent based on a quadratic utility function the generalized Jensen measure can easily be manipulated by the variation of a fund's engagement in riskless lending or borrowing. There are several possibilities in order to neutralize the influence of such manipulations. The most straightforward way seems to introduce normalized funds which are characterized by the same expected excess return $\mu^\circ > 0$. To this end, any fund f must be combined with riskless borrowing/lending by the investor in a certain way just rendering fund managers' endeavors to influence their Jensen measure by riskless borrowing/lending useless. To be precise, funds f are substituted by portfolios f° which consist of a fraction $x^\circ = (E[r_f] - \mu^\circ)/E[r_f]$ that is invested in riskless lending and a fraction $1 - x^\circ$ which is invested in the original fund f so that for $r_{f^\circ} = (1 - x^\circ) \cdot r_f$ we get $E[r_{f^\circ}] = \mu^\circ$. The resulting normalized funds f° may then be ranked according to the generalized Jensen measure of Definition 3. Since their expected return is identical for all funds the following transformation of their generalized Jensen measure is possible according to Appendix E: A fund g is better than a fund h if

$$\begin{aligned} & E[r_{g^\circ}] - \beta_{g^\circ P} \cdot E[r_P] \;>\; E[r_{h^\circ}] - \beta_{h^\circ P} \cdot E[r_P] \\ \Leftrightarrow \;& -\frac{1}{\frac{E[r_g]}{\beta_{gP}}} \;>\; -\frac{1}{\frac{E[r_h]}{\beta_{hP}}}. \end{aligned} \tag{13}$$

For quadratic utility the denominators of the fractions in the last line of (13) become the well-known Treynor ratio. The derivation of (13) thus suggests the following definition.

Definition 4. *The quantity*

$$E[r_f]/\beta_{fP} = GTM, \text{ say,} \tag{14}$$

with β_{fP} defined as in Definition 3 is called a generalized Treynor measure for fund f in the case of HARA utility.

Proposition 3. *The generalized Jensen measure of Definition 3 for normalized funds leads to a ranking of funds according to the negative inverse of the generalized Treynor measure of the original funds. If we assume beta coefficients to be greater than zero then a direct ranking according to the generalized Treynor measure evolves.*

Proof. *See (13). The last line can easily be transformed to* $E[u_g]/\beta_{gP} > E[u_h]/\beta_{hP}$*, if both beta coefficients (and expected excess returns) are positive.*

Proposition 3 in connection with Proposition 2 gives us a remarkable justification for the application of the original as well as the generalized Treynor measure. It turns out to be an adequate performance measure in the case of (exogenously given) marginal funds engagements when assuring invariance of ranking with respect to funds' riskless lending or borrowing. To be precise, (original) funds f should be ranked according to the negative inverse of the generalized Treynor measure if we postulate $y = \varepsilon$ for all (normalized) funds $f°$. In particular, this implies that all funds with *negative* performance measure GTM are better than all those with positive signs for GTM and each of these subsets of funds can be separately ranked according to GTM. The reason for the superiority of negative Treynor measures is that for positive expected excess returns they coincide with negative beta values so that their contribution to total portfolio risk is negative and therefore advantageous.

Once again, the (negative inverse of the) generalized Treynor measure of Definition 4 belongs to the class of period weighting measures according to Grinblatt and Titman(1989), but only with respect to our newly defined normalized funds $f°$ as is shown in Appendix D.

3. Performance evaluation in the special case of cubic HARA utility

Though quite general, the performance measures developed in section 2. lack some transparency. This is in particular true for the generalized Sharpe measure of a combination of a fund f with the reference portfolio P. Jobson and Korkie(1984) showed that in the case of quadratic utility and absence of short sales restrictions the performance measure according to Proposition 1 can be reduced to a ranking by the square of the Treynor/Black measure. Unfortunately, there seems to be no straightforward way to extend their results to the general case of HARA utility. For more transparent results, the set of admissible preferences must therefore be narrowed.

The most natural way to generalize traditional mean variance analysis is by additionally allowing for skewness preferences. In connection with HARA utility this implies considering cubic utility functions. The assumption of cubic utility enables us to give a more specific description of the Sharpe measure of the optimal combination of a fund f and equity portfolio P as described in Proposition 1. Moreover, we are able to relate the generalized Treynor measure for cubic HARA utility to the performance measure developed by Prakash and Bear(1986) on the basis of the Three-Moment CAPM by Kraus and Litzenberger(1976). Thereby, we give an example for the analysis of possible connections between a partial-analytical framework for the development of performance measures and a capital-market oriented equilibrium approach to performance measurement.

3.1. The Sharpe measure for optimal fund engagements

For cubic HARA utility the corresponding transformed[1] utility function U is

$$U(r(y,z)) = (z \cdot (y \cdot r_f + (1-y) \cdot r_P) - 1)^3. \tag{15}$$

In this situation we get the following formula for $E[U(r(y,z))]$ which is proven in Appendix F.

$$\begin{aligned} E[U(r(y,z))] &= (z \cdot \mu_q - 1)^3 + 3 \cdot (z \cdot \mu_q - 1) \cdot z^2 \cdot \sigma_q^2 + z^3 \cdot \gamma_q^3 \\ &= \Phi(\mu_q, \sigma_q^2, \gamma_q^3), \text{ say.} \end{aligned} \tag{16}$$

μ_q, σ_q^2, and γ_q^3 denote the corresponding first three moments of $\tilde{r}_q$. The third central moment γ^3 characterizes the skewness of a random variable and its relevance is a direct implication of the assumption of a cubic rather than of a quadratic utility function. It is easy to show that formula (16) is decreasing in σ_q^2. Moreover, as long as we have x < 1, i.e. no riskless lending financed by short sales of risky assets, (16) is increasing in μ_q and γ_q^3. A proof of these assertions and an explanation of the relevance of skewness considerations as a consequence of the assumption of a cubic rather than a quadratic utility function are given in Appendix G. With equation (16) we can prove the following proposition.

Proposition 4. *For cubic HARA utility the Sharpe measure of optimal fund engagements according to Definition 1 in a situation without short sales restrictions can be simplified to the following special cubic performance measure*

$$CSM^*(\alpha_1,\alpha_2) = \frac{2\cdot(1-\alpha_1^2-\alpha_1\cdot\alpha_2^3)^{1.5}-\alpha_2^3\cdot(\alpha_2^3-\alpha_1^3+3\cdot\alpha_1)-2\cdot(3\cdot\alpha_1^2+1)}{(\alpha_1^3+\alpha_2^3+3\cdot\alpha_1)^2} \quad (17)$$

with α_1 defined as μ_{q*}/σ_{q*} and α_2 as γ_{q*}/σ_{q*}. In this context q^* identifies the optimal structure of the overall portfolio's risk component with excess return $r_{q*} = y^*\cdot r_f + (1-y^*)\cdot r_P$. The same holds true if we take short sales restrictions explicitly into account. For such a situation, once again all optimal variables are denoted by two asterisks (**) so that in that case we especially write $CSM^{**}(\alpha_1, \alpha_2)$.

Proof. *See Appendix H.*

The performance measure according to Proposition 4 is lengthy, but its calculation is not difficult, because it only depends on two arguments. The first, $\alpha_1 = \mu_{q*}/\sigma_{q*}$, is just the simple quadratic Sharpe measure of the optimal portfolio q^*. The performance measure of Proposition 4 may be viewed as a generalization of this quadratic measure to the cubic case and is thus called the cubic Sharpe measure CSM^* of the optimal portfolio.

Since we already know that preference values ceteris paribus are increasing with higher values for μ_{q*} and smaller values for σ_{q*}, we can immediately conclude that the generalized performance measure of Proposition 4 is increasing in α_1 as well. This implies that in the case of constant values for γ_{q*} which corresponds with decision making on the basis of pure mean-variance preferences the performance measure of Proposition 4 is equivalent to a funds ranking on the basis of $\alpha_1 = \mu_{q*} / \sigma_{q*}$. As mentioned earlier, the relevance of this measure for performance evaluation in the case of simple mean-variance preferences was demonstrated in Jobson and Korkie(1984).

As an extension to the results in Jobson and Korkie(1984) the second argument α_2 determines the assessment of any fund f and relates the skewness of the optimal portfolio's return to its standard deviation. Once again, we know that the performance measure must be increasing in this argument because it is increasing in γ_{q*} (if we abstract from the somewhat pathological case $x^* > 1$ mentioned above in which an investor sells risky assets to finance riskless lending). A fund g with $\alpha_1 = \alpha_{g,1}$ and $\alpha_2 = \alpha_{g,2}$ is unambiguously better than a fund h with $\alpha_1 = \alpha_{h,1}$ and $\alpha_2 = \alpha_{h,2}$ if we have $\alpha_{g,1} > \alpha_{h,1}$ and $\alpha_{g,2} > \alpha_{h,2}$. No unambiguous relationship can be derived for cases where we have $\alpha_{g,1} > \alpha_{h,1}$ and $\alpha_{g,2} < \alpha_{h,2}$ or $\alpha_{g,1} < \alpha_{h,1}$ and $\alpha_{g,2} > \alpha_{h,2}$. In fact, as presented in Appendix I we may easily find examples where such scenarios lead to higher or to lower performance evaluations when switching from fund g to fund h. All these relationships hold still true if we refer explicitly to situations with short sales restrictions.

3.2. Cubic Treynor measure and Prakash and Bear(1986)

If all investors are risk averse and possess cubic utility functions of the HARA type, for any risky asset f according to Kraus and Litzenberger(1976) the following relationship must hold in capital market equilibrium:

$$E[r_f] = \lambda_1 \cdot \mathrm{cov}[r_f, r_M] + \lambda_2 \cdot E[(r_f - E[r_f]) \cdot (r_M - E[r_M])^2] \tag{18}$$

λ_1 and λ_2 are market constants which depend on the aggregated risk preferences of all investors in the capital market. Since risk averse investors with cubic HARA utility functions are variance averters but skewness lovers, λ_1 is positive and λ_2 negative as has been shown e.g. by Kraus and Litzenberger(1976), p. 1088. Equation (18) can be considered a generalized cubic security market line because when coskewness $E[(r_f - E[r_f]) \cdot (r_M - E[r_M])^2]$ or λ_2 is zero, the security market line given by the standard CAPM evolves. Following, Prakash and Bear(1986) we define η (< 0) as the quotient λ_1/λ_2 and rewrite (18) as

$$\begin{aligned} & E[r_f] = \lambda_2 \cdot (\eta \cdot \mathrm{cov}[r_f, r_M] + E[(r_f - E[r_f]) \cdot (r_M - E[r_M])^2]\ ) \\ \Leftrightarrow\ & \frac{E[r_f]}{(\eta \cdot \mathrm{cov}[r_f, r_M] + E[(r_f - E[r_f]) \cdot (r_M - E[r_M])^2]\ )} = \lambda_2, \end{aligned} \tag{19}$$

Let us now allow for possible deviations of funds f from the cubic security market line (19). On this basis Prakash and Bear(1986) suggested to rank funds according to the fraction on the left-hand side of the last equation in (19).

Definition 6. *The left-hand side of the last equation of (19) is called the Prakash/Bear performance measure PBM.*

In Appendix J it is shown that in the special case of a cubic (HARA-) utility function the funds ranking according to Proposition 3 can be transformed in the following way

$$\frac{E[(r_g - E[r_g]) \cdot (r_P - E[r_P])^2] + 2 \cdot \left(E[r_P] - \frac{2}{z(0)}\right) \cdot \text{cov}[r_P, r_g]}{E[\tilde{u}_g]}$$
$$> \frac{E[(r_h - E[r_h]) \cdot (r_P - E[r_P])^2] + 2 \cdot \left(E[r_P] - \frac{2}{z(0)}\right) \cdot \text{cov}[r_P, r_h]}{E[r_h]}, \tag{20}$$

which corresponds to the inverse of the Prakash/Bear performance measure of Definition 6 if we use the market portfolio M as the portfolio P of equity shares and apply the definition

$$2 \cdot (E[r_M] - 2 / z(0)) = \eta, \text{ say.} \tag{21}$$

Thereby, it is worth mentioning that the consideration of cubic utility of the HARA type in connection with only marginal fund engagements is consistent with the validity of the generalized security market line (18) for all *equity shares*. Capital market equilibrium will not be disturbed by funds deviating from the generalized security market line as long as their importance is negligible. Because of the two-funds separation by Hakansson(1969) and Cass and Stiglitz(1970), the consideration of investors with marginal fund engagements, cubic HARA utility, and homogeneous expectations (regarding equity shares) immediately leads to the Three-Moment CAPM by Kraus and Litzenberger(1976) as characterized by (18) and (19) with respect to equity shares. In such a scenario η as defined by (21) must indeed be identical to the fraction λ_1 / λ_2 as implied by the generalized security market line (19). In this sense, capital-market oriented equilibrium approaches to performance measurement turn out to be special cases of the partial-analytical approach favored in this paper. Moreover, it seems to us that the partial-analytical framework is much more transparent than the equilibrium approach.

The parameter z(0) has to be determined by the implicit definition $\partial\Phi / \partial y = 0$ for the special case y = 0. From Appendix K we get

$$z(0) = 2 \cdot \frac{E[r_M^2] - \sqrt{E^2[r_M^2] - E[r_M] \cdot E[r_M^3]}}{E[r_M^3]} \tag{22}$$

with

$$E[r_M^2] = var[r_M] + E^2[r_M] \tag{23}$$

and

$$E[r_M^3] = E[(r_M - E[r_M])^3] + 3 \cdot E[r_M] \cdot var[r_M] + E^3[r_M]\,. \tag{24}$$

Thus, by (22) and (21) it is easy to determine η empirically.

We summarize our findings of this subsection in the following proposition.

Proposition 5. *In the case of cubic HARA utility the negative inverse of the generalized Treynor measure leads to the same funds ranking as the inverse of the Prakash/Bear performance measure.*

Proof. *See derivation above.*

Among other things, Proposition 5 implies that a ranking according to the original Prakash/Bear performance measure typically totally reverses funds ranking according to the negative inverse of the generalized Treynor measure and thus selects that fund which (after normalization) leads to the *lowest* increase in utility when marginally added to the market portfolio M (or, more generally, to P). We consider such a ranking not very reasonable and therefore favor the performance measures derived in this paper.

4. Empirical Example

To exemplify our results, we consider a German investor who is planning to select one fund investing in German (British, French) equity shares and to combine this fund optimally with a given naive diversified direct investment on the German (British, French) capital market. We focus on the comparison of funds rankings based on quadratic and cubic (HARA) utility and generally speak of quadratic and cubic performance measures, respectively. Specifically, we consider quadratic as well as cubic Sharpe, Jensen, Treynor and optimized performance measures with exclusion of short sales and the quadratic Treynor/Black performance measure. Thereby, when applying optimized performance measures, inferior funds are separately ranked via (the negative inverse of) their Treynor measures. We refrain from computing optimized performance measures without short sales restrictions because of the obvious high practical importance of this limitation and because otherwise inferior funds would become very attractive only because of the possibility of being sold short.

The starting point of our analysis is monthly (post tax) return data for 45 German, 36 British, and 24 French funds over a period from June 1994 to July 1999 which are calculated on using the respective monthly repurchase prices (in Deutschmarks) per share. This means that possible selling markups are not taken into account. In this respect, the performance of funds generally tends to be overestimated when compared to the performance of any reference index. However, the determination here (in accordance with many other approaches) of gross performance measures allows at least some conclusions to be made with regard to the sensitivity of ranking when different types of performance measures are applied. Exactly this aspect forms the central issue of this paper.

We assume that all earnings paid out to the investors by a fund f are reinvested in this fund. As proxies of diversified direct capital markets investments we use the DAX 100 for Germany, the FTSE 100 for the UK, and the France CAC 40 for France. The DAX 100 (listed until 03/21/2003) consisted of 100 continuously traded shares of German companies including the 30 blue chips of the DAX 30 and the 70 midcap-stocks of the MDAX. Based on special criteria (e.g. that at least 25 % of the stock in issue must be publicly available for investment and must not be in the hands of a single party or parties acting in concert) the FTSE 100 comprises the largest 100 UK companies ranked by market value. A sample of 40 French stocks listed on the so-called Monthly Settlement Market (also known as the RM or the Règlement Mensuel) constitutes the CAC 40 index. Moreover, it is noteworthy that for the time period under consideration there existed index certificates with respect to all three indices thus making a monetary engagement in them indeed rather easy.[2]

The riskless interest rate R is approximated by the expected return of German time deposit running for one month and covering the respective period of time to be observed. Funds of each country are analyzed separately.

At the end of each month from July 1997 to July 1999 we estimate expectation values, variances, covariances, skewnesses and co-skewnesses on the basis of historical return data for the preceding 36 months and use these estimators in order to determine a ranking of funds for the following month as investment period and given performance measure. This gives 25 different funds rankings for the funds and for any performance measure under consideration. We thus allow for the problem of time-varying moments of return distributions.

We refrain from considering a more recent time interval as in the aftermath of the global stock market crash in 2000 there have been too few funds being able to earn positive average excess returns. Historical return data would not be suitable for the estimation of return moments in such a situation. As our aim is to give an example of the consequences of different measures for fund performance and not to develop new methods of return estimation we focus on pre-crash stock market data.

We determine the average ranking position of every fund f (and the respective reference portfolio P of direct capital market engagements) for all quadratic or cubic performance measures under consideration. As an example, for German funds these average ranking positions are presented in Table 1. Corresponding tables for British and French funds are available from the authors on request.

Table 1

Average ranking positions of German funds according to several performance measures

Since investors are mainly interested in superior funds as defined above and the optimized performance measures with border solutions of inferior funds reduce to simple rankings according to the (negative inverse of the) Treynor measure, further analysis focuses on the best ten German (British, French) funds (possibly including the corresponding reference portfolio P) according to the optimized cubic performance measure with short sales restrictions. In Table 1, German top ten funds are shaded.

Based on the average ranking positions of these top ten funds we are able to calculate ranking correlation coefficients between any pair of the performance measures under consideration as Table 2 displays. Moreover, as in practical applications funds are very often ranked according to their average past relative wealth increase (net of fund inflows and outflows) we allowed for funds rankings according to their simple average excess return. Henceforth, we call this the risk neutral performance measure and Table 2 presents average ranking correlation coefficients for the risk neutral performance measure as well.

Table 2

Ranking correlation coefficients between various performance measures for German, British, and French funds (top ten funds)

As Table 2 shows there may be considerable differences in funds ranking according to the optimized quadratic performance measure and the simple quadratic Sharpe, Treynor, or Jensen measure. The same holds true with respect to cubic measures. These findings are indicated by the shaded numbers at the intersection of the lines belonging to SM^{**} (quadratic utility) and SM^{**} (cubic utility) and the columns for the Sharpe, Jensen, and Treynor measure.

Moreover, we can identify similarly significant differences in funds rankings according to the (quadratic or cubic) optimized performance measure and the Treynor/Black measure as is indicated by the shaded numbers in the column belonging to the Treynor/Black measure. This is not too surprising since the latter one has been derived from portfolio optimization without short sales restrictions and thus can lead to considerable deviations from funds rankings which explicitly allow for such kind of restrictions. Summarizing, the consideration of optimized performance measures can be recommended for this example, because this requires similar return information than the corresponding Sharpe, Treynor, and Jensen measure, but additionally leads to a portfolio-theoretically based funds ranking.

In addition, Table 2 can be used for a comparison between rankings according to the quadratic performance measures and their respective cubic counterparts. In fact, as can be seen by examining the relationship between quadratic and cubic Treynor as well as Jensen measure for British funds, it is obvious that deviations in ranking can be of a similar size as the differences between rankings according to (quadratic, cubic) optimized and (quadratic, cubic) Sharpe, Treynor, and Jensen measures. Furthermore, optimized funds rankings for quadratic and cubic optimized performance measure differ to some degree, too, as can be seen by the three shaded numbers at the intersections of the row belonging to SM^{**} (quadratic utility) and the column belonging to SM^{**} (cubic utility). Because of these findings, it seems to be indeed reasonable to explicitly recognize skewness preferences in funds rankings as well. This conclusion is in line with several other analyses regarding portfolio optimization which were mentioned in the introduction.

Finally, risk neutral funds rankings differ considerably from funds rankings according to the optimized cubic utility. This is verified by the shaded cells at the intersection of the rows belonging to linear utiliy and the columns belonging to SM^{**} (cubic utility) and may indicate significant welfare losses resulting from fund selection according to their historical average excess return instead of a selection on the basis of the optimized cubic performance measure when utility is in fact cubic and of the HARA type.

Nevertheless, a simple comparison of funds rankings according to different performance measures does not reveal the precise amount of possible welfare losses from the application of an unsuitable performance measure. To do so, an analysis of attainable certainty equivalents is required. Thereby, for the case of German funds we want to compute relative losses in certainty equivalents for an investor with cubic HARA utility who acts according to an inadequate portfolio selection rule. In order to calculate expected utility levels and certainty equivalents it is not sufficient to determine which fund is chosen according to different performance measures. Additionally, it must also be fixed how this fund is combined with the reference portfolio P and riskless lending or borrowing. To this end, we identify the application of the quadratic Sharpe measure with a situation in which the investor assumes a setting $y = 1$ and combines a fund f with riskless lending or borrowing based on a quadratic utility function. The quadratic Jensen measure as well as the quadratic Treynor measure can be interpreted as situations with a restriction $y = \varepsilon > 0$, but small, and – once again – quadratic utility. Cubic Sharpe, Jensen and Treynor measure describe decision situations with corresponding settings for y but cubic utility. For all 25 portfolio selection problems from July 1997 to July 1999 we determine optimal portfolios based on the rules just described and compute resulting certainty equivalents for an investor whose utility function is actually cubic (and of the HARA type). Certainly, there are greater certainty equivalents achievable by portfolio selection according to the optimized cubic performance measure and with a restriction $y \in [0, 1]$ instead of $y = 1$ or $y = \varepsilon$ so that we express all resulting certainty equivalents as percentages of this attainable maximum value. Besides quadratic and cubic Sharpe, Jensen, and Treynor measure we also consider the optimized quadratic measure for which we assume portfolio selection with a restriction $y \in [0, 1]$ based on quadratic utility. Furthermore, we consider portfolio selection based on expected excess returns. Since risk neutrality would not lead to an inner solution for an investor's riskless lending or borrowing we assume risk neutral fund selection and the choice $y = 1$ but a quadratic utility for the determination of the amount of the riskless investment.

Unfortunately, in the case of linear and quadratic utility approximation resulting relative certainty equivalents depend on the fraction a/W_0 and thus are not independent of preference parameter a and initial wealth W_0 any longer. Therefore, average relative certainty equivalents over 25 periods each are presented in Table 3 for three cases of small, medium-level and high values of a/W_0. Thereby, increasing values of a/W_0 imply ceteris paribus greater risk tolerance. Details are presented in Appendix L. Because of the two-funds separation relative risk discounts resulting from cubic performance measures are independent of the investor's risk tolerance.

The procedure just described relies on the ex ante determination of certainty equivalents. For each of 25 portfolio selection problems over the whole time period return distributions for the computation of certainty equivalents are assumed as given ex ante based on the results of the corresponding past 36 months.

Nevertheless, it is also possible to estimate relative certainty equivalents based on ex post performance of different portfolio selection rules. For illustrative purposes we also compute the relative certainty equivalent of the optimized quadratic performance measure based on ex post performance. This means we use the 25 ex post return realizations resulting from 25 applications of the optimized quadratic performance measure as an estimator for the investor's portfolio return distribution in the case of such a portfolio selection behavior. The same is done for the application of the optimized cubic performance measure leading to a relative certainty equivalent for the optimized quadratic performance measure based on ex post performance results of only about $1/7 \approx 14.29$ % for low investor's risk tolerance expressing a welfare loss of about $1-0.1429 = 85.71$ %. Apparently, this result based on ex post performance complies very well with the corresponding number in Table 3 based on ex ante distributions.

Table 3

Average relative certainty equivalents attainable by the application of different performance measures when utility is actually cubic and of the HARA type

In fact, as can be learnt from Table 3 both the risk neutral performance measure and all quadratic ones may lead to considerable welfare losses for an investor with actual cubic HARA utility. The same holds true for the application of the cubic Jensen and Treynor measure. Welfare losses from linear or quadratic utility approximation are increasing with ceteris paribus smaller risk tolerance. In fact, this specific result corresponds perfectly to the findings by other authors like Levy and Markowitz(1979) who have been cited in the introduction.

Nevertheless, (in contrast to the cubic Treynor and Jensen measure) the simple cubic Sharpe measure together with the ad hoc restriction of $y = 1$ does a good job since in the case of (only) short sales restrictions the setting $y = 1$ is indeed typically the optimal solution for the top ten funds under consideration. After all, this examination of certainty equivalents seems to verify the relevance of cubic performance measures instead of the approximative use of decision rules based on risk neutral or quadratic preferences. If an investor is only interested in the determination of the best fund out of a set of F funds, the application of the cubic Sharpe measure might be used as an approximation of the optimized cubic performance measure. Nevertheless, the utilization of both performance measures requires the same information if the investor wants to optimize the overall portfolio consisting of a fund f, the reference portfolio P and riskless lending or borrowing. From this point of view we recommend once again the direct application of the optimized cubic performance measure.

5. Conclusion

This paper was motivated by the question how the portfolio-theoretic approach by Jobson and Korkie(1984) for performance evaluation could be extended to allow for short sales restrictions and be broadened in order to allow for preferences beyond mean-variance. By doing so, we were able to generalize the performance measures of Treynor(1965), Sharpe(1966), and

Jensen(1968) to the case of HARA utility and give a portfolio-theoretic foundation for all of these measures. Moreover, we related our work to the approaches by Prakash and Bear(1986) and Grinblatt and Titman(1989). We extended the notion by Grinblatt and Titman(1989) of a period weighting measure to identify superior funds to all classes of HARA utility functions and arbitrary return distributions. We also showed that for the special purpose of funds ranking period weighting measures are only apt to rank inferior funds. The ranking of the more interesting other funds cannot be based on a period weighting measure. Finally, we presented a brief empirical application of the performance measures under consideration which indicates the relevance of portfolio-theoretically founded performance measures recognizing skewness preferences.

Certainly, we have to admit that the performance measures developed in this paper are based on purely theoretical considerations and that there is a gap between the methods that are newly derived in academic journals and those that are often used in practice. For this reason we took into account the risk neutral performance measure which is based on historical average fund returns and which seems to be the most relevant criterion for fund selection in practical application. Obviously, for an investor with cubic HARA utility and a one-period time horizon the utilization of this simple risk-neutral performance measure may lead to considerable welfare losses. As a consequence, we believe there are indeed situations in which the performance measures developed in this paper may add value to investors. However, the main contribution of this paper is new theory while the task of making the methods developed in this paper fully operational in practice certainly is a separate project which should be addressed by future work.

[1] Notice, that $U_{(1,-0.5)}(.)$ and $1.5 \cdot U_{(1,-0.5)}(.)$ are equivalent because of the cardinality of Von Neumann-Morgenstern utility functions. Because of a similar reason, it is here possible to redefine z for the derivation of Proposition 4 as one half of the original decision variable z (i.e. $0.5 \cdot z$ is simply replaced by z).

[2] For further information on these indices see e.g. http://www.finix.at and particular for the former DAX 100 see Deutsche Boerse Group(2003), p. 6.

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Appendices

Appendix A: Proof of (6)

Case b = 0:

$$\begin{aligned} U_{(a,b)}(W_1) &= -\exp\left(-\frac{1}{a}\cdot W_1\right) \underset{(2)}{=} -\exp\left(-\frac{1}{a}\cdot W_0\cdot(1+R+(1-x)\cdot(y\cdot r_f+(1-y)\cdot r_P))\right) \\ &= -\exp\left(-\frac{1}{a}\cdot W_0\cdot(1+R)\right)\cdot\left(-\exp\left(-\frac{1}{a}\cdot W_0\cdot(1-x)\cdot(y\cdot r_f+(1-y)\cdot r_P)\right)\right) \\ &\underset{(5)}{=} -\exp\left(-\frac{1}{a}\cdot W_0\cdot(1+R)\right)\cdot\left(-\exp\left(-z\cdot(y\cdot r_f+(1-y)\cdot r_P)\right)\right) \\ &= -\exp\left(-\frac{1}{a}\cdot W_0\cdot(1+R)\right)\cdot U_{(1,b)}(z\cdot(y\cdot r_f+(1-y)\cdot r_P)). \end{aligned} \tag{A.A1}$$

Case b = 1:

$$\begin{aligned} U_{(a,b)}(W_1) &= \ln(a+W_1) \underset{(2)}{=} \ln(a+W_0\cdot(1+R+(1-x)\cdot(y\cdot r_f+(1-y)\cdot r_P))) \\ &= \ln\left((a+W_0\cdot(1+R))\cdot\left(1+\frac{W_0\cdot(1-x)\cdot(y\cdot r_f+(1-y)\cdot r_P)}{a+W_0\cdot(1+R)}\right)\right) \\ &\underset{(5)}{=} \ln(a+W_0\cdot(1+R))+\ln(1+z\cdot(y\cdot r_f+(1-y)\cdot r_P)) \\ &= \ln(a+W_0\cdot(1+R))+U_{(1,b)}(z\cdot(y\cdot r_f+(1-y)\cdot r_P)). \end{aligned} \tag{A.A2}$$

Case b ∈ $\Re$ \ {0, 1}:

$$\begin{aligned} U_{(a,b)}(W_1) &= \frac{1}{b-1}\cdot(a+b\cdot W_1)^{1-\frac{1}{b}} \underset{(2)}{=} \frac{1}{b-1}\cdot(a+b\cdot W_0\cdot(1+R+(1-x)\cdot(y\cdot r_f+(1-y)\cdot r_P)))^{1-\frac{1}{b}} \\ &= (a+b\cdot W_0\cdot(1+R))^{1-\frac{1}{b}}\cdot\frac{1}{b-1}\cdot\left(1+b\cdot\frac{W_0\cdot(1-x)\cdot(y\cdot r_f+(1-y)\cdot r_P)}{a+b\cdot W_0\cdot(1+R)}\right)^{1-\frac{1}{b}} \\ &\underset{(5)}{=} (a+b\cdot W_0\cdot(1+R))^{1-\frac{1}{b}}\cdot\frac{1}{b-1}\cdot\left(1+b\cdot z\cdot(y\cdot r_f+(1-y)\cdot r_P)\right)^{1-\frac{1}{b}} \\ &= (a+b\cdot W_0\cdot(1+R))^{1-\frac{1}{b}}\cdot U_{(1,b)}(z\cdot(y\cdot r_f+(1-y)\cdot r_P)). \end{aligned} \tag{A.A3}$$

Appendix B: Proof of the Lemma

First of all, the investor faces the optimization problem

$$E[U(z\cdot(y\cdot r_f+(1-y)\cdot r_P))] \;\rightarrow\; \max_{z,\,y}.! \qquad \text{(A.B1)}$$

Thus, the corresponding necessary conditions are as follows:

$$E[U'(z\cdot(y\cdot r_f+(1-y)\cdot r_P))\cdot(y\cdot r_f+(1-y)\cdot r_P)]=0, \qquad \text{(A.B2)}$$

$$E[U'(z\cdot(y\cdot r_f+(1-y)\cdot r_P))\cdot z\cdot(r_f-r_P)]=0. \qquad \text{(A.B3)}$$

(i):

A value of zero for z leads to a modification of (A.B2) as follows: $E[U'(0)\cdot(y\cdot r_f+(1-y)\cdot r_P)]$ = 0. This equality is fulfilled if and only if $y\cdot E[r_f]+(1-y)\cdot E[r_P]=0$. This in turn is equivalent to $y=E[r_P]/(E[r_P]-E[r_f])$ and thus the statement of (i) is proven.

(ii):

By the use of the necessary condition (A.B2) z is implicitly defined. Since we have $E[V_1\cdot V_2]$ = $cov[V_1,V_2]+E[V_1]\cdot E[V_2]$ for arbitrary random variables V_1 and V_2, (A.B2) is equivalent to

$$\begin{aligned}&cov[U'(z(y)\cdot(y\cdot r_f+(1-y)\cdot r_P)),\, y\cdot r_f+(1-y)\cdot r_P]\\ &=-E[U'(z(y)\cdot(y\cdot r_f+(1-y)\cdot r_P))]\cdot E[y\cdot r_f+(1-y)\cdot r_P].\end{aligned} \qquad \text{(A.B4)}$$

As a consequence of positive marginal utility we have $E[U'(.)]>0$. Therefore, the right-hand side of (A.B4) has the opposite sign of $y\cdot E[r_f]+(1-y)\cdot E[r_P]$. Since U' is a decreasing function, the sign of the left-hand side of (A.B4) is positive for $z(0) < 0$ and negative for $z(0) > 0$. This means that it is the opposite of the sign of z(0). Since the signs of the left-hand side and the right-hand side of (A.B4) must be equal, so must be the signs of z(0) and of $y\cdot E[r_f]+(1-y)\cdot E[r_P]$.

(iii):

In the case $E[r_P] > E[r_f]$ statement (i) leads to $\hat{y}_f = E[r_P]/(E[r_P] - E[r_f]) > 1$ and in the case $E[r_P] < E[r_f]$ the inequality $\hat{y}_f = E[r_P]/(E[r_P] - E[r_f]) < 0$ is true. Moreover, with

$$E[U'(z\cdot(y\cdot r_f + (1-y)\cdot r_P))\cdot(y\cdot r_f + (1-y)\cdot r_P)] = F(y,z), \text{ say} \qquad \text{(A.B5)}$$

we get from (A.B2) under consideration of $z(\hat{y}) = 0$ the equality $F(\hat{y}, z(\hat{y})) = F(\hat{y}, 0) = 0$. Application of the Implicit Function Theorem[1] (with r(y) as a shortcut for $z(y)\cdot(y\cdot r_f + (1-y)\cdot r_P)$) yields

$$\left.\frac{\partial z}{\partial y}\right|_{\hat{y}} = -\frac{\left.\partial F/\partial y\right|_{(y,z)=(\hat{y},0)}}{\left.\partial F/\partial z\right|_{(y,z)=(\hat{y},0)}} = -\frac{E[U''(r(\hat{y}))\cdot(r_f - r_P)\cdot r(\hat{y}) + U'(r(\hat{y}))\cdot(r_f - r_P)]}{E[U''(r(\hat{y}))\cdot(\hat{y}\cdot r_f + (1-\hat{y})\cdot r_P)^2]}$$
$$\underset{r(\hat{y})=0}{=} \frac{U''(0)\cdot 0 + U'(0)\cdot(E[r_f] - E[r_P])}{-E[U''(0)\cdot(\hat{y}\cdot r_f + (1-\hat{y})\cdot r_P)^2]}. \qquad \text{(A.B6)}$$

Since the denominator and $U'(0)$ are positive, the sign of $\partial z/\partial y|_{y=\hat{y}}$ corresponds with the sign of $E[r_f] - E[r_P]$. Finally, following (i) z has a unique zero at $\hat{y}$ implying all further statements of (iii).

(iv):

From portfolio theory we know[2] that the problem

$$E[U(\xi_f\cdot r_f + \xi_P\cdot r_P)] \;\rightarrow\; \max_{\xi_f,\ \xi_P}.! \qquad \text{(A.B7)}$$

owns a unique maximum and no other local extrema. This problem is equivalent to problem (A.B1) if the potential solution of no risky engagement (z = 0) is excluded. For $z \neq 0$ the sum $\xi_f + \xi_P$ corresponds with z and $\xi_f/(\xi_f + \xi_P)$ equals y. Thus, there are only two candidates for a local extremum: the unique maximum of the problem mentioned above and (y,z) = $(\hat{y}, 0)$.

To analyze the second potential extremum we have to form the first and the second partial derivative of $GSM(y, z(y)) = E[U(z(y) \cdot (y \cdot r_f + (1-y) \cdot r_P))]$:

$$\begin{aligned}
&\frac{\partial GSM(y, z(y))}{\partial y} \\
&= E[U'(z(y) \cdot (y \cdot r_f + (1-y) \cdot r_P)) \cdot ((\partial z / \partial y) \cdot (y \cdot r_f + (1-y) \cdot r_P) + z(y) \cdot (r_f - r_P))] \quad \text{(A.B8)} \\
&\underset{\text{(A.B2)}}{=} E[U'(z(y) \cdot (y \cdot r_f + (1-y) \cdot r_P)) \cdot z(y) \cdot (r_f - r_P)]
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 GSM(y, z(y), y)}{\partial y^2} \\
&= E[U''(z(y) \cdot (y \cdot r_f + (1-y) \cdot r_P)) \cdot ((\partial z / \partial y) \cdot (y \cdot r_f + (1-y) \cdot r_P) + z(y) \cdot (r_f - r_P)) \quad \text{(A.B9)} \\
&\quad \cdot z(y) \cdot (r_f - r_P)] + E[U'(z(y) \cdot (y \cdot r_f + (1-y) \cdot r_P)) \cdot (\partial z / \partial y) \cdot (r_f - r_P)].
\end{aligned}$$

Obviously we get $\partial GSM(y, z(y)) / \partial y \big|_{y=\hat{y}} = 0$. Moreover,

$$\frac{\partial^2 GSM(y, z(y))}{\partial y^2}(\hat{y}) = U''(0) \cdot 0 + U'(0) \cdot (\partial z / \partial y)\big|_{y=\hat{y}} \cdot (E[r_f] - E[r_P]) \underset{\text{(iii)}}{>} 0, \quad \text{(A.B10)}$$

so that $GSM(z(y), y)$ has a local minimum in $\hat{y}$. Since there are only two potential extrema the local minimum as well as the local maximum are unique.

(v):

To prove this statement we distinguish between two cases.

Case 1: $z(y^*) > 0$

Let $y^* < 0$ ($\Rightarrow$ $z(y^*) \cdot y^* < 0$). Firstly, let $E[r_P] > E[r_f]$. Following (iii) we get $\hat{y} > 1$ and thus $y^* < \hat{y}$. According to (iv) $GSM(y, z(y))$ is strictly decreasing on the interval $[y^*, \hat{y}] \supset [0, 1]$.

Secondly, let $E[r_P] < E[r_f]$. Part (iii) leads to $\hat{y} < 0$. We do not have $y^* < \hat{y}$ since this statement corresponds with $z(y^*) < 0$ according to (iii) and is therefore not treated in Case 1. Using (iv) again we get that $GSM(y, z(y))$ is strictly decreasing on $[y^*, \infty) \supset [0, 1]$. Hence, we have characterized the monotonic behavior of $GSM(y, z(y))$ and shown that $y^{**} = 0$ is the optimal restricted engagement.

Now look at $y^* > 1$ ($\Rightarrow z(y^*) \cdot (1 - y^*) < 0$). Again we firstly analyze $E[r_P] > E[r_f]$ so that (iii) leads to $\hat{y} > 1$. Analogously to the proof above we get $y^* < \hat{y}$ according to (iii) and Case 1 since $y^* > \hat{y}$ does not correspond with Case 1. Using (iv) $GSM(y, z(y))$ is strictly increasing on $(-\infty, y^*] \supset [0, 1]$.

Let secondly $E[r_P] < E[r_f]$. From (iii) we know $\hat{y} < 0$ and thus $y^* < \hat{y}$. According to (iv) $GSM(y, z(y))$ is strictly increasing on the interval $[\hat{y}, y^*] \supset [0, 1]$. Consequently, the derivative of $GSM(y, z(y))$ has a positive sign on $(0, y^*)$ and this leads to $y^{**} = 1$ to be the optimal restricted engagement.

Case 2: $z(y^*) < 0$

We immediately get from (ii) $y^* \cdot E[r_f] + (1 - y^*) \cdot E[r_P] < 0$ and $y^* \notin [0, 1]$ respectively. Especially $z(y^*) \cdot y^* < 0$ corresponds with $y^* > 1$ and $z(y^*) \cdot (1 - y^*) < 0$ complies with $y^* < 0$.

Firstly, the case $y^* < 0$ is treated ($\Rightarrow z(y^*) \cdot (1 - y^*) < 0$). The assumption $E[r_P] > E[r_f]$ leads to $\hat{y} > 1$ and consequently $y^* < \hat{y}$ which in turn implies $z(y^*) > 0$ according to (iii). This is inconsistent with Case 2.

Consequently, we have $E[r_P] < E[r_f]$. This also implies $\hat{y} < 0$ and because of $z(y^*) < 0$ and (iii) it follows $y^* < \hat{y}$. Thus, $GSM(y, z(y))$ is strictly increasing on $[\hat{y}, \infty) \supset [0, 1]$ and $y^{**} = 1$ is the optimal engagement.

Secondly, we suppose $y^* > 1$ ($\Rightarrow z(y^*) \cdot y^* < 0$). From $E[r_P] > E[r_f]$ it follows $\hat{y} > 1$ using (iii) and because of $z(y^*) < 0$ and (iii) also the statement $\hat{y} < y^*$ is true. Thus, $GSM(y, z(y))$ is strictly decreasing on $(-\infty, \hat{y}] \supset [0, 1]$ and $y^{**} = 0$ is optimal in the restricted case.

Finally, $E[r_P] < E[r_f]$ cannot occur, since (iii) leads to $\hat{y} < 0$ and thus $y^* > \hat{y}$ and according to (iii) especially to $z(y^*) > 0$, which is not consistent with the assumption of Case 2.

Appendix C: Proof of (11)

From (10) we know that fund g is better than fund h for $y_g = y_h = \varepsilon \to 0$ if

$$E[U'(z(0) \cdot r_P) \cdot r_g] > E[U'(z(0) \cdot r_P) \cdot r_h]. \tag{A.C1}$$

Because of $cov[V_1,V_2] = E[V_1 \cdot V_2] - E[V_1] \cdot E[V_2]$ for arbitrary random variables V_1 and V_2, we have $cov[U'(z(0) \cdot r_P), r_f] = E[U'(z(0) \cdot r_P) \cdot r_f] - E[U'(z(0) \cdot r_P)] \cdot E[r_f]$. Inequality (AC.1) is therefore equivalent to

$$\begin{aligned} &E[U'(z(0) \cdot r_P)] \cdot E[r_g] + cov[U'(z(0) \cdot r_P), r_g] \\ &> E[U'(z(0) \cdot r_P)] \cdot E[r_h] + cov[U'(z(0) \cdot r_P), r_h]. \end{aligned} \tag{A.C2}$$

Since $E[U'(z(0) \cdot r_P)]$ is positive inequality (A.C2) can be transformed into the following form

$$E[r_g] + \frac{cov[U'(z(0) \cdot r_P), r_g]}{E[U'(z(0) \cdot r_P)]} > E[r_h] + \frac{cov[U'(z(0) \cdot r_P), r_h]}{E[U'(z(0) \cdot r_P)]}. \tag{A.C3}$$

Further, we know $\mathrm{cov}[U'(z(0)\cdot r_P), r_P] = E[U'(z(0)\cdot r_P)\cdot r_P] - E[U'(z(0)\cdot r_P)]\cdot E[r_P]$ and (from the definition of z (0)) $E[U'(z(0)\cdot r_P)\cdot r_P] = 0$, so that (A.C3) is equivalent to

$$E[r_g] - \frac{\mathrm{cov}[U'(z(0)\cdot r_P), r_g]}{\mathrm{cov}[U'(z(0)\cdot r_P), r_P]}\cdot E[r_P] \;>\; E[r_h] - \frac{\mathrm{cov}[U'(z(0)\cdot r_P), r_h]}{\mathrm{cov}[U'(z(0)\cdot r_P), r_P]}\cdot E[r_P]. \qquad \text{(A.C4)}$$

Appendix D: Proof that the generalized Sharpe measure of optimal portfolio structure (y^*, z^*) (Proposition 1) and the Treynor measure (Definition 4) do not belong to the class of period weighting measures

Let $r_{P,t}$ and $r_{f,t}$ be the excess returns of portfolio P and of fund f, respectively, from time t−1 to t. Following Grinblatt and Titman(1989) the period weighting measure PWM of a fund f can then be defined as follows:

$$\mathrm{PWM} = p - \lim_{T\to\infty} \sum_{t=1}^{T} \omega(r_{P,t}, T)\cdot r_{f,t}\,. \qquad \text{(A.D1)}$$

In this context the terms $\omega(r_{P,t},T)$ characterize arbitrary weights that only depend on $r_{P,t}$ and T and thus are independent of the realizations $r_{f,t}$ for any fund f at any point in time t. It is sufficient to show that the quadratic optimized Sharpe measure does not belong to the class of period weighting measures. According to Jobson/Korkie(1984) (in the case without short sales restrictions) this measure is equivalent to the (square of the) Treynor/Black measure. It thus suffices to show that the latter one does not belong to the class of period weighting measures. This in turn can be explained by using the quadratic Jensen measure. The Jensen measure is a period weighting measure as was shown by Grinblatt and Titman(1989)[3] and the corresponding period weights are

$$\omega(r_{P,t}, T) \;=\; \frac{\hat{\sigma}_P^2(T) - (r_{P,t} - \bar{r}_P(T))\cdot \bar{r}_P(T)}{T\cdot \hat{\sigma}_P^2(T)}\,, \qquad \text{(A.D2)}$$

with $\overline{r}_P(T)$ and $\hat{\sigma}_P^2(T)$ defined as $(1/T)\cdot\sum_{t=1}^{T} r_{P,t}$ and $(1/T)\cdot\sum_{t=1}^{T}(r_{P,t}-\overline{r}_P(T))^2$, respectively, as the estimators of the expected excess return and the excess return variance. The (quadratic) Treynor/Black measure is defined as the quotient of the quadratic Jensen measure and the variance $var[\varepsilon_{fP}]$ of the error term from a linear regression of r_f on r_P thus depending on the fund's excess return r_f. These statements and the definition of the period weighting measure lead to a dependency of potential weights $\hat{\omega}$ for forming the Treynor/Black measure on the excess returns $r_{f,t}$ of fund f so that a representation like (AD.1) (with weights $\hat{\omega}$ being independent of r_f) is not possible for this measure.

The same arguments can be given for the quadratic Treynor measure since the following equality holds in the general case:

$$GTM = \frac{GJM}{\beta_{fP}} + E[r_P]. \qquad \text{(A.D3)}$$

Consequently, since weights according to (AD.2) are independent of r_f potential weights for the quadratic Treynor measure must be depending on r_f.

Appendix E: Proof of (13)

If we rank normalized funds according to the generalized Jensen measure a fund g is better than a fund h if

$$
\begin{aligned}
& E[r_{g^\circ}] - \beta_{g^\circ P} \cdot E[r_P] \;>\; E[r_{h^\circ}] - \beta_{h^\circ P} \cdot E[r_P] \\
\underset{E[r_{g^\circ}]=E[r_{h^\circ}]}{\Leftrightarrow}\; & \beta_{g^\circ P} \;<\; \beta_{h^\circ P} \\
\Leftrightarrow\; & \frac{\text{cov}[U'(z(0)\cdot r_P),(1-\overset{\circ}{x}_g)\cdot r_g]}{\text{cov}[U'(z(0)\cdot r_P), r_P]} \;<\; \frac{\text{cov}[U'(z(0)\cdot r_P),(1-\overset{\circ}{x}_h)\cdot r_h]}{\text{cov}[U'(z(0)\cdot r_P), r_P]} \\
\Leftrightarrow\; & \underbrace{(1-\overset{\circ}{x}_g)}_{=\frac{\mu^\circ}{E[r_g]}} \cdot \underbrace{\frac{\text{cov}[U'(z(0)\cdot r_P), r_g]}{\text{cov}[U'(z(0)\cdot r_P), r_P]}}_{=\beta_{gP}} \;<\; \underbrace{(1-\overset{\circ}{x}_h)}_{=\frac{\mu^\circ}{E[r_h]}} \cdot \underbrace{\frac{\text{cov}[U'(z(0)\cdot r_P), r_h]}{\text{cov}[U'(z(0)\cdot r_P), r_P]}}_{=\beta_{hP}} \qquad \text{(A.E1)} \\
\Leftrightarrow\; & \frac{\mu^\circ}{E[r_g]} \cdot \beta_{gP} \;<\; \frac{\mu^\circ}{E[r_h]} \cdot \beta_{hP} \\
\underset{\mu^\circ>0}{\Leftrightarrow}\; & -\frac{1}{\frac{E[r_g]}{\beta_{gP}}} \;>\; -\frac{1}{\frac{E[r_h]}{\beta_{hP}}}.
\end{aligned}
$$

Appendix F: Proof of (16)

Using the (exact[4]) Taylor series expansion around $E[r(y,z)] = \mu_r$, say, we get the following formula for $U(r(y,z)) = (z \cdot (y \cdot r_f + (1-y) \cdot r_P) - 1)^3$:

$$
\begin{aligned}
U(r(y,z)) = {} & U(\mu_r) + U'(\mu_r) \cdot (r(y,z) - \mu_r) \\
& + \frac{1}{2} \cdot U''(\mu_r) \cdot (r(y,z) - \mu_r)^2 + \frac{1}{6} \cdot U'''(\mu_r) \cdot (r(y,z) - \mu_r)^3.
\end{aligned} \qquad \text{(A.F1)}
$$

Based on (AF.1) it follows:

$$
\begin{aligned}
E[U(r(y,z))] & = U(\mu_r) + \frac{1}{2} \cdot U''(\mu_r) \cdot \sigma_r^2 + \frac{1}{6} \cdot U'''(\mu_r) \cdot \gamma_r^3 \\
& = (\mu_r - 1)^3 + 3 \cdot (\mu_r - 1) \cdot \sigma_r^2 + \gamma_r^3 \\
& = (z \cdot \mu_q - 1)^3 + 3 \cdot (z \cdot \mu_q - 1) \cdot z^2 \cdot \sigma_q^2 + z^3 \cdot \gamma_q^3 \\
& = \Phi(\mu_q, \sigma_q^2, \gamma_q^3).
\end{aligned} \qquad \text{(A.F2)}
$$

Appendix G: Proof of the relevance of skewness in the case of cubic rather than quadratic utility and monotonic dependency of (16) regarding to μ_q, σ_q^2 and γ_q^3

If we assume a quadratic (HARA) utility function U (i.e. $b = -1$) expected utility can be calculated as $E[U(r(y,z))] = -E[(z\cdot(y\cdot r_f + (1-y)\cdot r_P) - 1)^2]$. Again, using the (exact) Taylor series expansion around $\mu_r = E[r(y,z)]$ leads to

$$\begin{aligned} E[U(r(y,z))] &= U(\mu_r) + \frac{1}{2}\cdot U''(\mu_r)\cdot\sigma_r^2 \\ &= -(\mu_r - 1)^2 - \frac{1}{2}\cdot 2\cdot\sigma_r^2 \\ &= -(z\cdot\mu_q - 1)^2 - z^2\cdot\sigma_q^2. \end{aligned} \tag{A.G1}$$

Thus, in contrast to the cubic case skewness plays no role in situations with quadratic utility. In addition we get the following partial derivatives of (16):

$$\begin{aligned} \frac{\partial\Phi(\mu_q,\sigma_q^2,\gamma_q^3)}{\partial\mu_q} &= 3\cdot z\cdot(z\cdot\mu_q - 1)^2 + 3\cdot z^3\cdot\sigma_q^2 = z\cdot(3\cdot(z\cdot\mu_q - 1)^2 + 3\cdot z^2\cdot\sigma_q^2), \\ \frac{\partial\Phi(\mu_q,\sigma_q^2,\gamma_q^3)}{\partial\sigma_q^2} &= 3\cdot(z\cdot\mu_q - 1)\cdot z^2, \\ \frac{\partial\Phi(\mu_q,\sigma_q^2,\gamma_q^3)}{\partial\gamma_q^3} &= z^3. \end{aligned} \tag{A.G2}$$

Further, $0 > U''(\mu_r) = 6\cdot(\mu_r - 1) = 6\cdot(z\cdot\mu_q - 1),$ so that the second equality of (A.G2) implies $\Phi(\mu_q,\sigma_q^2,\gamma_q^3)$ to be decreasing in σ_q^2. For x < 1, i.e. no riskless lending financed by short sales of risky assets, (5) leads to z > 0. Under consideration of this property the first and the third equality of (A.G2) result in $\partial\Phi(\mu_q,\sigma_q^2,\gamma_q^3)/\partial\mu_q > 0$ and $\partial\Phi(\mu_q,\sigma_q^2,\gamma_q^3)/\partial\gamma_q^3 > 0$.

Appendix H: Proof of Proposition 4

For given optimal value y^* we can determine the corresponding solution for z by deriving (16) with respect to z for $y = y^*$. The necessary condition $\partial\Phi / \partial z\big|_{y=y^*} = 0$ leads to the following solution:

$$\begin{aligned} &\frac{\partial\Phi}{\partial z} = 3\cdot(z\cdot\mu_{q*}-1)^2\cdot\mu_{q*} + 3\cdot\mu_{q*}\cdot z^2\cdot\sigma_{q*}^2 + 3\cdot(z\cdot\mu_{q*}-1)\cdot 2\cdot z\cdot\sigma_{q*}^2 + 3\cdot z^2\cdot\gamma_{q*}^3 \overset{!}{=} 0 \\ &\Leftrightarrow z^{\pm} = \frac{\mu_{q*}^2 + \sigma_{q*}^2 \pm \sqrt{\sigma_{q*}^4 - \sigma_{q*}^2\cdot\mu_{q*}^2 - \gamma_{q*}^3\cdot\mu_{q*}}}{\mu_{q*}^3 + 3\cdot\sigma_{q*}^2\cdot\mu_{q*} + \gamma_{q*}^3}. \end{aligned} \tag{A.H1}$$

Φ is a polynomial of third order in z with leading coefficient $\mu_{q*}^3 + 3\cdot\sigma_{q*}^2\cdot\mu_{q*} + \gamma_{q*}^3$, so that we have to distinguish between a positive sign and a negative sign of this term. In the first case the local maximum of Φ obviously is smaller than the local minimum which implies z^- to determine the local maximum. In the second case the local minimum of Φ is smaller than the local maximum. But in this case the denominator of (A4) has a negative sign whereby $z^- > z^+$ and again the local maximum is characterized by z^-. Substitution of z^- in (16) gives[5]

$$\begin{aligned} &\Phi(\mu_{q*}, \sigma_{q*}, \gamma_{q*}) \\ &= \frac{(-\gamma_{q*}^6 + \gamma_{q*}^3\cdot\mu_{q*}^3 - 3\cdot\gamma_{q*}^3\cdot\sigma_{q*}^2\cdot\mu_{q*}) - 2\cdot(3\cdot\sigma_{q*}^4\cdot\mu_{q*}^2 + \sigma_{q*}^6) + 2\cdot(\sigma_{q*}^4 - \sigma_{q*}^2\cdot\mu_{q*}^2 - \gamma_{q*}^3\cdot\mu_{q*})^{1.5}}{(\mu_{q*}^3 + 3\cdot\sigma_{q*}^2\cdot\mu_{q*} + \gamma_{q*}^3)^2} \\ &= \frac{-\gamma_{q*}^3\cdot\sigma_{q*}^3\cdot\left(\frac{\gamma_{q*}^3}{\sigma_{q*}^3} - \frac{\mu_{q*}^3}{\sigma_{q*}^3} + 3\cdot\frac{\mu_{q*}}{\sigma_{q*}}\right) - 2\cdot\sigma_{q*}^6\cdot\left(3\cdot\frac{\mu_{q*}^2}{\sigma_{q*}^2} + 1\right) + 2\cdot(\sigma_{q*}^4)^{1.5}\cdot\left(1 - \frac{\mu_{q*}^2}{\sigma_{q*}^2} - \frac{\gamma_{q*}^3\cdot\mu_{q*}}{\sigma_{q*}^4}\right)^{1.5}}{(\sigma_{q*}^3)^2\cdot\left(\frac{\gamma_{q*}^3}{\sigma_{q*}^3} + \frac{\mu_{q*}^3}{\sigma_{q*}^3} + 3\cdot\frac{\mu_{q*}}{\sigma_{q*}}\right)^2} \\ &= \frac{-(\frac{\gamma_{q*}}{\sigma_{q*}})^3\cdot((\frac{\gamma_{q*}}{\sigma_{q*}})^3 - (\frac{\mu_{q*}}{\sigma_{q*}})^3 + 3\cdot\frac{\mu_{q*}}{\sigma_{q*}}) - 2\cdot(3\cdot(\frac{\mu_{q*}}{\sigma_{q*}})^2 + 1) + 2\cdot(1 - (\frac{\mu_{q*}}{\sigma_{q*}})^2 - \frac{\mu_{q*}}{\sigma_{q*}}\cdot(\frac{\gamma_q}{\sigma_q})^3)^{1.5}}{((\frac{\gamma_{q*}}{\sigma_{q*}})^3 + (\frac{\mu_{q*}}{\sigma_{q*}})^3 + 3\cdot\frac{\mu_{q*}}{\sigma_{q*}})^2} \end{aligned} \tag{A.H2}$$

With $\alpha_1 = \mu_{q*} / \sigma_{q*}$ and $\alpha_2 = \gamma_{q*} / \sigma_{q*}$ we immediately get the measure of Proposition 4. The same holds true for $y = y^{**}$ and thus $q = q^{**}$, that is, in the case of the explicit recognition of short sales restrictions.

Appendix I: Examples for increasing (scenario 1) and decreasing (scenario 2) optimal cubic performance when switching from a fund g with $\alpha_{g,1}$ and $\alpha_{g,2}$ to a fund h with $\alpha_{h,1} > \alpha_{g,1}$ and $\alpha_{h,2} < \alpha_{g,2}$.

fund	μ_{q*}	σ_{q*}	γ_{q*}	α_1	α_2	CSM*
g	10 %	10 %	–20 %	1.0	–2.0	–0.672
h (scenario 1)	20 %	10 %	–50 %	2.0	–5.0	–0.660
h (scenario 2)	20 %	10 %	–60 %	2.0	–6.0	–0.719

When we compare fund g and fund h in both scenarios, we have $\alpha_{g,1} < \alpha_{h,1}$ and $\alpha_{g,2} > \alpha_{h,2}$. But in the first case the performance measure rises from approximately –0.672 to –0.66 while it declines from approximately –0.672 to –0.719 in the second case.

Appendix J: Derivation of (20)

As z(0) is positive and the second derivative of U is negative, we have a negative (and fund-independent) value of $\text{cov}[U'(z(0) \cdot r_P), r_P]$ as well. Using the definition of β_{fP} inequality (13) can thus be transformed to

$$\frac{\text{cov}[U'(z(0) \cdot r_P), r_g]}{E[r_g]} > \frac{\text{cov}[U'(z(0) \cdot r_P), r_h]}{E[r_h]}. \qquad \text{(A.J1)}$$

In the special case of the cubic (HARA-) utility function[6] $U(W) = (W-2)^3$ we get $U'(W) = 3 \cdot (W-2)^2$ and consequently

$$\operatorname{cov}[U'(z(0)\cdot r_P), r_f] = 3\cdot z(0)^2\cdot \operatorname{cov}[r_P^2, r_f] - 12\cdot z(0)\cdot \operatorname{cov}[r_P, r_f]. \tag{A.J2}$$

In addition we have

$$\begin{aligned}
&\operatorname{cov}[r_P^2, r_f]\\
&= E[(r_f - E[r_f])\cdot(r_P^2 - E[r_P^2])]\\
&= E[(r_f - E[r_f])\cdot(r_P^2 - 2\cdot r_P\cdot E[r_P] + E^2[r_P])]\\
&\quad + E[(r_f - E[r_f])\cdot(2\cdot r_P\cdot E[r_P] - E^2[r_P])] - E[r_P^2]\cdot \underbrace{E[r_f - E[r_f]]}_{=0}\\
&= E[(r_f - E[r_f])\cdot(r_P - E[r_P])^2] + 2\cdot E[r_P]\cdot E[(r_f - E[r_f])\cdot(r_P - E[r_P])]\\
&= E[(r_f - E[r_f])\cdot(r_P - E[r_P])^2] + 2\cdot E[r_P]\cdot \operatorname{cov}[r_f, r_P].
\end{aligned} \tag{A.J3}$$

Summarizing, for $U(W) = (W-2)^3$ (A.J1) is equivalent to

$$\begin{aligned}
&\frac{3\cdot z(0)^2\cdot(E[(r_g - E[r_g])\cdot(r_P - E[r_P])^2] + 2\cdot E[r_P]\cdot \operatorname{cov}[r_g, r_P]) - 12\cdot z(0)\cdot \operatorname{cov}[r_P, r_g]}{E[r_g]}\\
&> \frac{3\cdot z(0)^2\cdot(E[(r_h - E[r_h])\cdot(r_P - E[r_P])^2] + 2\cdot E[r_P]\cdot \operatorname{cov}[r_h, r_P]) - 12\cdot z(0)\cdot \operatorname{cov}[r_P, r_h]}{E[r_h]}\\
&\Leftrightarrow \frac{E[(r_g - E[r_g])\cdot(r_P - E[r_P])^2] + 2\cdot\left(E[r_P] - \frac{2}{z(0)}\right)\cdot \operatorname{cov}[r_P, r_g]}{E[r_g]}\\
&> \frac{E[(r_h - E[r_h])\cdot(r_P - E[r_P])^2] + 2\cdot\left(E[r_P] - \frac{2}{z(0)}\right)\cdot \operatorname{cov}[r_P, r_h]}{E[r_h]}.
\end{aligned} \tag{A.J4}$$

Appendix K: Derivation of (22), (23), and (24)

The derivation of $GSM(y,z) = E[U(z\cdot(y\cdot r_f + (1-y)\cdot r_P))]$ with respect to z for given value y leads to the following first-order necessary condition for z(y):

$$E[U'(z\cdot(y\cdot r_f + (1-y)\cdot r_P))\cdot(y\cdot r_f + (1-y)\cdot r_P)] = 0. \tag{A.K1}$$

The parameter z(0) is thus determined by the implicit definition (A.K1) for the special case y = 0. With $y = 0$ and $U(W) = (W-2)^3$ formula (A.K1) becomes:

$$\begin{aligned}
& E[U'(z(0)\cdot r_P)\cdot r_P]=0\\
\Leftrightarrow\ & E[(z(0)\cdot r_P-2)^2\cdot r_P]=0\\
\Leftrightarrow\ & z^2(0)\cdot E[r_P^3]-4\cdot z(0)\cdot E[r_P^2]+4\cdot E[r_P]\ =\ 0\\
\Leftrightarrow\ & z^{\pm}(0)=2\cdot\frac{E[r_P^2]\pm\sqrt{E^2[r_P^2]-E[r_P]\cdot E[r_P^3]}}{E[r_P^3]}.
\end{aligned}\tag{A.K2}$$

Since $z(0)$ characterizes a maximum of $E[U(z\cdot r_P)]$ for give value y = 0, the second-order condition for $z(0)$ is

$$\begin{aligned}
& E[U''(z(0)\cdot r_P)\cdot r_P^2]\le 0\\
\Leftrightarrow\ & E[(z(0)\cdot r_P-2)\cdot r_P^2]\le 0\\
\Leftrightarrow\ & z(0)\cdot E[r_P^3]-2\cdot E[r_P^2]\le 0\\
\Leftrightarrow\ & z(0)\begin{cases}\le\ 2\cdot\dfrac{E[r_P^2]}{E[r_P^3]}, & \text{if}\quad E[r_P^3]>0,\\[2ex] \ge\ 2\cdot\dfrac{E[r_P^2]}{E[r_P^3]}, & \text{if}\quad E[r_P^3]<0.\end{cases}
\end{aligned}\tag{A.K3}$$

Because of the requirement (A.K3), $z(0)$ is obviously always identical to $z^-(0)$ in (A.K2). With $E[r_P^2]=var[r_P]+E^2[r_P]$ and (A.J3) we get $E[r_P^3]=cov[r_P^2,r_P]+E[r_P^2]\cdot E[r_P]$ $=E[(r_P-E[r_P])^3]+3\cdot E[r_P]\cdot var[r_P]+E^3[r_P]$. Now using the market portfolio M as reference portfolio P leads to the postulated results.

Appendix L: Determination of small, medium-level and great values of an investor's risk tolerance and computation of relative certainty equivalents according to Table 3

First of all we have to analyze the connection between parameter a in the quadratic case (a_{quadr}) and parameter a in the cubic case (a_{cub}). If we look at cubic preferences we are allowed to consider the transformed cubic utility function

$$\begin{aligned}
& 1.5\cdot U_{(a_{cub},-0.5)}(W_1)=-(a_{cub}-0.5\cdot W_1)^3\\
& =-a_{cub}^3+3\cdot a_{cub}^2\cdot 0.5\cdot W_1-3\cdot a_{cub}\cdot 0.25\cdot W_1^2+0.125\cdot W_1^3.
\end{aligned}\tag{A.L1}$$

In addition we assume that the considered approximative quadratic utility function will be generated from cubic utility by deleting the cubic terms in (A.L1). Thus, the relevant quadratic utility function has the form

$$\begin{aligned} U(W_1) &= 3\cdot a_{cub}^2\cdot 0.5\cdot W_1 - 3\cdot a_{cub}\cdot 0.25\cdot W_1^2 \\ &= -0.75\cdot a_{cub}\cdot[(a_{cub}-W_1)^2 - a_{cub}^2]. \end{aligned} \qquad \text{(A.L2)}$$

Because of the cardinality of utility functions we are able to substitute the utility function (A.L2) by $U(W_1) = -(a_{cub}-W_1)^2$ which is a quadratic HARA utility function with a_{quadr} = a_{cub}. Thus we have to identify a_{cub} with a_{quadr} = a.

For given value of y according to the decision rule under consideration denote z_{quad} as the optimal value for z in the case of quadratic utility. In addition, define z_{cub} as the corresponding value for z in the case of cubic HARA utility which would lead to the same riskless investment as for z_{quad} and quadratic utility. From (5) we know

$$z_{quad} = ((1-x)\cdot W_0)/(a_{quad} - W_0\cdot(1+R)),\ z_{cub} = ((1-x)\cdot W_0)/(a_{cub} - 0.5\cdot W_0\cdot(1+R)). \quad \text{(A.L3)}$$

Consequentially, the relationship between z_{quad} and z_{cub} can be described in the following way:

$$\begin{aligned} &\frac{z_{cub}}{z_{quad}} = \left(\frac{1-x}{(a/W_0)-0.5\cdot(1+R)}\right)\Big/\left(\frac{1-x}{(a/W_0)-(1+R)}\right) \\ \Leftrightarrow\ &z_{cub} = 2\cdot z_{quad}\cdot\left(\frac{(a/W_0)-(1+R)}{2\cdot(a/W_0)-(1+R)}\right). \end{aligned} \qquad \text{(A.L4)}$$

From this we get $z_{cub} = 2\cdot z_{quad}$ for $a/W_0 = 0$ and $z_{cub} = z_{quad}$ for $a/W_0 \to \infty$. However, we have to allow for the restriction $a/W_0 > 1+R$ so that the term in brackets on the right-hand side of (A.L4) can only obtain values between 0 and 0.5 and is monotone increasing in a/W_0. As $a/W_0 \to 1+R$ implies the optimality of sole riskless lending, we consider − somewhat arbitrarily − situations with $z_{cub} = 2 \cdot z_{quad} \cdot 0.05 = 0.1 \cdot z_{quad}$, $z_{cub} = 2 \cdot z_{quad} \cdot 0.25 = 0.5 \cdot z_{quad}$ and $z_{cub} = 2 \cdot z_{quad} \cdot 0.45 = 0.9 \cdot z_{quad}$. Thereby, we implicitly define risk tolerance measured by a/W_0 for given value of R. The greater z_{cub}/z_{quad}, the greater is ceteris paribus an investor's risk tolerance. Based on y and z_{cub} we are then able to compute the resulting investor's certainty equivalent according to the transformed utility function (AL.1). We come to relative certainty equivalents according to Table 3 by dividing the certainty equivalents resulting from the approximative solution y and z_{cub} by maximum achievable certainty equivalents when directly maximizing expected cubic (HARA) utility (AL.1). As we consider 25 periods of fund selection, all values of Table 3 are average results based on twenty-five different relative certainty equivalents.

Endnotes (relating to the appendices)

[1] See e.g. Ingersoll, J. E., Jr., (1987), *Theory of Financial Decision Making* (Rowman and Littlefield Publishers Inc., Maryland), p. 3.

[2] See again e.g. Ingersoll, (1987), p. 65.

[3] See Grinblatt and Titman(1989), p. 407 in connection with formula (11) on p. 405.

[4] The Taylor series expansion is exact because all derivatives of the utility function of fourth or higher order are zero.

[5] The first equality can be checked with a Software like Mathematica or Maple.

[6] Again, as in connection with formula (15) (see endnote 1 of our paper), we use an equivalent utility function $U(.) = 12 \cdot U_{(1,-0.5)}(.)$.

Table 1

Average ranking positions of German funds according to several performance measures

Ranking positions are presented for different performance measures. Performance measures under consideration comprise the Sharpe, Treynor, and Jensen measure for quadratic as well as cubic (HARA) utility, the quadratic Treynor/Black measure (TB) and the optimized quadratic or cubic performance measure (SM**) in the case of short sales restrictions. Top ten funds are shaded.

		quadratic utility					cubic utility			
	fund name	**Sharpe**	**Treynor**	**Jensen**	**TB**	**SM****	**Sharpe**	**Treynor**	**Jensen**	**SM****
1	Aberdeen Global German Eq	46	46	46	44	46	46	46	46	46
2	AC Deutschland	27	24	24	24	24	26	24	24	24
3	ADIFONDS	21	22	22	22	22	20	21	22	21
4	Baer Multistock German Stk A	6	6	8	8	6	6	6	6	6
5	Baring German Growth	11	4	2	9	7	9	4	3	7
6	BBV Invest Union	7	9	9	7	8	7	9	9	8
7	CB Lux Portaolio Euro Aktien	35	38	38	40	38	33	37	38	38
8	Concentra	26	27	28	29	28	23	27	27	27
9	CS EF (Lux) Germany	39	40	40	38	40	39	40	40	40
10	DekaFonds	30	33	33	34	33	31	31	32	32
11	DELBRÜCK Aktien UNION-Fonds	45	45	45	45	45	45	45	45	45
12	Dexia Eq L Allemagne C	40	41	41	41	41	40	41	40	40
13	DIT Fonds für Vermögensbildung	31	28	27	26	27	29	28	28	28
14	DIT Wachstumfonds	5	8	7	4	5	5	8	6	5
15	DVG Fonds SELECT INVEST	8	10	10	5	9	8	10	10	9
16	DWS Deutschland	13	14	14	14	14	13	14	14	14
17	EMIF Germany Index plus B	43	44	44	46	44	43	44	44	44
18	Fidelity Fds Germany	22	21	21	21	21	22	22	21	22
19	Flex Fonds	44	43	43	43	43	44	43	43	43
20	Frankfurter Sparinvest Deka	24	25	25	25	25	24	25	25	25
21	FT Deutschland Dynamik Fonds	18	18	18	17	17	18	17	17	17
22	Gerling Deutschland Fonds	38	32	29	30	30	38	34	30	30
23	HANSAeffekt	25	26	26	27	26	27	26	26	26
24	Hauck Main I Universal Fonds	34	34	32	33	32	34	32	33	33
25	Incofonds	1	1	1	2	1	1	1	1	1
26	Interselex Equity Germany B	29	30	31	36	31	30	30	31	31
27	INVESCO GT German Growth C	14	2	4	13	10	15	3	8	10
28	Investa	12	13	13	11	13	12	13	13	13
29	Köln Aktienfonds DEKA	23	23	23	23	23	25	23	23	23
30	Lux Linea	33	30	35	32	34	36	32	36	36
31	Metallbank Aktienfonds DWS	4	3	6	6	4	4	2	4	4
32	MK Alfakapital	36	36	39	35	39	35	36	39	39
33	MMWI PROGRESS Fonds	28	29	29	30	29	28	29	29	29
34	Oppenheim Select	37	39	37	37	37	37	39	37	37
35	Parvest Germany C	20	19	19	20	19	21	20	20	20
36	Plusfonds	10	12	11	12	12	11	11	11	11
37	Portfolio Partner Universal G	41	37	34	28	36	41	38	34	35
38	Ring Aktienfonds DWS	9	11	12	10	11	10	12	12	12
39	SMH Special UBS Fonds 1	16	16	16	15	16	14	15	15	15
40	Thesaurus	31	35	36	39	35	32	35	35	34
41	Trinkaus Capital Fonds INKA	14	15	15	16	15	16	16	16	16
42	UniFonds	19	17	17	18	18	19	18	18	18
43	Universal Effect Fonds	42	42	42	42	42	42	42	42	42
44	VERI VALEUR Fonds	3	5	3	3	3	2	5	2	2
45	VICTORIA Eurokapital	2	7	5	1	2	3	7	5	3
P	DAX XETRA 100	17	20	20	19	20	17	19	19	19

Table 2

Ranking correlation coefficients between various performance measures for German, British, and French funds (top ten funds)

Performance measures under consideration comprise the Sharpe, Treynor, and Jensen measure for quadratic as well as cubic (HARA) utility, the quadratic Treynor/Black measure (TB) and the optimized quadratic or cubic performance measure (SM^{**}) in the case of short sales restrictions. Moreover, funds are ranked according to historical average excess return μ.

		linear utility	quadratic utility					cubic utility			
		μ	Sharpe	Treynor	Jensen	TB	SM**	Sharpe	Treynor	Jensen	SM**
German funds											
linear utility	μ	100.00%	53.37%	10.30%	41.24%	49.09%	41.82%	52.73%	20.00%	43.03%	45.54%
quadratic utility	Sharpe	53.37%	100.00%	33.96%	62.38%	83.69%	95.82%	95.82%	40.02%	54.58%	91.61%
	Treynor	10.30%	33.96%	100.00%	88.54%	-6.67%	55.15%	43.03%	98.79%	89.09%	63.52%
	Jensen	41.24%	62.38%	88.54%	100.00%	27.90%	77.62%	66.71%	90.96%	97.03%	83.22%
	TB	49.09%	83.69%	-6.67%	27.90%	100.00%	73.33%	74.55%	-4.24%	18.79%	62.32%
	SM**	41.82%	95.82%	55.15%	77.62%	73.33%	100.00%	92.73%	58.79%	68.48%	95.87%
cubic utility	Sharpe	52.73%	95.82%	43.03%	66.71%	74.55%	92.73%	100.00%	49.09%	62.42%	95.87%
	Treynor	20.00%	40.02%	98.79%	90.96%	-4.24%	58.79%	49.09%	100.00%	91.52%	68.31%
	Jensen	43.03%	54.58%	89.09%	97.03%	18.79%	68.48%	62.42%	91.52%	100.00%	79.09%
	SM**	45.54%	91.61%	63.52%	83.22%	62.32%	95.87%	95.87%	68.31%	79.09%	100.00%
British funds											
linear utility	μ	100.00%	26.06%	20.87%	20.87%	10.30%	4.36%	4.91%	-16.58%	-16.36%	-3.03%
quadratic utility	Sharpe	26.06%	100.00%	13.51%	13.51%	40.61%	59.18%	94.54%	43.82%	39.39%	62.42%
	Treynor	20.87%	13.51%	100.00%	100.00%	78.58%	77.99%	0.50%	66.47%	71.22%	62.62%
	Jensen	20.87%	13.51%	100.00%	100.00%	78.58%	77.99%	0.50%	66.47%	71.22%	62.62%
	TB	10.30%	40.61%	78.58%	78.58%	100.00%	86.58%	22.10%	71.07%	70.91%	72.12%
	SM**	4.36%	59.18%	77.99%	77.99%	86.58%	100.00%	50.23%	83.75%	82.85%	92.81%
cubic utility	Sharpe	4.91%	94.54%	0.50%	0.50%	22.10%	50.23%	100.00%	44.87%	40.52%	58.94%
	Treynor	-16.58%	43.82%	66.47%	66.47%	71.07%	83.75%	44.87%	100.00%	99.49%	91.20%
	Jensen	-16.36%	39.39%	71.22%	71.22%	70.91%	82.85%	40.52%	99.49%	100.00%	89.09%
	SM**	-3.03%	62.42%	62.62%	62.62%	72.12%	92.81%	58.94%	91.20%	89.09%	100.00%
French funds											
linear utility	μ	100.00%	26.06%	29.70%	41.82%	28.48%	28.48%	45.54%	34.75%	44.24%	36.97%
quadratic utility	Sharpe	26.06%	100.00%	64.85%	67.27%	86.67%	86.67%	88.68%	68.31%	68.48%	75.76%
	Treynor	29.70%	64.85%	100.00%	97.58%	89.09%	89.09%	88.68%	98.27%	96.36%	95.15%
	Jensen	41.82%	67.27%	97.58%	100.00%	91.52%	91.52%	91.08%	97.07%	98.79%	97.58%
	TB	28.48%	86.67%	89.09%	91.52%	100.00%	100.00%	95.87%	92.28%	92.73%	96.36%
	SM**	28.48%	86.67%	89.09%	91.52%	100.00%	100.00%	95.87%	92.28%	92.73%	96.36%
cubic utility	Sharpe	45.54%	88.68%	88.68%	91.08%	95.87%	95.87%	100.00%	91.71%	92.28%	94.67%
	Treynor	34.75%	68.31%	98.27%	97.07%	92.28%	92.28%	91.71%	100.00%	98.27%	97.07%
	Jensen	44.24%	68.48%	96.36%	98.79%	92.73%	92.73%	92.28%	98.27%	100.00%	98.79%
	SM**	36.97%	75.76%	95.15%	97.58%	96.36%	96.36%	94.67%	97.07%	98.79%	100.00%

Table 3

Average relative certainty equivalents attainable by the application of different performance measures when utility is actually cubic and of the HARA type

Attainable average fractions of maximum certainty equivalents for an investor with cubic HARA utility are displayed for the case that the investor deviates from the best portfolio consisting of a fund f, the reference portfolio P and riskless lending or borrowing implied by cubic HARA utility. Deviations are caused by the application of another performance measure than the optimized cubic performance measure CSM^{**} (identical to SM^{**} in the line of Table 3 belonging to cubic utility) and comprise the selection of other funds than the best one and the suboptimal combination of this fund with reference portfolio P and riskless lending and borrowing. Moreover, the extent of the investor's welfare loss caused by linear or quadratic utility approximation depends on the investor's risk tolerance.

	Investor's risk tolerance	**low**	**medium**	**high**
linear utility	**μ**	15.145%	62.167%	86.758%
quadratic utility	**Sharpe**	16.511%	68.006%	95.110%
	Treynor	11.032%	44.660%	60.448%
	Jensen	11.032%	44.660%	60.448%
	SM^{**}	16.536%	68.095%	95.093%
cubic utility	**Sharpe**		99.997%	
	Treynor		61.800%	
	Jensen		61.800%	
	SM^{**}		100.000%	