Balancedness of Permutation Games and Envy-free Allocations in Indivisible Good Economies

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Abstract: We present a simple proof of the balancedness of permutation games. In the proof we use the existence of envy-free allocations in economies with indivisible objects, quasi-linear utility functions, and an amount of money.

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1 Introduction

In this paper we provide an alternative proof for the balancedness (i.e. the non-emptiness of the core) of permutation games as introduced by Tijs *et al.* (1984). For this result some proofs already exist. Tijs *et al.* (1984) obtained the result by using the extreme points theorem of Birkhoff-von Neumann on doubly stochastic matrices, and Curiel and Tijs (1986) by using an equilibrium existence theorem of Gale (1984) for a discrete exchange economy with money.

Our alternative proof is interesting because it relates the core conditions of permutation games with the properties of envy-freeness and Pareto-efficiency in economies with indivisible objects, quasi-linear utility functions, and an amount of money. Moreover, our proof does not rely on deep mathematical theorems. In an economy an allocation is called envy-free (cf. Foley (1967)) if every agent in the economy likes his own bundle at least as well as that of anyone

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else. Alkan *et al.* (1991) and Svensson (1983) showed that in a large class of economies the set of envy-free allocations is non-empty and that it is a subset of the set of Pareto-efficient allocations. Aragones (1995) gave another proof of the existence of envy-free allocations in the particular case of economies with indivisible objects, a fixed amount of money, and agents with quasi-linear utility functions. Finally, Klijn (1999) provided an algorithm consisting of two natural procedures for finding envy-free allocations in quasi-linear economies. To prove that permutation games are balanced, we invoke the existence result on envy-free allocations in quasi-linear economies to construct a core allocation.

2 The balancedness of permutation games

Let us start by recalling the definition of permutation games. After that, we will recall a result on envy-free allocations for our particular class of economies. And finally, we will provide the proof on the balancedness of permutation games by relating the core conditions with the properties of envy-freeness and Pareto-efficiency.

Permutation games, introduced by Tijs *et al.* (1984) (cf. Curiel (1997) and Quint (1996)), describe a situation in which n persons all have one job to be processed and one machine on which each job can be processed. No machine is allowed to process more than one job. Sidepayments between the players are allowed. If player i processes his job on the machine of player j the processing costs are a_{ij}. Let $N := \{1, \ldots, n\}$ be the set of players. The *permutation game* (N, c) with costs a_{ij} is the cooperative TU-game defined by

$$\begin{aligned} c(\emptyset) &:= 0 \quad \text{and} \\ c(S) &:= \min_{\pi_S \in \Pi_S} \sum_{i \in S} a_{i\pi_S(i)} \quad \text{for all } S \in 2^N \backslash \{\emptyset\}, \end{aligned}$$

where Π_S is the class of all S-permutations and 2^N the collection of all subsets of N. The number $c(S)$ denotes the minimum costs for coalition S to process its jobs on its own machines under the restriction that each machine processes exactly one job.

Tijs *et al.* (1984) proved, using the Birkhoff-von Neumann theorem on doubly stochastic matrices, that permutation games are balanced. This means that there is an efficient payoff vector x for which no coalition has an incentive to separate from the grand coalition. Formally, there is a vector $x \in \mathbb{R}^N$ such that $\sum_{i \in S} x_i \leq c(S)$ for all $S \in 2^N \backslash \{\emptyset\}$ and $\sum_{i \in N} x_i = c(N)$. Curiel and Tijs (1986) gave another proof of the balancedness of permutation games. They used an equilibrium existence theorem of Gale (1984) for a discrete exchange economy with money, thereby showing a relation between assignment games (cf. Shapley and Shubik (1972)) and permutation games.

Next, let us turn to the specific result on envy-free allocations that we will need for our proof. Here, an economy is represented by an ordered triple $E = (N, Q, U)$, where $N = \{1, \ldots, n\}$ is a finite set of agents and $Q = \{1, \ldots, n\}$ a set of indivisible objects, and U the utility matrix which will be defined next. Each agent $i \in N$ is assumed to be endowed with a quasi-linear utility function $u_i : Q \times \mathbb{R} \to \mathbb{R}$:

$$u_i(j, x) = u_{ij} + x \qquad (j \in Q, x \in \mathbb{R}),$$

where u_{ij} can be any real number. The number $u_i(j, x)$ is interpreted as the utility that agent $i \in N$ derives when he receives an object $j \in Q$ and an amount of money $x \in \mathbb{R}$. Now, we define the utility matrix U by letting u_{ij} be its ij-th entry ($i \in N$, $j \in Q$). Let $E = (N, Q, U)$ be an economy. A feasible allocation for the economy E is a pair $(\sigma, x) \in \Pi_N \times \mathbb{R}^n$ such that $\sum_{i=1}^{n} x_i = 0$. A feasible allocation (σ, x) gives object $\sigma(i)$ and the amount $x_{\sigma(i)}$ of money to agent i.

We are interested in so called envy-free allocations, which satisfy the following notion of equity: no agent prefers the bundle of any other agent to his own. Formally, a feasible allocation (σ, x) is *envy-free* (cf. Foley (1967)) if

$$u_{i\sigma(i)} + x_{\sigma(i)} \geq u_{i\sigma(j)} + x_{\sigma(j)} \qquad \text{for all } i, j \in N.$$

Another property that is often used in the selection of normatively appealing allocations is Pareto-efficiency. In our model a feasible allocation is Pareto-efficient if and only if there is no other feasible allocation that makes all agents strictly better off. The proof of the next result for a more general class of preferences can be found in Alkan *et al.* (1991) and Svensson (1983). Proofs of the existence of envy-free allocations for quasi-linear economies can be found in Aragones (1995) and Klijn (1999). These proofs are based on elementary mathematics.

Theorem 2.1 *For every economy E, there exists an envy-free allocation. Moreover, all envy-free allocations are Pareto-efficient.*

We are now ready to give our alternative proof of the balancedness of permutation games.

Theorem 2.2 *Permutation games are balanced.*

Proof. Let (N,c) be a permutation game. We prove that (N,c) is balanced. Let a_{ij} be the processing costs that correspond with (N,c). Let U be the utility matrix defined by $u_{ij} := -a_{ij}$, and let E be the economy defined by $E := (N,Q,U)$, where $Q = N = \{1,\ldots,n\}$. From theorem 2.1 it follows that there is an envy-free allocation, say (σ,x). Define

$$y_i := -u_{i\sigma(i)} - x_{\sigma(i)} + x_i \quad \text{for all } i \in N.$$

Then $y := (y_i)_{i\in N}$ is a core element of (N,c). This can be seen as follows.

For $S \in 2^N \backslash \{\emptyset\}$, with $\tau_S \in \Pi_S$ such that

$$c(S) = \min_{\pi_S\in\Pi_S} \sum_{i\in S} -u_{i\pi_S(i)} = \sum_{i\in S} -u_{i\tau_S(i)},$$

it holds that

$$\sum_{i\in S} y_i = \sum_{i\in S} (-u_{i\sigma(i)} - x_{\sigma(i)} + x_i) = \sum_{i\in S} (-u_{i\sigma(i)} - x_{\sigma(i)} + x_{\tau_S(i)}) \leq \sum_{i\in S} -u_{i\tau_S(i)} = c(S),$$

where the inequality follows from the envy-freeness of (σ,x). Now the theorem follows from the remark that for $S = N$ the inequality is an equality since theorem 2.1 implies that (σ,x) is Pareto-efficient, i.e. maximizes the sum of utilities among all feasible allocations. □

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