A note on link formation[a]

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March 3, 1998

Abstract

In this note we study the endogenous formation of cooperation structures. According to several equilibrium concepts the full cooperation structure will form or some structure that is payoff-equivalent to the full cooperation structure. As a by-product we find a class of games in strategic form where several equilibrium concepts coincide.

JOURNAL OF ECONOMIC LITERATURE classification numbers: C71, C72.

1 Introduction

In the past few years several papers have modelled the process of distributing the payoffs in a cooperative situation as a two-stage game. In the first stage, the players negotiate on the cooperation structure. The second stage then determines the payoffs, usually according to some exogenously given allocation rule.

In this note we will follow *Dutta*, *Nouweland*, and *Tijs* (1998). They analyze the link formation games introduced by *Myerson* (1991), which were also studied by *Qin* (1996). *Dutta et al.* (1998) find that given a superadditive game and an allocation rule satisfying some appealing properties, the full cooperation structure will form or a structure resulting in the same payoffs as the full cooperation structure. These results are shown for two equilibrium concepts, undominated Nash equilibria and coalition proof Nash equilibria. We will extend these results for several other equilibrium concepts, specifically strictly proper, proper, weakly proper, strictly perfect and perfect equilibria. As a by-product we find a class of games in strategic form where several equilibrium concepts coincide.

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The plan of this paper is as follows. Section 2 deals with games in strategic form, equilibrium concepts for such games, and the relation between these equilibrium concepts. In section 3 we describe and analyze the link formation games. We show that according to the equilibrium concepts described in section 2 the full cooperation structure will form or a cooperation structure that results in the same payoff as the full cooperation structure.

2 Games in strategic form

In this section we introduce some notation for games in strategic form. We also show some relationships between equilibrium concepts, which will be used in the subsequent section on link formation games. For a survey of equilibrium concepts for games in strategic form we refer to *Van Damme* (1991).

Let $\Gamma = (N; (S_i)_{i \in N}; (f_i)_{i \in N})$ be a game in strategic form, where $N = \{1, \ldots, n\}$ denotes the player set, S_i the strategy space of player i, and f_i the payoff function of player i, which assigns to every tuple $s \in S = \prod_{i \in N} S_i$ a payoff $f_i(s) \in \mathbb{R}$. Denote $f = (f_i)_{i \in N}$.

The first equilibrium concept we define is the *undominated Nash equilibrium.* Recall that a strategy profile is a Nash equilibrium if no player can improve his payoff by a unilateral deviation. For any $i \in N$, s_i *dominates* s_i' if for all $s_{-i} \in S_{-i} := \prod_{j \in N \setminus \{i\}} S_i$, $f_i(s_i, s_{-i}) \geq f_i(s_i', s_{-i})$ with the inequality being strict for some $s_{-i} \in S_{-i}$. Now, $s_i^* \in S_i$ is an undominated strategy if there is no $s_i \in S_i$ such that s_i dominates s_i^*. Let $S_i^u(\Gamma)$ be the set of undominated strategies for i in Γ and define $S^u(\Gamma) := \prod_{i \in N} S_i^u(\Gamma)$. If no confusion on the underlying game can arise we simply write S_i^u and S^u. A strategy tuple s is an undominated Nash equilibrium if s is a Nash equilibrium and $s \in S^u$.

A strategy s_i of player i is a *weakly dominant strategy* if for all $s_{-i} \in S_{-i}$ and all $s_i' \in S_i$, $f_i(s_i, s_{-i}) \geq f_i(s_i', s_{-i})$. Denote the set of weakly dominant strategies of player i by $S_i^w(\Gamma)$ and define $S^w(\Gamma) := \prod_{i \in N} S_i^w(\Gamma)$. We also write S_i^w and S^w if no confusion on Γ can arise. Note that every $s \in S^w$ is a Nash equilibrium and that every weakly dominant strategy is undominated.

The following lemma shows that if a player has a weakly dominant strategy, then all his undominated strategies are weakly dominant.

Lemma 2.1 Let Γ be a game in strategic form. If $S_i^w \neq \emptyset$ then $S_i^w = S_i^u$.

Proof: Obviously, $S_i^w \subseteq S_i^u$. Assume $S_i^w \neq \emptyset$, so there exists $s_i \in S_i^w$. Let $s_i' \in S_i^u$. We will show that s_i' is a weakly dominant strategy. Since s_i is a weakly dominant strategy it holds for all $s_{-i} \in S_{-i}$ that $f_i(s_i, s_{-i}) \geq f_i(s_i', s_{-i})$. But s_i' is undominated and hence this inequality holds with equality for all $s_{-i} \in S_{-i}$. Since s_i is a weakly dominant strategy this implies that s_i' is a weakly dominant strategy and hence, $S_i^w \supseteq S_i^u$. This completes the proof. $\square$

Remark 2.1 Note that if $s_i, s_i' \in S_i^w(\Gamma)$ then for all $s_{-i} \in S_{-i}$ it holds that $f_i(s_i, s_{-i}) = f_i(s_i', s_{-i})$.

From now on assume that the strategy space of every player is finite, i.e. $|S_i| < \infty$ for all $i \in N$. A mixed strategy p_i of player i is a probability distribution on S_i. The probability player i assigns to strategy $k \in S_i$ will be denoted by p_i^k. Hence, the set of mixed strategies of player i is described by[1]

$$P_i := \left\{ p_i \in \mathbb{R}^{S_i} \mid \sum_{k \in S_i} p_i^k = 1,\ p_i^k \geq 0 \text{ for all } k \in S_i \right\}.$$

Denote $\Gamma^p = (N; (P_i)_{i \in N}; (f_i')_{i \in N})$, the *mixed extension* of Γ, where $f_i'(p)$ denotes the expected payoff to player i according to mixed strategy profile $p = (p_i)_{i \in N} \in P := \prod_{i \in N} P_i$ and the original payoff function f_i, i.e.

$$f_i'(p) = \sum_{s \in S} \prod_{j \in N} p_j^{s_j}\ f_i(s).$$

For notational convenience we define for all $i \in N$ and all $s_{-i} \in S_{-i}$ the probability that the players in $N \backslash \{i\}$ play s_{-i} by,

$$p(s_{-i}) = p((s_j)_{j \in N \backslash \{i\}}) := \prod_{j \in N \backslash \{i\}} p_j^{s_j}\ .$$

For all $i \in N$ and all $k \in S_i$ denote the mixed strategy associated with pure strategy k of player i by $e_{i,k}$. So,

$$e_{i,k}^l = \begin{cases} 1 & \text{, if } l = k \\ 0 & \text{, otherwise} \end{cases}.$$

Furthermore, we denote for all $s \in S$, $e_s = (e_{i,s_i})_{i \in N}$.

Before we can define strictly proper equilibria we need some more notation. For all $\eta_i = (\eta_i^k)_{k \in S_i} \in \mathbb{R}_{++}^{S_i}$ we define

$$P_i(\eta_i) := \left\{ p_i \in P_i \mid p_i^k \geq \eta_i^k,\ \text{for all } k \in S_i \right\}.$$

For $\eta = (\eta_i)_{i \in N} \in \prod_{i \in N} \mathbb{R}_{++}^{S_i}$ the set of Nash equilibria of the game $(N; P_1(\eta_1), \ldots, P_n(\eta_n); f')$ is denoted by $E(\Gamma^p, \eta)$. This game is called a *perturbed game.*

[1] For notational convenience we will simply write P_i in stead of $P(S_i)$.

In such a perturbed game every player plays each of his strategies with at least some prespecified positive probability. For all $\hat{\eta} \in \prod_{i\in N} \mathbb{R}^{S_i}_{++}$ define $U_{\hat{\eta}} := \{\eta \in \prod_{i\in N} \mathbb{R}^{S_i}_{++}; \eta < \hat{\eta}\}$.

Now, we can describe the *strictly proper equilibria* of a strategic form game Γ. A strategy profile $p \in P$ is a strictly proper equilibrium of Γ if there exists some $\hat{\eta} \in \prod_{i\in N} \mathbb{R}^{S_i}_{++}$ and a continuous map $\eta \to p(\eta)$ from $U_{\hat{\eta}}$ to $P = \prod_{i\in N} P_i$ such that $p(\eta) \in E(\Gamma^p, \eta)$ for all η and $\lim_{\eta\downarrow 0} p(\eta) = p$. The set of strictly proper Nash equilibria in Γ will be denoted by StrProp(Γ). Note that by definition a strictly proper equilibrium of a strategic form game is a mixed strategy of that game. This strategy does not necessarily correspond to a pure strategy.

In the following lemma we show that every weakly dominant strategy in the mixed extension of a game puts positive weights on strategies that are weakly dominant in the original game.

Lemma 2.2 Let Γ be a game in strategic form. Then

$$S_i^w(\Gamma^p) = \mathrm{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}.^2$$

Proof: First we will show that $S_i^w(\Gamma^p) \subseteq \mathrm{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}$. Let $p_i \in S_i^w(\Gamma^p)$ and suppose $p_i \notin \mathrm{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}$. Note that every mixed strategy is a convex combination of mixed strategies associated with pure strategies, $p_i = \sum_{k\in S_i} p_i^k e_{i,k}$.

Since $p_i \notin \mathrm{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}$ there exists $l \in S_i$ with $l \notin S_i^w(\Gamma)$ and $p_i^l > 0$. Since $l \notin S_i^w(\Gamma)$ there exists $\hat{l} \in S_i$ and $s_{-i} \in S_{-i}$ such that $f_i(\hat{l}, s_{-i}) > f(l, s_{-i})$. Now, define

$$\hat{p}_i^k = \begin{cases} p_i^k & , \text{ for all } k \in S_i\backslash\{l, \hat{l}\} \\ p_i^k + p_i^l & , k = \hat{l} \\ 0 & , k = l \end{cases} .$$

Let $p_{-i} = (p_j)_{j\in N\backslash\{i\}}$ be the mixed strategy profile of $N\backslash\{i\}$ associated with s_{-i}, i.e. $p_{-i} = (e_{j,s_j})_{j\in N\backslash\{i\}}$. If player i plays $\hat{p}_i$ instead of p_i against p_{-i} he improves his payoff, since

$$f_i'(\hat{p}_i, p_{-i}) - f_i'(p) = p_i^l \left(f_i(\hat{l}, s_{-i}) - f_i(l, s_{-i})\right) > 0.$$

So, $p_i \notin S_i^w(\Gamma^p)$, a contradiction with $p_i \in S_i^w(\Gamma^p)$. Hence,

$$S_i^w(\Gamma^p) \subseteq \mathrm{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}.$$

Secondly, we will show that all $p_i \in \mathrm{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}$ belong to $S_i^w(\Gamma^p)$. Therefore, let $p_i \in \mathrm{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}$. Let $p_{-i} \in P_{-i} := \prod_{j\in N\backslash\{i\}} P_j$ and let $\hat{p}_i \in P_i$.

[2] Conv$\{A\}$ denotes the set of all convex combinations of elements of A, where $\mathrm{conv}\{\emptyset\} := \emptyset$.

Then

$$\begin{aligned} f_i'(p_i, p_{-i}) &= \sum_{k\in S_i^w(\Gamma)} p_i^k f_i'(e_{i,k}, p_{-i}) \\ &= \sum_{k\in S_i^w(\Gamma)} p_i^k \sum_{s_{-i}\in S_{-i}} p(s_{-i}) f_i(k, s_{-i}) \\ &\geq \sum_{k\in S_i^w(\Gamma)} p_i^k \sum_{s_{-i}\in S_{-i}} p(s_{-i}) \sum_{l\in S_i} \hat{p}_i^l f_i(l, s_{-i}) \\ &= \sum_{k\in S_i^w(\Gamma)} p_i^k f_i'(\hat{p}_i, p_{-i}) \\ &= f_i'(\hat{p}_i, p_{-i}), \end{aligned}$$

where the equalities follow by definition of the strategies. The inequality follows since $f_i(k, s_{-i}) \geq f_i(l, s_{-i})$ for all $l \in S_i$ and $\sum_{l\in S_i} \hat{p}_i^l = 1$.

So, p_i is a weakly dominant strategy in Γ^p. Hence,

$$S_i^w(\Gamma^p) \supseteq \text{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}.$$

This completes the proof. □

Before we can prove the main result of this section, we need two more lemmas. First we show that every weakly dominant mixed strategy profile is a strictly proper Nash equilibrium.

Lemma 2.3 Let Γ be a game in strategic form. Then

$$S^w(\Gamma^p) \subseteq \text{StrProp}(\Gamma).$$

Proof: Let $p \in S^w(\Gamma^p)$. We have to show that $p \in \text{StrProp}(\Gamma)$. By lemma 2.2 it holds for all $i \in N$ that p_i is a convex combination of mixed strategies associated with weakly dominant pure strategies,

$$p_i = \sum_{k\in S_i^w(\Gamma)} p_i^k e_{i,k}.$$

Let $m := \sum_{i\in N} |S_i|$ and let $\hat{\eta} \in \prod_{i\in N} \mathbb{R}_{++}^{S_i}$ with $\hat{\eta}_i^k = \frac{1}{m}$ for all $i \in \{1, \ldots, n\}$ and all $k \in S_i$. For all $\eta \in U_{\hat{\eta}}$, all $i \in \{1, \ldots, n\}$, and all $k \in S_i^w(\Gamma)$ let

$$q_{i,k}^l(\eta) = \begin{cases} \eta_i^l & , \text{ for all } l \in S_i \backslash \{k\} \\ 1 - \sum_{r\in S_i\backslash\{k\}} \eta_i^r & , \ l = k \end{cases} . \tag{1}$$

Furthermore, let $q_i(\eta) = \sum_{k\in S_i^w(\Gamma)} p_i^k q_{i,k}(\eta)$ for all $i \in N$. Note that $q_i(\eta) \in P_i(\eta_i)$ for all $i \in N$ since $q_{i,k}^l(\eta) \geq \eta_i^l$ for all $k \in S_i^w(\Gamma)$, all $l \in S_i$, and all $i \in N$.

Let $i \in N$. For all $k \in S_i^w(\Gamma)$ the map $\eta \to q_{i,k}(\eta)$ from $U_{\hat{\eta}}$ to P_i is continuous, with $q_{i,k}(\eta) \in P_i(\eta_i)$ for all η, and $\lim_{\eta \downarrow 0} q_{i,k}(\eta) = e_{i,k}$. Then it follows immediately that $\eta \to q(\eta) = (q_i(\eta))_{i \in N}$ is a continuous map from $U_{\hat{\eta}}$ to P with $\lim_{\eta \downarrow 0} q(\eta) = p = (p_i)_{i \in N}$.

It remains to show that $q(\eta) \in E(\Gamma^p, \eta)$, for all $\eta \in U_{\hat{\eta}}$. Therefore, let $\eta \in U_{\hat{\eta}}$ and consider a possible deviation of player $i \in \{1, \dots, n\}$, $u_i \in P_i(\eta_i)$. The change in payoff for player i by deviating from $q_i(\eta)$ to u_i is equal to

$$
\begin{aligned}
& f_i'(u_i, q_{-i}(\eta)) - f_i'(q(\eta)) \\
= & \sum_{k \in S_i^w(\Gamma)} p_i^k \left(f_i'(u_i, q_{-i}(\eta)) - f_i'(q_{i,k}(\eta), q_{-i}(\eta))\right) \\
= & \sum_{k \in S_i^w(\Gamma)} p_i^k \left[\sum_{l \in S_i \setminus \{k\}} (u_i^l - \eta_i^l) f_i'(e_{i,l}, q_{-i}(\eta)) \right. \\
& \left. \quad + \left((1 - \sum_{l \in S_i \setminus \{k\}} u_i^l) - (1 - \sum_{l \in S_i \setminus \{k\}} \eta_i^l) \right) f_i'(e_{i,k}, q_{-i}(\eta)) \right] \\
= & \sum_{k \in S_i^w(\Gamma)} p_i^k \left[\sum_{l \in S_i \setminus \{k\}} (u_i^l - \eta_i^l) \left(f_i'(e_{i,l}, q_{-i}(\eta)) - f_i'(e_{i,k}, q_{-i}(\eta))\right) \right] \\
\leq & \; 0,
\end{aligned}
$$

where the equalities follow by definition of the strategies. The inequality holds since for all $k \in S_i^w(\Gamma)$ and all $l \in S_i \setminus \{k\}$ it holds that $u_i^l \geq \eta_i^l$ and since for all $k \in S_i^w(\Gamma)$, all $l \in S_i \setminus \{k\}$, and all $t_{-i} \in S_{-i}$, $f_i(k, t_{-i}) - f_i(l, t_{-i}) \geq 0$.

This completes the proof. □

The following result is taken from *Van Damme* (1991).

Lemma 2.4 For a game in strategic form Γ

$$\mathrm{StrProp}(\Gamma) \subseteq S^u(\Gamma^p).$$

Proof: See *Van Damme* (1991). □

We can now state the main result of this section.

Theorem 2.1 Let Γ be a game in strategic form. If $S^w(\Gamma) \neq \emptyset$ then

$$S^u(\Gamma^p) = \mathrm{StrProp}(\Gamma) = S^w(\Gamma^p) = \prod_{i \in N} \mathrm{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}.$$

Proof: Assume $S^w(\Gamma) \neq \emptyset$. Then

$$\mathrm{StrProp}(\Gamma) \subseteq S^u(\Gamma^p) = S^w(\Gamma^p) \subseteq \mathrm{StrProp}(\Gamma)$$

by lemmas 2.4, 2.1, and 2.3 respectively. Lemma 2.2 completes the proof. □

Recall that every profile of weakly dominant strategies is a Nash equilibrium. Hence, by lemma 2.1, if every player has a weakly dominant strategy then every strategy profile consisting only of undominated strategies is a Nash equilibrium. There are several equilibrium concepts that result in supersets of the set of strictly proper Nash equilibria and subsets of the set of undominated Nash equilibria in mixed strategies. For a survey see *Van Damme* (1991). He shows that for a strategic form game with a finite strategy space for all players the sets of (i) proper equilibria, (ii) weakly proper equilibria, (iii) strictly perfect equilibria, and (iv) perfect equilibria are all supersets of the set of strictly proper equilibria and subsets of the set of undominated Nash equilibria in mixed strategies. Using this and the theorem above the following corollary results for strategic form games with a weakly dominant strategy for all players.

Corollary 2.1 Let Γ be a game in strategic form. If $S^w(\Gamma) \neq \emptyset$ then the following sets of equilibria coincide with $S^w(\Gamma^p)$ and the set of undominated Nash equilibria in mixed strategies : strictly proper, proper, weakly proper, strictly perfect, and perfect equilibria.

3 Link formation

In this section we will describe and analyze a class of link formation games, introduced by *Myerson* (1991) and also studied by *Qin* (1996) and *Dutta et al.* (1998).

A communication situation is a triple (N, v, L), with (N, v) a cooperative game and (N, L) a cooperation graph (N, L). So, $N = \{1, \ldots, n\}$ denotes the player set, v the characteristic function that assigns to every subset of N a value, and L a set of pairs of players in N, describing the cooperation possibilities between the players.

The pair (N, L) is an undirected (communication) graph. A link in the graph indicates that the players forming this link can cooperate with each other directly. If two players are not connected directly but there is a path in the graph between the players, then these two players can communicate with each other indirectly via the players on the path. The notion of connectedness induces a partition of the player set into communication components, where i and j are in the same component if and only if $i = j$ or i and j can communicate with each other, directly or indirectly. The resulting partition will be denoted by N/L.

An allocation rule γ assigns to every communication situation (N, v, L) a payoff vector $\gamma(N, v, L) \in \mathbb{R}^N$. Here, we will restrict ourselves to the same class of allocation rules as studied by *Dutta et al.* (1998). This class is described by the following properties.

- **Component efficiency (CE):** For all communication situations (N, v, L) and all communication components $C \in N/L$ it holds that $\sum_{i \in C} \gamma_i(N, v, L) = v(C)$.
- **Weak link symmetry (WLS):** For all communication situations (N, v, L) and all $i, j \in N$, if $\gamma_i(N, v, L \cup \{\{i, j\}\}) > \gamma_i(N, v, L)$ then $\gamma_j(N, v, L \cup \{\{i, j\}\}) > \gamma_j(N, v, L)$.
- **Improvement property (IP):** For all communication situations (N, v, L) and all $i, j \in N$, if there exists $k \in N \backslash \{i, j\}$ with $\gamma_k(N, v, L \cup \{\{i, j\}\}) > \gamma_k(N, v, L)$, then $\gamma_i(N, v, L \cup \{\{i, j\}\}) > \gamma_i(N, v, L)$ or $\gamma_j(N, v, L \cup \{\{i, j\}\}) > \gamma_j(N, v, L)$.

The following lemma was proven by *Dutta et al.* (1998).

Lemma 3.1 Let γ be an allocation rule that satisfies CE, WLS, and IP and (N, v, L) a communication situation with (N, v) superadditive.[3] For all $i, j \in N$ it holds that

$$\gamma_i(N, v, L \cup \{\{i, j\}\}) \geq \gamma_i(N, v, L). \tag{2}$$

Proof: See *Dutta et al.* (1998) □

The property incorporated in equation (2) will be called *link monotonicity*.

We will now describe the class of link formation games. Let γ be an allocation rule and (N, v) a cooperative game. The link formation game $\Gamma(N, v, \gamma)$ is described by the tuple $(N; (S_i)_{i \in N}; (f_i^\gamma)_{i \in N})$ where for all $i \in N$ the set $S_i = 2^{N \backslash \{i\}}$ represents the strategy set of player i. A strategy of player i is an announcement of the set of players he wants to form communication links with. A communication link between two players will only form if both players want to form the link. The set of links that will form according to strategy profile $s \in S = \prod_{i \in N} S_i$ will be denoted by

$$L(s) := \{\{i, j\} \subseteq N \mid i \in s_j,\ j \in s_i\}.$$

The payoff function $f^\gamma = (f_i^\gamma)_{i \in N}$ is defined as the allocation rule γ applied to the communication situation $(N, v, L(s))$, i.e.

$$f^\gamma(s) = \gamma(N, v, L(s)).$$

In the following lemma we show that the link formation games described above have a weakly dominant strategy profile. Moreover, this strategy profile results in the full cooperation structure (i.e. every player cooperates directly with every other player). This strategy profile is denoted by $\bar{s}$, i.e. $\bar{s}_i = N \backslash \{i\}$ for all $i \in N$.

[3]The game (N, v) is superadditive if for all $S, T \in 2^N$ with $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$.

Lemma 3.2 Let γ be an allocation rule that satisfies CE, WLS, and IP and (N, v) a superadditive cooperative game. Then $\overline{s}$ is a weakly dominant strategy profile in the associated link formation game $\Gamma := \Gamma(N, v, \gamma)$.

Proof: Let $i \in N$, $s_i \in S_i$ and $s_{-i} \in S_{-i}$. Define the following sets of links: $L^1 = L(\overline{s}_i, s_{-i})$ and $L^2 = L(s_i, s_{-i})$. Since $s_i \subseteq \overline{s}_i$ it holds that $L^2 \subseteq L^1$. Furthermore, $L^1 \backslash L^2 \subseteq \{\{i,j\} \mid j \in N \backslash \{i\}\}$, since only the strategy of player i has been changed. If we apply lemma 3.1 for all $\{i,j\} \in L^1 \backslash L^2$ then

$$f_i^\gamma(\overline{s}_i, s_{-i}) = \gamma_i(N, v, L^1) \geq \gamma_i(N, v, L^2) = f_i^\gamma(s_i, s_{-i}). \tag{3}$$

We conclude that $\overline{s}_i \in S_i^w(\Gamma)$ and hence, $\overline{s} \in S^w(\Gamma)$. □

Lemma 3.2 was not proven explicitly in *Dutta et al.* (1998). However, they showed it implicitly in showing that $\overline{s}$ is an undominated Nash equilibrium.

Now that we have showed the existence of a weakly dominant strategy profile we can use the results of the previous section to give some relations between equilibrium concepts for mixed extensions of link formation games.

Theorem 3.1 Let γ be an allocation rule that satisfies CE, WLS, and IP and let (N, v) be a superadditive cooperative game. Then the following relations between several equilibrium refinements hold for the corresponding link formation game $\Gamma := \Gamma(N, v, \gamma)$:

$$\{e_{\overline{s}}\} \subseteq \prod_{i \in N} \text{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\} = S^u(\Gamma^p) = \text{StrProp}(\Gamma) = S^w(\Gamma^p). \tag{4}$$

Proof: From lemma 3.2 it follows that $\overline{s} \in S^w(\Gamma)$. Then it follows from theorem 2.1 that

$$S^u(\Gamma^p) = \text{StrProp}(\Gamma) = S^w(\Gamma^p) = \prod_{i \in N} \text{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}.$$

Since $\overline{s}$ is a weakly dominant strategy it holds that

$$e_{\overline{s}} \in \prod_{i \in N} \text{conv}\{e_{i,k} \mid k \in S_i^w(\Gamma)\}.$$

This completes the proof. □

Note that the result in theorem 3.1 depends only on the assumption that γ satisfies link monotonicity, which is implied by CE, WLS, and IP.

Remark 3.1 Obviously we can also extend the theorem above to include proper, weakly proper, strictly perfect and perfect equilibria (see corollary 2.1).

Remark 3.2 It can be shown that if γ satisfies CE, WLS, and IP then $f_i^\gamma(s) = f_i^\gamma(s_i', s_{-i})$ for some $i \in N$ implies that $f^\gamma(s) = f^\gamma(s_i', s_{-i})$ (This follows directly from lemmas 1 and 2 in *Dutta et al.* (1998)). So, if γ satisfies CE, WLS, and IP then this implies that all weakly dominant strategy profiles result in the same payoff. *Dutta et al.* (1998) call structures that lead to identical payoffs *payoff-equivalent.* Furthermore, they call a structure *essentially complete* if it is payoff-equivalent to the full cooperation structure. The structures that can result according to any $p \in S^u(\Gamma(N, v, \gamma)^p) = \mathrm{StrProp}(\Gamma(N, v, \gamma))$ are obviously all essentially complete. We cannot speak of the structure that will result since p is a mixed strategy profile.

4 References

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