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Abstract: In this paper it is shown that a well known procedure (GEKS) of transitivizing a bilateral system of productivity comparisons is implicitly a way of imposing a homothetic structure onto the data. The main implication of this result is that deviations between the bilateral and the multilateral (GEKS) indexes can be interpreted as a measure of local deviation from the homothetic assumption. This establishes an additional link between homotheticity and transitivity.

Keywords: multilateral comparisons, productivity, EKS, homotheticity, transitivity
JEL Classification: D24

1. Introduction

Commonly used index numbers for quantity and price comparisons do not satisfy the transitivity property. The building of a multilateral system of comparison is a quite demanding task as testified by the number of impossibility theorems that relate to transitivity (see Balk, 2008 and Veelen, 2002, 2009). Since the seminal work of Caves et al (1982) a number of ways of transitivizing bilateral index numbers have been proposed (for a review see Balk, 2006, 2008, 2009). Among these the so called Gini-Elteto-Koves-Szulc (GEKS) procedure has come to prominence in the last round (2005) of the international comparison program (see Prasada Rao, 2009). The GEKS procedure has been also proposed by Forsund (2002) in order to transitivize the Malmquist productivity index in a primal production setting. Despite its spread use, an economic interpretation (from a theoretical perspective) of the procedure is still missing. In fact, the GEKS is justified on the ground of its least squares statistical properties (Rao and Banerjee, 1986), its axiomatic properties (Balk, 2009) and for its capacity of building-up superlative index numbers (Fox, 2003).

It has become usual practice to talk about transitivity and circularity as the same property. However, as pointed out by Balk and Althin (1996), strictly speaking transitivity and circularity are different properties. Being $\mathrm{I}(\mathrm{A},\mathrm{B})$ an index comparing situation A with situation B, transitivity requires:

$$\mathrm{I}(\mathrm{A},\mathrm{B})\cdot\mathrm{I}(\mathrm{B},\mathrm{C})=\mathrm{I}(\mathrm{A},\mathrm{C})$$

Now, transitivity implies that $\mathrm{I}(\mathrm{A},\mathrm{B})\cdot\mathrm{I}(\mathrm{B},\mathrm{A})=\mathrm{I}(\mathrm{A},\mathrm{A})$ that is a weaker form of the time reversal property. Multiplying both terms by $\mathrm{I}(\mathrm{C},\mathrm{A})$ gives:

$$\mathrm{I}(\mathrm{A},\mathrm{B})\cdot\mathrm{I}(\mathrm{B},\mathrm{C})\cdot\mathrm{I}(\mathrm{C},\mathrm{A})=\mathrm{I}(\mathrm{A},\mathrm{C})\cdot\mathrm{I}(\mathrm{C},\mathrm{A})=\mathrm{I}(\mathrm{A},\mathrm{A})$$

This is the definition of circularity if and only if the index satisfies the identity property $\mathrm{I}(\mathrm{A},\mathrm{A})=1$:

$$\mathrm{I}(\mathrm{A},\mathrm{B})\cdot\mathrm{I}(\mathrm{B},\mathrm{C})\cdot\mathrm{I}(\mathrm{C},\mathrm{A})=1$$

Therefore circularity and transitivity are different notions (unless identity is satisfied). A quite famous example of an index number that is transitive but not circular is, in fact, the GEKS transitive index number. In this paper the word transitivity is not used interchangeably with the word circularity.

In a general production context the definition of a multilateral input, output and productivity measure corresponds to the transitive comparison of K firms facing K different technologies. In this paper it is shown that the GEKS procedure corresponds to a peculiar way of imposing a homothetic structure onto an otherwise non-homothetic technology. In section 2 bilateral Malmquist quantity indexes are introduced and it is shown that these are transitive if and only if the technology is (input and output) homothetic. Section 3 is dedicated to show two main procedures for imposing homotheticity of the technology. It is shown that these two procedures are the underpinning of the GEKS procedure. It follows that deviations between the bilateral and the multilateral (GEKS) indexes are due to an underlying deviation between the actual technology and the enlarged homothetic technology.

2. Technology

Let consider a production process that produces $\mathbf{y} \in R_+^M$ outputs by means of $\mathbf{x} \in R_+^N$ inputs. The production set or technology set is the set of all the feasible production plans of a technology t: $T(t) = \{(\mathbf{x}, \mathbf{y}) \in R_+^N x R_+^M : \mathbf{x} \text{ can produce } \mathbf{y} \text{ with techno}\ell\text{ogy } t\}$. It follows that a given production plan $(\mathbf{x}, \mathbf{y})$ can be feasible with a technology $T(t)$ but unfeasible with another technology $T(q)$. The output set is the collection of all the output vectors producible by a given input quantity vector $P(\mathbf{x}, t) = \{\mathbf{y} \in R_+^M : (\mathbf{x}, \mathbf{y}) \in T(t)\}$ and can be represented in a functional form by the output distance function:

$$D_o(\mathbf{x}, \mathbf{y}, t) = \min\left\{\theta > 0 : \left(\mathbf{y}/\theta\right) \in P(\mathbf{x}, t)\right\}$$

The boundary of the output set is called output isoquant and it represents all the weakly efficient output vectors for a given input vector and a given technology:

$$\text{Isoq}(\mathbf{x}, t) = \left\{\mathbf{y} \in P(\mathbf{x}, t) : \left(\mathbf{y}/\theta\right) \notin P(\mathbf{x}, t), \quad 0 < \theta < 1\right\}$$

The input set is the collection of all the input vectors able to produce a given output quantity vector $L(\mathbf{y}, t) = \{\mathbf{x} \in R_+^N : (\mathbf{x}, \mathbf{y}) \in T(t)\}$ and can be represented in a functional form by the input distance function:

$$D_i(\mathbf{x}, \mathbf{y}, t) = \max\left\{\lambda > 0 : \left(\mathbf{x}/\lambda\right) \in L(\mathbf{y}, t)\right\}$$

The boundary of the input set is called the input isoquant and it represents the set of weakly efficient input vectors:

$$\mathrm{Isoq}(\mathbf{y},t)=\left\{\mathbf{x}\in L(\mathbf{y},t):\left(\mathbf{x}/\theta\right)\notin L(\mathbf{y},t),\quad \theta>1\right\}$$

The technology satisfies input and output homotheticity if (respectively):

$$D_i(\mathbf{x},\mathbf{y},t)=\frac{1}{H(\mathbf{y})}D_i(\mathbf{x},1,t)$$

$$D_o(\mathbf{x},\mathbf{y},t)=\frac{1}{G(\mathbf{x})}D_o(1,\mathbf{y},t)$$

where $H(\mathbf{y})$ and $G(\mathbf{x})$ are consistent with the technology axioms. Output Hicks neutrality is a form of separability (homotheticity) between time and the output vector and is defined as:

$$D_o(\mathbf{x},\mathbf{y},t)=A(\mathbf{x},t)D_o(\mathbf{x},\mathbf{y},1)$$

Input Hicks neutrality is a form of separability (homotheticity) between time and the input vector and is defined as:

$$D_i(\mathbf{x},\mathbf{y},t)=B(\mathbf{y},t)D_i(\mathbf{x},\mathbf{y},1)$$

Global Hicks neutrality implies separability of time from the input and the output vector and can be defined both in terms of the output distance function ($D_o(\mathbf{x},\mathbf{y},t)=A(t)D_o(\mathbf{x},\mathbf{y},1)$) or in terms of the input distance function ($D_i(\mathbf{x},\mathbf{y},t)=B(t)D_i(\mathbf{x},\mathbf{y},1)$). It is important to emphasize that Hicks neutrality is a form of homotheticity since involve the separability of t from the input-output vectors. In other words, Hicks homotheticity implies that technological differential between pairs of observations are neutra.

Definition 1: a technology that is output homothetic and output Hicks neutral is called output Hicks homothetic.

Definition 2: a technology that is input homothetic and input Hicks neutral is called input Hicks homothetic.

From these two definitions it follows that a technology satisfying joint output Hicks homotheticity and input Hicks homotheticity is inversely homothetic (Fare and Primont, 1995) and Hicks neutral, thus inversely Hicks homothetic.

3. Bilateral indexes

Bilateral input and output quantity comparisons between $\left(\mathbf{x}^t, \mathbf{y}^t, t\right)$ and $\left(\mathbf{x}^q, \mathbf{y}^q, q\right)$ may be done using the Malmquist quantity indexes. The base period Malmquist output quantity index is defined as:

$$Y^L(t,q) = \frac{D_o\left(\mathbf{x}^t, \mathbf{y}^q, t\right)}{D_o\left(\mathbf{x}^t, \mathbf{y}^t, t\right)}$$

Since this index of output quantity change fixes the input vector and the technology to the base period value, it follows a Laspeyres logic and it will be called the "Laspeyres-Malmquist" output quantity index. The comparison period Malmquist output quantity index is defined as:

$$Y^P(t,q) = \frac{D_o\left(\mathbf{x}^q, \mathbf{y}^q, q\right)}{D_o\left(\mathbf{x}^q, \mathbf{y}^t, q\right)}$$

Since this index fixes the output vector and the technology to the comparison period value, it responds to the logic of a "Paasche-Malmquist" index. One can also take the geometric average of these two indexes obtaining a "Fisher-Malmquist" output quantity index:

$$Y^F(t,q) = \left[Y^L(t,q) Y^P(t,q)\right]^{\frac{1}{2}}$$

Similar indexes can be defined for input quantity change using the input distance function. The Laspeyres-Malmquist input quantity index fixes the output vector and the technology to the base period value and is defined as:

$$X^L(t,q) = \frac{D_i\left(\mathbf{x}^q, \mathbf{y}^t, t\right)}{D_i\left(\mathbf{x}^t, \mathbf{y}^t, t\right)}$$

The Paasche-Malmquist input quantity index fixes the technology and the output vector to the comparison period values:

$$X^P(t,q) = \frac{D_i\left(\mathbf{x}^q, \mathbf{y}^q, q\right)}{D_i\left(\mathbf{x}^t, \mathbf{y}^q, q\right)}$$

Finally, taking the geometric mean of the previous two indexes, one obtains the Fisher-Malmquist input quantity change index:

$$X^F(t,q) = \left[X^L(t,q)X^P(t,q)\right]^{\frac{1}{2}}$$

Finally, associated to the Laspeyres-, Paasche- and Fisher-Malmquist input and output quantity indexes, it is possible to define three total factor productivity indexes (TFP). The ratio of the Laspeyres-Malmquist output index to the Laspeyres-Malmquist input index returns the Laspeyres-Hicks Moorsteen productivity index:

$$TFP^L(t,q) = \frac{Y^L(t,q)}{X^L(t,q)}$$

The ratio of the Paasche-Malmquist output index to the Paasche-Malmquist input index returns the Paasche-Hicks Moorsteen productivity index:

$$TFP^P(t,q) = \frac{Y^P(t,q)}{X^P(t,q)}$$

Finally, the ratio of the Fisher-Malmquist output index to the Fisher-Malmquist input index returns the Fisher-Hicks Moorsteen productivity index:

$$TFP^F(t,q) = \frac{Y^F(t,q)}{X^F(t,q)}$$

The input and output quantity indexes date back at least to Caves et al. (1982) and Diewert (1992). The TFP indexes above were explicitly discussed by Bjurek (1996) and named as Hicks-Moorsteen productivity indexes by Fare et al (1996). All these Malmquist (input, output and productivity) quantity indexes do not satisfy transitivity. Transitivity has been regarded as an important property in cross-country quantity comparisons where a natural ordering of the observations does not exist. More recently Daskovska et al (2010) showed that also in time series settings transitivity is a desirable requirement at the purpose of forecasting.

Proposition 1: The bilateral Laspeyres-Malmquist productivity index is transitive if and only if the technology is inversely Hicks homothetic.

Proof: To show that inverse Hicks homotheticity is sufficient for transitivity is easy. To show it is necessary, let start with the definition of transitivity:

$$\frac{D_o(\mathbf{x}^t,\mathbf{y}^q,t)}{D_o(\mathbf{x}^t,\mathbf{y}^t,t)}\frac{D_o(\mathbf{x}^q,\mathbf{y}^z,q)}{D_o(\mathbf{x}^q,\mathbf{y}^q,q)}\frac{D_i(\mathbf{x}^t,\mathbf{y}^t,t)}{D_i(\mathbf{x}^q,\mathbf{y}^t,t)}\frac{D_i(\mathbf{x}^q,\mathbf{y}^q,q)}{D_i(\mathbf{x}^z,\mathbf{y}^q,q)}=\frac{D_o(\mathbf{x}^t,\mathbf{y}^z,t)}{D_o(\mathbf{x}^t,\mathbf{y}^t,t)}\frac{D_i(\mathbf{x}^t,\mathbf{y}^t,t)}{D_i(\mathbf{x}^z,\mathbf{y}^t,t)}$$

$$\frac{D_o(\mathbf{x}^t,\mathbf{y}^q,t)}{D_o(\mathbf{x}^t,\mathbf{y}^z,t)}\frac{D_o(\mathbf{x}^q,\mathbf{y}^z,q)}{D_o(\mathbf{x}^q,\mathbf{y}^q,q)}\frac{D_i(\mathbf{x}^z,\mathbf{y}^t,t)}{D_i(\mathbf{x}^q,\mathbf{y}^t,t)}\frac{D_i(\mathbf{x}^q,\mathbf{y}^q,q)}{D_i(\mathbf{x}^z,\mathbf{y}^q,q)}=1$$

This must be true for any $\mathbf{y}^z$, then consider $\mathbf{y}^z \neq \mathbf{y}^{z'}$:

$$\frac{D_o(\mathbf{x}^t,\mathbf{y}^q,t)}{D_o(\mathbf{x}^t,\mathbf{y}^{z'},t)}\frac{D_o(\mathbf{x}^q,\mathbf{y}^{z'},q)}{D_o(\mathbf{x}^q,\mathbf{y}^q,q)}\frac{D_i(\mathbf{x}^z,\mathbf{y}^t,t)}{D_i(\mathbf{x}^q,\mathbf{y}^t,t)}\frac{D_i(\mathbf{x}^q,\mathbf{y}^q,q)}{D_i(\mathbf{x}^z,\mathbf{y}^q,q)}=1$$

Taking the two expressions together:

$$\frac{D_o(\mathbf{x}^q,\mathbf{y}^{z'},q)}{D_o(\mathbf{x}^t,\mathbf{y}^{z'},t)}=\frac{D_o(\mathbf{x}^q,\mathbf{y}^z,q)}{D_o(\mathbf{x}^t,\mathbf{y}^z,t)}$$

That defines output Hicks homotheticity. The equation must hold also for any $\mathbf{x}^{z'} \neq \mathbf{x}^z$, then

$$\frac{D_o(\mathbf{x}^t,\mathbf{y}^q,t)}{D_o(\mathbf{x}^t,\mathbf{y}^z,t)}\frac{D_o(\mathbf{x}^q,\mathbf{y}^z,q)}{D_o(\mathbf{x}^q,\mathbf{y}^q,q)}\frac{D_i(\mathbf{x}^{z'},\mathbf{y}^t,t)}{D_i(\mathbf{x}^q,\mathbf{y}^t,t)}\frac{D_i(\mathbf{x}^q,\mathbf{y}^q,q)}{D_i(\mathbf{x}^{z'},\mathbf{y}^q,q)}=1$$

Taking the last two expressions together gives:

$$\frac{D_i(\mathbf{x}^{z'},\mathbf{y}^t,t)}{D_i(\mathbf{x}^{z'},\mathbf{y}^q,q)}=\frac{D_i(\mathbf{x}^z,\mathbf{y}^t,t)}{D_i(\mathbf{x}^z,\mathbf{y}^q,q)}$$

That is the definition of input Hicks homotheticity. Joint input and output Hicks homotheticity imply inverse Hicks homotheticity.

□

Proposition 2: The bilateral Paasche-Malmquist productivity index is transitive if and only if the technology is inversely Hicks homothetic.

Proof: The proof is similar to proposition 1 and is here omitted for reasons of space.

□

As a corollary to the previous two propositions it is easy to verify that homotheticity is sufficient and necessary also for the Laspeyres-Malmquist and Paasche-Malmquist input and output quantity indexes. It is easy to verify that homotheticity is only sufficient and not necessary for the Fisher type indexes to be transitive. In fact, due to its geometric mean nature, it could happen that Fisher type indexes are numerically transitive with non-homothetic structures.

4. The enlarged homothetic technologies

From the previous discussion it is clear that a way of building transitive index numbers is to impose a homothetic structure onto the data. Indeed this is also necessary in order to get Laspeyres- and Paasche-Malmquist transitive index numbers. For the Fisher type indexes homotheticity is not necessary but still sufficient. Therefore the importance of building homothetic technologies is crucial to any primal input, output and productivity multilateral comparison and it is useful to start by the explicit building of these technologies. A useful insight is given by the building of CRS technologies. For any given actual technology $\mathrm{T}(\mathrm{t})$ it is possible to define a virtual CRS technology as the enlargement of the actual technology:

$$\mathrm{T}_{\mathrm{CRS}}(\mathrm{t})=\{(\lambda\mathbf{x},\lambda\mathbf{y}):(\mathbf{x},\mathbf{y})\in\mathrm{T}(\mathrm{t}),\quad \lambda>0\}$$

It should be noted that this technology, although widely used, is not always well defined, in the sense that under some conditions it can collapse to the all positive orthant. In any case, the main lesson here is that from an actual technology $\mathrm{T}(\mathrm{t})$ it is possible to build a virtual CRS technology $\mathrm{T}_{\mathrm{CRS}}(\mathrm{t})$ as an enlargement of the original technology and this procedure is unique. One may ask if a similar procedure exists in the case of homotheticity, i.e. if it is possible to build a virtual homothetic technology $\mathrm{T}_{\mathrm{hom}}(\mathrm{t})$ starting from an actual technology $\mathrm{T}(\mathrm{t})$.

As a preliminary result let follow Primont and Primont (1994) by defining the following test for output Hicks homotheticity at the K observed data points:

$$\frac{\mathrm{D}_{\mathrm{o}}(\mathbf{x}^{\mathrm{k}},\mathbf{y}^{\mathrm{q}},\mathrm{k})}{\mathrm{D}_{\mathrm{o}}(\mathbf{x}^{\mathrm{k}},\mathbf{y}^{\mathrm{t}},\mathrm{k})}=\frac{\mathrm{D}_{\mathrm{o}}(\mathbf{x}^{\mathrm{j}},\mathbf{y}^{\mathrm{q}},\mathrm{j})}{\mathrm{D}_{\mathrm{o}}(\mathbf{x}^{\mathrm{j}},\mathbf{y}^{\mathrm{t}},\mathrm{j})},\quad \forall \mathrm{k},\mathrm{j},\mathrm{t},\mathrm{q}=1,\ldots,\mathrm{K}$$

This basically means, in the current framework, that for any pair of output quantity vectors the binary comparison is invariant to the choice of the reference technology and the reference input vector (this is stronger than the test in Primont and Primont, 1994). In other words it is possible to find an aggregator function $\mathrm{Y}(\mathbf{y})$ for the output vector that is independent from the input vector and the technology:

$$\frac{D_o(\mathbf{x}^j, \mathbf{y}^q, j)}{D_o(\mathbf{x}^j, \mathbf{y}^t, j)} = \frac{Y(\mathbf{y}^q)}{Y(\mathbf{y}^t)}, \quad \forall j, t, q = 1, \ldots, K$$

As pointed out by Primont and Primont (1994) this is necessary and sufficient for output Hicks homotheticity to hold on all the observed data points. With similar interpretation, the test for global input Hicks homotheticity is:

$$\frac{D_i(\mathbf{x}^q, \mathbf{y}^k, k)}{D_i(\mathbf{x}^t, \mathbf{y}^k, k)} = \frac{D_i(\mathbf{x}^q, \mathbf{y}^j, j)}{D_i(\mathbf{x}^t, \mathbf{y}^j, j)} = \frac{X(\mathbf{x}^q)}{X(\mathbf{x}^t)}, \quad \forall k, j, t, q = 1, \ldots, K$$

There are basically two methods for imposing homotheticity onto the data.

Method 1

For a given set of observations K, Primont and Primont (1994) proposed to impose output homotheticity onto the observed data points as follow. Pick-up an output isoquant $\mathrm{Isoq}(\mathbf{x}^k, k)$ and impose that all the rest of observed output isoquants are parallel to this base isoquant along all the observed possible output rays. This in formulas means that it is possible to associate the following virtual homothetic output distance function $D_{oh}(\mathbf{x}, \mathbf{y}, t)$ to the observed output distance function $D_o(\mathbf{x}, \mathbf{y}, t)$:

$$D_{oh}^k(\mathbf{x}^t, \mathbf{y}^q, t) = D_o(\mathbf{x}^t, \mathbf{y}^t, t)\frac{D_o(\mathbf{x}^k, \mathbf{y}^q, k)}{D_o(\mathbf{x}^k, \mathbf{y}^t, k)}, \quad \forall t, q = 1, \ldots, K$$

where the subscript "h" emphasizes that the virtual distance function is homothetic and superscript 'k' emphasizes that the k-th output isoquant has been chosen as the base one. This virtual homothetic distance function satisfies output Hicks homotheticity at all the observed data points. Moreover it is easy to see that $D_{oh}(\mathbf{x}^t, \mathbf{y}^t, t) = D_o(\mathbf{x}^t, \mathbf{y}^t, t)$; therefore the virtual output homothetic distance function equalizes the actual observed output distance function at all data points, but it differs along hypothetical points such as $\left(\mathbf{x}^{t},\mathbf{y}^{q},t\right)$. In other words, the virtual homothetic output distance function returns the observed technical efficiency of each data point.

A major shortcoming of the Primont and Primont (1994) procedure is its non-invariance to the choice of the base isoquant. Choosing another isoquant, say $\mathrm{Isoq}\left(\mathbf{x}^{j}, j\right)$, would return a different virtual homothetic distance function and this simple fact basically establishes that the procedure for imposing homotheticity onto a set of observations is not unique.

Now, suppose that the choice of the base isoquant has been made; then one may use the virtual homothetic distance function to build a Malmquist output quantity index:

$$\frac{D_{oh}^{k}\left(\mathbf{x}^{t},\mathbf{y}^{q},t\right)}{D_{oh}^{k}\left(\mathbf{x}^{t},\mathbf{y}^{t},t\right)}=Y^{k}(t,q)$$

The output quantity index is now invariant to the choice of the input vector and the reference technology and it is transitive. The invariance derives from the fact that the output distance function we are using is output Hicks homothetic:

$$\frac{D_{oh}^{k}\left(\mathbf{x}^{t},\mathbf{y}^{q},t\right)}{D_{oh}^{k}\left(\mathbf{x}^{t},\mathbf{y}^{t},t\right)}=\frac{D_{oh}^{k}\left(\mathbf{x}^{q},\mathbf{y}^{q},q\right)}{D_{oh}^{k}\left(\mathbf{x}^{q},\mathbf{y}^{t},q\right)}$$

This in turn implies that it is possible to build an aggregator function for the output vector that is independent from the input vector and the reference technology

$$Y^{k}(t,q)=\frac{Y_{oh}^{k}\left(\mathbf{y}^{q}\right)}{Y_{oh}^{k}\left(\mathbf{y}^{t}\right)}$$

Nonetheless the major shortcoming of this method is its dependence on the choice of the base isoquant $\mathrm{Isoq}\left(\mathbf{x}^{k},k\right)$. Therefore, one can obtain as many transitive output quantity indexes as many choices for the base output isoquant.

Method 2

Although the previous procedure is quite appealing it is indeed not the only possibility. An alternative way of building a virtual homothetic output distance function is to choose an output ray $\mathbf{y}^{j}$ along which measuring the distance between the two output isoquants $\mathrm{Isoq}(\mathbf{x}^t, t)$ and $\mathrm{Isoq}(\mathbf{x}^q, q)$. This can be done defining the following virtual homothetic distance function:

$$D_{oh}^{j}(\mathbf{x}^t, \mathbf{y}^q, t) = D_o(\mathbf{x}^q, \mathbf{y}^q, q)\frac{D_o(\mathbf{x}^t, \mathbf{y}^j, t)}{D_o(\mathbf{x}^q, \mathbf{y}^j, q)}, \quad \forall t, q = 1, \ldots, K$$

This virtual homothetic distance function satisfies the definition of output Hicks homotheticity at all the observed data points. An output quantity comparison can now be made by building a Malmquist type index using the virtual homothetic distance function:

$$Y^{j}(t, q) = \frac{D_{oh}^{j}(\mathbf{x}^t, \mathbf{y}^q, t)}{D_{oh}^{j}(\mathbf{x}^t, \mathbf{y}^t, t)} = \frac{Y^P(t, k)}{Y^P(q, k)}$$

This output quantity index is transitive and is invariant to the choice of the input vector and the reference technology:

$$\frac{D_{oh}^{j}(\mathbf{x}^t, \mathbf{y}^q, t)}{D_{oh}^{j}(\mathbf{x}^t, \mathbf{y}^t, t)} = \frac{D_{oh}^{j}(\mathbf{x}^q, \mathbf{y}^q, q)}{D_{oh}^{j}(\mathbf{x}^q, \mathbf{y}^t, q)}$$

As in the previous procedure, imposing homotheticity is here not invariant to the choice of the output ray. Therefore, so far there are two methods of imposing homotheticity and both these methods provide K different ways of imposing homotheticity for a total of 2K possible ways of imposing homotheticity.

5. A revisitation of the GEKS procedure

One standard way of solving the problem of choosing the base isoquant in method 1 is to averaging across all the possibilities. Since there are K transitive output quantity indexes according to method one, the geometric average will return:

$$Y^{PEKS}(t, q) = \left[\prod_k Y^k(t, q)\right]^{\frac{1}{K}}$$

This geometric average index can also be written as:

$$\begin{aligned} Y^{PEKS}(t, q) &= \left[\prod_k Y^k(t, q)\right]^{\frac{1}{K}} = \\ &= \left[\prod_k \frac{D_o(\mathbf{x}^k, \mathbf{y}^q, k)}{D_o(\mathbf{x}^k, \mathbf{y}^t, k)}\frac{D_o(\mathbf{x}^k, \mathbf{y}^k, k)}{D_o(\mathbf{x}^k, \mathbf{y}^k, k)}\right]^{\frac{1}{K}} = \left[\prod_k \frac{Y^P(t, k)}{Y^P(q, k)}\right]^{\frac{1}{K}} \end{aligned}$$

Therefore the geometric mean index is an average of ratios of bilateral Paasche-Malmquist output quantity indexes. Associated to the $Y^{PEKS}(t,q)$ transitive output quantity index it is possible to define the implicit virtual homothetic technology as a geometric mean across all the possible choices of the base isoquant:

$$D_{oh}^{PEKS}(\mathbf{x}^t,\mathbf{y}^q,t) = D_o(\mathbf{x}^t,\mathbf{y}^t,t)\left(\prod_k \frac{D_o(\mathbf{x}^k,\mathbf{y}^q,k)}{D_o(\mathbf{x}^k,\mathbf{y}^t,k)}\right)^{\frac{1}{K}}$$

Since method 2 of imposing homotheticity gives K different transitive output quantity indexes, a geometric mean can be used to avoid the arbitrariness of choosing an output ray $\mathbf{y}^j$:

$$Y^{LEKS}(t,q) = \left(\prod_j Y^j(t,q)\right)^{\frac{1}{K}}$$

This geometric average index can be written as:

$$Y^{LEKS}(t,q) = \left(\prod_j \frac{D_o(\mathbf{x}^q,\mathbf{y}^q,q)}{D_o(\mathbf{x}^t,\mathbf{y}^t,t)}\frac{D_o(\mathbf{x}^t,\mathbf{y}^j,t)}{D_o(\mathbf{x}^q,\mathbf{y}^j,q)}\right)^{\frac{1}{K}} = \left(\prod_j \frac{Y^L(t,j)}{Y^L(q,j)}\right)^{\frac{1}{K}}$$

The last formula is a geometric mean of ratios of bilateral Laspeyres-Malmquist output quantity indexes. Associated to the transitive index $Y^{LEKS}(t,q)$ one can obtain the virtual homothetic distance function as a geometric mean of the underlying virtual distance functions:

$$D_{oh}^{LEKS}(\mathbf{x}^t,\mathbf{y}^q,t) = D_o(\mathbf{x}^q,\mathbf{y}^q,q)\left(\prod_k \frac{D_o(\mathbf{x}^t,\mathbf{y}^k,t)}{D_o(\mathbf{x}^q,\mathbf{y}^k,q)}\right)^{\frac{1}{K}}$$

Now, if the geometric mean of the two methods is taken, the following surprisingly simple result is obtained:

$$\left(\prod_k \frac{Y^P(t,k)}{Y^P(q,k)}\frac{Y^L(t,k)}{Y^L(q,k)}\right)^{\frac{1}{2K}} = \left(\prod_k \frac{Y^F(t,k)}{Y^F(q,k)}\right)^{\frac{1}{K}} = \left(\prod_k Y^F(t,k)Y^F(k,q)\right)^{\frac{1}{K}} = Y^{FEKS}(t,q)$$

Quite interestingly this is the exact definition of a GEKS procedure applied to the Fisher-Malmquist output quantity index. In other words the GEKS procedure of transitivizing Fisher-Malmquist output indexes is implicitly a procedure to impose homotheticity onto an otherwise non-homothetic technology. It follows that deviations between the GEKS Malmquist output quantity index and the bilateral Fisher-Malmquist output quantity index may be interpreted as deviations between the real technology and the virtual homothetic technology imposed onto the data by the GEKS procedure.

Input indexes

With similar passages it is possible to impose input Hicks homotheticity onto the input distance function and obtain the GEKS procedure applied to the Fisher-Malmquist input quantity index. The first method of imposing input Hicks homotheticity returns the following virtual homothetic distance function once the input isoquant $\mathrm{Isoq}(\mathbf{y}^k, k)$ is chosen as reference:

$$D_{ih}^{k}(\mathbf{x}^q, \mathbf{y}^t, t) = D_i(\mathbf{x}^t, \mathbf{y}^t, t)\frac{D_i(\mathbf{x}^q, \mathbf{y}^k, k)}{D_i(\mathbf{x}^t, \mathbf{y}^k, k)}$$

The use of this distance function returns the input quantity index:

$$X^k(t, q) = \frac{D_{ih}^{k}(\mathbf{x}^q, \mathbf{y}^t, t)}{D_{ih}^{k}(\mathbf{x}^t, \mathbf{y}^t, t)}$$

To avoid the arbitrariness of choosing the base isoquant, the geometric mean returns:

$$X^{PEKS}(t, q) = \left[\prod_k X^k(t, q)\right]^{\frac{1}{K}}$$

and its associated distance function:

$$D_{ih}^{PEKS}(\mathbf{x}^q, \mathbf{y}^t, t) = D_i(\mathbf{x}^t, \mathbf{y}^t, t)\left(\prod_k \frac{D_i(\mathbf{x}^q, \mathbf{y}^k, k)}{D_i(\mathbf{x}^t, \mathbf{y}^k, k)}\right)^{\frac{1}{K}}$$

The input index $X^{PEKS}(t, q)$ is a geometric average of ratios of bilateral Paasche-Malmquist input quantity indexes:

$$X^{PEKS}(t,q)=\left[\prod_k \frac{X^P(t,k)}{X^P(q,k)}\right]^{\frac{1}{K}}$$

The second strategy of imposing input Hicks homotheticity will return the following virtual homothetic distance function:

$$D_{ih}^{j}(\mathbf{x}^q,\mathbf{y}^t,t)=D_i(\mathbf{x}^q,\mathbf{y}^q,q)\frac{D_i(\mathbf{x}^j,\mathbf{y}^t,t)}{D_i(\mathbf{x}^j,\mathbf{y}^q,q)}$$

and its associated transitive input quantity index:

$$X^j(t,q)=\frac{D_{ih}^{j}(\mathbf{x}^q,\mathbf{y}^t,t)}{D_{ih}^{j}(\mathbf{x}^t,\mathbf{y}^t,t)}$$

Taking geometric mean:

$$X^{LEKS}(t,q)=\left[\prod_j X^j(t,q)\right]^{\frac{1}{K}}$$

The associated virtual homothetic distance function will be:

$$D_{ih}^{L}(\mathbf{x}^q,\mathbf{y}^t,t)=D_i(\mathbf{x}^t,\mathbf{y}^t,t)\left(\prod_k \frac{D_i(\mathbf{x}^k,\mathbf{y}^t,t)}{D_i(\mathbf{x}^k,\mathbf{y}^q,q)}\frac{D_i(\mathbf{x}^q,\mathbf{y}^q,q)}{D_i(\mathbf{x}^t,\mathbf{y}^t,t)}\right)^{\frac{1}{K}}$$

and it is easy to show that the index can be expressed as a geometric mean of ratios of bilater Paasche-Malmquist input quantity indexes:

$$X^{LEKS}(t,q)=\left(\prod_k \frac{X^L(t,k)}{X^L(q,k)}\right)^{\frac{1}{K}}$$

Taking the geometric average of these two indexes returns:

$$\left(\prod_k \frac{X^P(t,k)}{X^P(q,k)}\frac{X^L(t,k)}{X^L(q,k)}\right)^{\frac{1}{2K}}=\left(\prod_k \frac{X^F(t,k)}{X^F(q,k)}\right)^{\frac{1}{K}}=\left(\prod_k X^F(t,k)X^F(k,q)\right)^{\frac{1}{K}}=X^{FEKS}(t,q)$$

which is a GEKS procedure applied to the bilateral Fisher-Malmquist input quantity index. It follows that the GEKS procedure applied to the bilateral Fisher-Malmquist productivity index will be:

$$TFP^{FEKS}(t,q)=\left[\prod_{k} TFP^{F}(t,k) TFP^{F}(k,q)\right]^{\frac{1}{K}}=\left[\prod_{k}\frac{Y^{F}(t,k)}{X^{F}(t,k)}\frac{Y^{F}(k,q)}{X^{F}(k,q)}\right]^{\frac{1}{K}}=\frac{Y^{FEKS}(t,q)}{X^{FEKS}(t,q)}$$

This procedure is, all in all, a way of imposing an inversely Hicks homothetic structure to an otherwise non-homothetic technology. Therefore deviations between the bilateral and the multilateral productivity index can be interpreted as deviations between the actual technology and the virtual inversely Hicks homothetic technology.

6. Conclusion

This paper showed that the building of a multilateral productivity comparison system is logically equivalent to the construction of a virtual homothetic technology. Since the procedure to impose homotheticity is not unique, there are many ways of getting multilateral productivity comparisons. The GEKS procedure is one of them, where the underlying technology has the nature of geometric mean of primitive technologies. Given these relationship between transitivity and homotheticity it is quite useful to put more research in the building and the use of virtual homothetic distance functions.

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