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# Cones of Games Arising from Market Entry Problems

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**Abstract**

Market entry situations are modelled, where an entrepreneur has to decide for a collection of markets which market to enter and which not. The entrepreneur can improve his prior information by making use of a group of informants, each of them knowing the situation in one or more markets. For such a market entry situation a related cooperative game is introduced, which can be helpful in dealing with the question of how to share the reward of cooperation. The games arising turn out to be elements of the cone of information market games which were introduced for another economic context. This implies that the cooperative solutions of these games have interesting properties. Extra attention is paid to the subcone of information market games arising from market entry situations where for each market only one informant knows the state of the market.

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*Keywords:* cooperative games, information, market entry, information market games.

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# 1 Introduction

Cooperation in all kinds of economic situations leads not only to an optimization problem for the agents involved but also to a problem of sharing the benefits or costs. Often the second problem is tackled by constructing a suitable cooperative game and using the solutions developed in cooperative game theory, or by finding new solutions appealing for the class of problems at hand. There are many economic situations leading to interesting cones of games. Let us mention some.

(i) In Shapley and Shubik (1969) market situations are considered leading to market games which turn out to exhaust the cone of totally balanced games.

(ii) Flow situations (cf. Kalai and Zemel (1982)) and linear production situations (cf. Owen (1975)) lead to the cone of non–negative totally balanced games.

(iii) Problems of cooperation to make phoning in planes possible (cf. Nouweland (1996)) lead to the cone generated by unanimity games with two veto–players.

In this paper we consider in Section 2 simple market entry problems of an entrepreneur who has to decide for each member of a given collection of markets either to enter that market or not. He can cooperate with a network of informants who know the quality of certain markets, or in our simple case if the market is good or bad, meaning whether it is worthwhile to enter a market or not.

In Section 3 these market entry situations lead to the cone of information market games (cf. Muto et al. (1989)) for which many interesting properties are known. An interesting subcone, where the Shapley value, the $\tau$-value and the nucleolus coincide, is obtained by considering markets where for each market only one agent is informed.

# 2 Market entry situations

In this paper a *market entry situation* is a tuple

$$< \{0\}, M, N, \{(\mu_m, r_m; 1-\mu_m, -l_m) \mid m \in M\},\ K : M \longrightarrow N >$$

Here agent 0 is the entrepreneur interested in a non–empty finite set $M$ of markets, and $N = \{1, 2, ..., n\}$ is the set of possible informants, which the entrepreneur can consult before making a decision to enter a market $m$ or not. If the entrepreneur considers market $m \in M$ without consulting informants, then he expects with probability $\mu_m \in (0, 1)$ a good market with reward $r_m \in [0, \infty)$ and with probability $1 - \mu_m$ a bad market with loss $l_m$. The knowledge of the informants in $N$ about the markets is described by the correspondence (multi–function) $K$, where $K(m)$ is the non–empty set of agents in $N$ who know the true state of the market $m$.

To exclude trivialities and to make our mathematical life easier, we suppose throughout this paper the extra conditions ME.1 and ME.2 to be introduced now.

ME.1 (Each informant knows something) For each $i \in N$, there is a market $m$ such that $i \in K(m)$, i.e., $K : M \longrightarrow N$ is surjective.

ME.2 (Information is valuable for the entrepreneur) For each $m \in M$ the prior expectation $\mu_m r_m - (1 - \mu_m) l_m$ is non–positive.

Problems for the entrepreneur are: which informants to consult, which markets to enter and how to share the extra earnings with the informants?

Because of ME.1 and ME.2 it is reasonable to assume that player 0 decides to cooperate with all informants and to tackle his reward sharing problem by looking at the following cooperative game $< N \cup \{0\}, v >$.

For each coalition $S \subset N$ of players the worth $v(S) = 0$ and the worth $v(S \cup \{0\})$ is equal to the expected reward of the markets, given the fact that the knowledge of the informants in $S$ can avoid agent 0 to enter a bad market $m \in M(S)$ where $M(S) = \{m \in M | K(m) \cap S \neq \emptyset\}$. In formula

$$v(S \cup \{0\}) = \sum_{m \in M(S)} \mu_m r_m.$$

This worth can be reached by collecting from the informants in $S$ information about the states of the markets $M(S)$ and entering $m \in M(S)$ if the market is good, by not entering if the market's state is bad, and also not entering markets outside of $M(S)$.

In Section 3 we look more closely to games arising from market entry situations. But first let us give an example.

**Example 1.** Consider the market entry situation

$$< \{0\}, M, N, \{(\mu_m, r_m; 1 - \mu_m, -l_m) \mid m \in M\},\ K : M \longrightarrow N >$$

with $M = \{m_1, m_2\}$, $N = \{1, 2, 3\}$, $\mu_{m_1} = \mu_{m_2} = \dfrac{1}{2}$, $r_{m_1} = 20$, $l_{m_1} = 40$,$r_{m_2} = 100$, $l_{m_2} = 110$, and $K(m_1) = \{1, 2\}$, $K(m_2) = \{2, 3\}$. Then ME.1 and ME.2 hold. The corresponding market entry game $< \{0, 1, 2, 3\}, v >$ is given by $v(\{0\}) = 0$, $v(S) = 0$ if $S \subset \{1, 2, 3\}$, $v(\{0, 1\}) = 10$, $v(\{0, 3\}) = 50$, $v(\{0, 1, 3\}) = 60$ and $v(S) = 60$ for all $S$ with $\{0, 2\} \subset S$.

Now we want to describe a relation between market entry problems and information collecting (IC) situations, introduced in Brânzei, Tijs and Timmer (2000). Recall that an IC-situation is given by a tuple

$$< \{0\}, N, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, A, r : \Omega \times A \longrightarrow \mathbb{R} >.$$

Here 0 is the action taker with a finite action set $A$. The reward corresponding to action $a \in A$ is $r(\omega, a)$, so this reward depends on the state $\omega \in \Omega$ which appears with probability $\mu(\omega) \in (0, 1)$. $\Omega$ is a finite state space and $\sum_{\omega \in \Omega} \mu(\omega) = 1$. The action taker can, before choosing an action, collect information from the informants in $N = \{1, 2, ..., n\}$. The information that $i \in N$ has about the state is described by the information partition $\mathcal{I}_i$, which is a partition of $\Omega$.

The corresponding IC-game $<N \cup \{0\}, w>$ is described by $w(S) = 0$ for each $S \subset N$, $w(\{0\}) = 0$, and

$$w(S \cup \{0\}) = \sum_{I \in \mathcal{I}_S} \max_{a \in A} \sum_{\omega \in I} r(\omega, a)\mu(\omega) - \max_{a \in A} \sum_{\omega \in \Omega} r(\omega, a)\mu(\omega)$$

where $\mathcal{I}_S$ consists of non–empty intersections of the form $\bigcap_{i \in S} I_i$ with $I_i \in \mathcal{I}_i$ for each $i \in S$. Such a game is monotonic and player 0 is a veto–player (cf. Arin and Feltkamp (1997)).

A market entry situation can be related as follows to an IC-situation as above. In both situations $\{0\}$ and $N$ have the same meaning. Let $\Omega = \{0, 1\}^M$, where $\omega = (\omega_m)_{m \in M}$ corresponds to the markets' state. We denote by $G(\omega) = \{m \in M \mid \omega_m = 1\}$ the set of markets which are good when $\omega$ is the true state; then the markets in $M \setminus G(\omega)$ are bad. The probability that the true state is $\omega \in \Omega$, $\mu(\omega)$, is given by

$$\mu(\omega) = \prod_{m \in G(\omega)} \mu_m \prod_{m \in M \setminus G(\omega)} (1 - \mu_m).$$

For each $i \in N$ the corresponding information partition $\mathcal{I}_i$ has parts (atoms) of the form

$$I(x) = \{\omega \in \Omega \mid \omega_m = x_m \text{ for all } m \in M(\{i\})\}$$

where $x \in \{0, 1\}^{M(\{i\})}$. Such a part $I(x)$ corresponds to the situation where the informant $i$ knows that the state of market $m \in M(\{i\})$ is $x_m$. The action set $A$ is $\{0, 1\}^M$, where $a = (a_m)_{m \in M}$ corresponds to the strategy: enter the market $m$ if $a_m = 1$ and don't enter if $a_m = 0$. Finally, $r(\omega, a)$ is equal to

$$\sum_{m \in M, a_m = 1} (r_m \omega_m - l_m(1 - \omega_m)). \tag{2.1}$$

Note that for $S \subset N$, $\mathcal{I}_S$ is a partition of $\Omega$ with parts of the form

$$I(x) = \{\omega \in \Omega \mid \omega_m = x_m \text{ for all } m \in M(S)\}$$

where $x \in \{0, 1\}^{M(S)}$. Because of condition ME.2, it is for player 0 optimal to choose the following strategy when working together with $S \subset N$:

(i) do not enter markets outside of $M(S)$;

(ii) do not enter markets $m \in M(S)$ if $\omega \in I(x) \in \mathcal{I}_S$ and if $x_m = 0$;

(iii) enter a market $m \in M(S)$ if $\omega \in I(x) \in \mathcal{I}_S$ and if $x_m = 1$.

The expected reward of such a strategy is $\sum_{m \in M(S)} \mu_m r_m$, and this is also the worth $v(S \cup \{0\})$ in the market entry game. Thus we have the following proposition.

**Proposition 1.** *For each market entry situation and corresponding* IC–*situation the related games are the same.*

**Example 2.** The market entry situation of Example 1, where $M = \{m_1, m_2\}$, $\mu_{m_1} = \mu_{m_2} = 1/2$ corresponds to the following IC-situation

$$<\{0\}, N, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, A, r : \Omega \times A \longrightarrow \mathbb{R}>.$$

Here $N = \{1, 2, 3\}$, $\Omega = \{(0,0), (1,0), (0,1), (1,1)\}$, $\omega = (\omega_1, \omega_2) \in \Omega$ denotes the market state with $\omega_1 \in \{0,1\}$ the state of $m_1$, $\omega_2 \in \{0,1\}$ the state of $m_2$, and where 0 stands for bad market and 1 stands for good market, $\mu(\omega) = 1/4$ for all $\omega \in \Omega$, $\mathcal{I}_1 = \{\{(0,0), (0,1)\}, \{(1,0), (1,1)\}\}$, $\mathcal{I}_2$ is the discrete partition of $\Omega$ in singletons, $\mathcal{I}_3 = \{\{(0,0), (1,0)\}, \{(0,1), (1,1)\}\}$, $A = \{(a_1, a_2) \mid a_i \in \{0,1\}, i = 1, 2\}$, and $r(\omega, a)$ is as in formula (2.1).

# 3 Cones of games and market entry situations

Let $N = \{1, 2, ..., n\}$ and $T \subset N$. Let $<N \cup \{0\}, v>$ be the simple game such that for $S \subset N$ the worth $v(S) = 0$ and such that $v(S \cup \{0\}) = 1$ if $T \cap S \neq \emptyset$, and $v(S \cup \{0\}) = 0$ otherwise. Following Muto et al. (1989), we denote this game by $u^*_{T,0}$. Furthermore we denote by IG the convex cone generated by $\{u^*_{T,0} \mid T \subset N\}$. A subcone of this cone is $\text{IG}_\text{C}$, the cone generated by $\{u^*_{T,0} \mid T \subset N, |T| = 1\}$, which is equal to the cone generated by $\{u_{\{0,i\}} \mid i \in N\}$, where $u_{\{0,i\}}$ is the unanimity game with veto player set $\{0, i\}$, i.e., $u_{\{0,i\}}(S) \in \{0,1\}$ and $u_{\{0,i\}}(S) = 1$ if and only if $\{0, i\} \subset S$. Propositions 2 and 3 below show that these cones IG and $\text{IG}_\text{C}$ are interesting for our market entry situations.

**Proposition 2.**

(i) *Each market entry game is an element of* IG.

(ii) *Each* $v \in$ IG *is a game corresponding to a market entry situation.*

**Proof.** (i) For a market entry situation as in Section 2, the corresponding market entry game $<N \cup \{0\}, v>$ is given by $v(S) = 0$ if $0 \notin S$, $v(\{0\}) = 0$ and

$$v(S \cup \{0\}) = \sum_{m \in M(S)} \mu_m r_m, \text{ where } M(S) = \{m \in M \mid K(m) \cap S \neq \emptyset\}.$$

Obviously, this game can be rewritten as $v = \sum_{m \in M} \mu_m r_m u^*_{K(m),0}$. So $v \in$ IG.

(ii) Suppose that $v \in$ IG. Then there exists a number $t$ in $\{0, 1, 2, ...\}$ such that $v = \sum_{k=1}^{t} c_k u^*_{T_k,0}$, with $c_k > 0$ and $T_k \subset N$ for all $k \in \{1, 2, ..., t\}$. (We allow for an empty sum.) We consider the market entry problem

$$<\{0\}, M, N, \{(\mu_m, r_m; 1 - \mu_m, -l_m) \mid m \in M\}, K : M \longrightarrow N>$$

with $M = \{m_0, m_1, ..., m_t\}$, $\mu_m = 1/2$ for all $m \in M$, $r_{m_0} = \ell_{m_0} = 0$, $r_{m_k} = 2c_k$, $l_{m_k} = 3c_k$, $K(m_0) = N$ and $K(m_k) = T_k$ for all $k \in \{1, 2, ..., t\}$. Then the conditions ME.1 and ME.2 are satisfied and for the corresponding market entry game $v'$ we have $v' = 0 \cdot u^*_{N,0} + \sum_{k=1}^{t} c_k u^*_{T_k,0} = v$. ∎

**Example 3.** For the game $v$ in Example 1 we have

$$v = 10\, u^*_{\{1,2\},0} + 50\, u^*_{\{2,3\},0}.$$

Knowing now that the set of market entry games coincides with the cone IG we can use information in the literature, e.g., Muto et al. (1989) and Muto et al. (1988). A few of these results are summarized in the theorem below.

Recall the following properties. A game $<N \cup \{0\}, v>$

(M) is monotonic if $v(S) \leq v(T)$ for all $S \subset T \subset N \cup \{0\}$.

$(\mathrm{V}_0)$ has the 0–veto property if $v(S) = 0$ if $0 \notin S$ and $v(\{0\}) = 0$.

(U) has the union property if $v(N \cup \{0\}) - v(S) \geq \sum_{i \in N \cup \{0\} \setminus S} M_i(v)$ for all $S \subset N \cup \{0\}$ with $0 \in S$, where $M_i(v) := v(N \cup \{0\}) - v(N \cup \{0\} \setminus \{i\})$.

**Theorem 1.** *Let $<N \cup \{0\}, v>$ be a market entry game. Then*

(i) *$v$ has the properties* (M), $(\mathrm{V}_0)$ *and* (U).

(ii) *The core $C(v)$ of $<N \cup \{0\}, v>$ is given by*

$$C(v) = \left\{ x \in \mathbb{R}^{N \cup \{0\}} \,\middle|\, \sum_{i=0}^{n} x_i = v(N \cup \{0\}),\ 0 \leq x_i \leq M_i(v) \text{ for all } i \in N \right\}.$$

(iii) *The $\tau$-value* (Tijs (1981)) *is equal to the nucleolus* (Schmeidler (1969)) *and is given by*

$$\left( v(N \cup \{0\}) - \frac{1}{2} \sum_{i=1}^{n} M_i(v), \frac{1}{2} M_1(v), ..., \frac{1}{2} M_n(v) \right).$$

Let us now look at the subclass of games in IG with the property

$(C_N)$ $v(S \cup \{i\}) - v(S) \leq v(N \cup \{0\}) - v(N \cup \{0\} \setminus \{i\})$ for all $S \subset N \cup \{0\} \setminus \{i\}$.

Then we have

**Theorem 2.** *Let $<N \cup \{0\}, v>$ be an element of* IG. *Then the following assertions are equivalent*

(i) *$v$ satisfies property $(C_N)$.*

(ii) $v \in \mathrm{IG_C}$ *and* $v = \sum_{i=1}^{n} M_i(v) u^*_{\{i\},0}$.

(iii) *$v$ is convex i.e. $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subset T \subset N \cup \{0\} \setminus \{i\}$.*

**Proof.** ((i)$\Longrightarrow$(ii)). Suppose that $v$ has the property $(C_N)$. Take $S \subset N \cup \{0\}$ with $0 \in S$. Suppose that $N \cup \{0\} \setminus S = \{i_1, i_2, \ldots, i_k\}$. Then $(C_N)$ implies:

$$\begin{aligned} v(N \cup \{0\}) - v(S) &= \sum_{r=1}^{k} (v(S \cup \{i_1, \ldots, i_r\}) - v(S \cup \{i_1, \ldots, i_{r-1}\})) \\ &\leq \sum_{r=1}^{k} (v(N \cup \{0\}) - v(N \cup \{0\} \setminus \{i_r\})) = \sum_{i \in N \cup \{0\} \setminus S} M_i(v). \end{aligned}$$

Using also the property (U) for $v$ we obtain:

$$v(N \cup \{0\}) - v(S) = \sum_{i \in N \cup \{0\} \setminus S} M_i(v) \text{ for } 0 \in S \subset N \cup \{0\}.$$

Then

$$\begin{aligned} v(S) &= (v(N \cup \{0\}) - v(\{0\})) - (v(N \cup \{0\}) - v(S)) \\ &= \sum_{i \in N} M_i(v) - \sum_{i \in N \cup \{0\} \setminus S} M_i(v) = \sum_{i \in S \setminus \{0\}} M_i(v) \text{ for } S \ni 0. \end{aligned}$$

Since $v(T) = 0$ for all $T$ with $0 \notin T$, we can conclude that

$$v = \sum_{i=1}^{n} M_i(v) u^*_{\{i\},0}.$$

So, $v \in \mathrm{IG_C}$.

((ii)$\Longrightarrow$(iii)). It is well–known that $u_{\{0,i\}} = u^*_{\{i\},0}$ is a convex game for each $i \in N$. Then $v$ is also convex.

((iii)$\Longrightarrow$(i)). Obviously, a convex game satisfies $(C_N)$. ■

From this theorem and the proof of Proposition 2, (ii), we can conclude that the next proposition holds.

**Proposition 3.** *Games in* $\mathrm{IG_C}$ *correspond to market entry situations, where each market only has one informant.*

Furthermore, it follows from Theorem 4.3 in Nouweland et al. (1996) that for games in $\mathrm{IG_C}$ the Shapley value (cf. Shapley (1953)), the $\tau$-value and the nucleolus coincide and are equal to $(\sum_{i=1}^{n} c_i/2, c_1/2, \ldots, c_n/2)$ if $v = \sum_{i=1}^{n} c_i u_{\{0,i\}} \in \mathrm{IG_C}$.

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