Market Games with Multiple Trading Posts

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Abstract

We study market games with multiple posts per commodity. We provide some facts that characterize prices of commodities across posts and show the following results: (i) As the number of agents increases, the price variability across posts for a commodity becomes smaller and it becomes zero when the number of agents becomes infinite, irrespectively of the distribution of characteristics in the economy. (ii) The set of equilibrium prices and allocations of a market game is a subset of the set of equilibria of another game with more trading posts per commodity. (iii) We demonstrate via an example that the inclusion can be strict, as there are equilibria with price disparities across posts for a commodity which cannot be captured with less trading posts. (iv) One can pass from an equilibrium of a market game into an equilibrium of a game with less trading posts per commodity, by consolidating posts where the price of a commodity is uniform.

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1 Introduction

The standard setup of market games -see [1] [7] [8] [3] [4]- is based on the concept of *trading posts* where individuals submit orders for purchases and sales of commodities. The orders that reach each trading post are aggregated and commodities are distributed to individuals in proportion to their orders. Thus, a structure of trading posts must be postulated at the outset of a market game, so that we have a way to match orders for purchases and sales. In the absence of some friction, such as transaction costs etc, the particular structure of trading posts might at first seem irrelevant. However, as it has been demonstrated in [2] the strategic market game is not robust with respect to the structure of trading posts postulated at the outset of the game: with more trading posts there are equilibrium profiles of strategies, allocations and prices that cannot be captured by the single trading post model. Surprisingly enough this result does not depend on liquidity constraints or any other structural elements of the market game. It is clearly due to the lack of perfect competition, since such phenomena never arise in the competitive framework. But why does the lack of perfect competition, manifested by the small number of traders with non negligible effects on price formation, render some relevance to the structure of markets? This question has motivated the study in this paper.

In this paper we provide some general analysis of the multiple posts model. We first provide a simple characterization of equilibria with multiple posts, which clarifies the reason for the relevance of the structure of trading posts: equilibria are compatible with the presence of limited arbitrage opportunities across posts. Thus, the introduction of additional posts is not redundant because it gives rise to further arbitrage opportunities. The concepts of 'arbitrage' and 'equilibrium' might perhaps sound like a contradiction in terms. However, as we show via an example, as in [2], it is possible that arbitrage opportunities exist in equilibrium but they cannot be profitably exploited by individuals. In the example featured in this paper there is an equilibrium where commodities are exchanged simultaneously in two posts at different prices. However, any contemplated effort to exploit the price difference by shifting orders across posts (e.g. from the expensive to the cheap) does not provide any benefit. The reason for this is as follows: due to the imperfectly competitive nature of the model any such effort to exploit arbitrage opportunities, induces changes in prices. The key idea is that since agents are also sellers, the change in prices causes income effects that are detrimental for the benefits of this shift. Hence, we conclude that equilibria are indeed compatible with the presence of arbitrage.

The natural question that arises is what limits the extend of arbitrage that can exist in equilibrium. In other words, is it possible to obtain an estimate of the arbitrage opportunities that can occur in equilibrium as a function of the data of the economy? Further, what can be said about arbitrage in equilibrium asymptotically as the number of agents becomes large? The main result of this paper is that 'large' price disparities across posts for a commodity, are compatible only with 'small' sets of agents. In particular, when the number of agents tends to infinity, prices across posts converge to a common limit regardless of the characteristics present in the economy. We believe that this is the appropriate result in this context because the relevance of the structure of trading posts, which is only an institutional matter, should not hinge on the characteristics of the economy. In this way we can conclude that in a frictionless context, the lack of perfect competition is the *only* source of arbitrage in equilibrium. This conclusion is important for yet another reason: in a way this constitutes a proof that the correspondence 'one commodity-one price' is indeed a *law* in the sense that it obtains asymptotically irrespectively of the sequence of economies and the corresponding limit. In order to keep things in perspective, our result has no implication whatsoever regarding the nature of the common limit of prices. In particular, this common limit need not be a competitive price. Indeed, in the context of market games the common limit could be zero, which typically is not compatible with competitive equilibria. In conclusion, our result shows that an arbitrarily large number of agents implies that equilibria are characterized by no arbitrage. However, the lack of arbitrage itself does not suffice to characterize perfect competition. Apparently some further qualifications are necessary for this. On the other hand, this result allows us to conjecture the possibility of some asymptotic equivalence result between market games with multiple posts and competitive equilibria.

Finally, we show two results regarding the number of trading posts in a market game: First, that the equilibria of the single trading post model are equilibria for the model with multiple posts and more generally, the set of equilibria of a model with any number of posts is embedded into any model with more trading posts. Thus, by augmenting the number of trading posts we do not 'lose' any equilibria. Second, by consolidating trading posts where prices of a commodity are equal we can obtain an equilibrium for a model with less trading posts. In particular, every equilibrium in a model with many posts where the 'law of one price' holds (namely, prices are uniform across all posts for each commodity) is an equilibrium for the single trading post model. However, as our example shows, the inclusion of the set of equilibria of the standard game into market games with more trading posts can be strict.

We proceed with our analysis as follows: Section two introduces a multiple posts version of the model in [3] and in [4]. In section three we establish some conditions that characterize the distribution of prices across posts for each commodity. In section four we develop an example and provide a theorem on the relationship between the numer of agents, the number of trading posts for each commodity and the maximal possible price discrepancy across posts for a commodity. Section five features some results on the structure of equilibria where the law of one price is valid. Finally, some concluding remarks follow in section six.

2 The model

Let H be a finite set of agents. There are L commodity types in the economy and the consumption set of each agent is identified with $\Re^L_+$. Each individual is characterized by a preference relation, which is representable by a utility function $u_h : \Re^L_+ \to \Re$, and an initial endowment $e_h \in \Re^L_+$. An economy is defined as $\mathcal{E} = \{(\Re^L_+, u_h, e_h) : h \in H\}$. Throughout the rest of the paper I will employ the following assumptions:

Assumption 2.1 *$e_h >> 0$ for each $h \in H$.*

Assumption 2.2 *Preferences are convex, C^2, differentiably strictly monotone*[1] *and indifference surfaces through the endowment do not intersect the axis.*

Trade in this economy is organized via a system of trading posts where individuals offer commodities for sale and place bids for purchases of commodities. A possible scenario for the rules of exchange which is attributed to [3] is presented below. The interested reader should consult the original source for a detailed account of this formulation.

2.1 Trade via inside money

In this setup bids are placed in terms of a unit of account. Let K_i be a positive integer denoting the number of trading posts for each commodity $i = 1, 2, \ldots, L$. in this way the structure of the market game underlying this economy is characterized by the vector $\mathbf{k} = (K_i)_{i=1}^L$.

The strategy set of each agent consists of buy and/or sell actions in each trading post:

$$S_h^{\mathbf{k}} = \{(b_h, q_h) \in \prod_{i=1}^{L} \Re_+^{K_i} \times \prod_{i=1}^{L} \Re_+^{K_i} \ : \ \sum_{s=1}^{K_i} q_h^{i,s} \leq e_h^i,\ i = 1, 2, \ldots, L\}.$$

Given a strategy profile let $B^{i,s} = \sum_{h \in H} b_h^{i,s}$ and $Q^{i,s} = \sum_{h \in H} q_h^{i,s}$. Also for each $h \in H$ define $B_{-h}^{i,s} = \sum_{n \neq h} b_n^{i,s}$, $Q_{-h}^{i,s} = \sum_{n \neq h} q_n^{i,s}$. Transactions in each trading post clear through the price $p^{i,s} = B^{i,s}/Q^{i,s}$. A commodity allocation is determined as follows: For each $h \in H$ and $i = 1, 2, \ldots, L$:

$$x_h^i = \begin{cases} e_h^i \Leftrightarrow \sum_{s=1}^{K_i} q_h^{i,s} + \sum_{s=1}^{K_i} \frac{b_h^{i,s}}{p^{i,s}} & \text{if } \sum_{i=1}^{L} \sum_{s=1}^{K_i} p^{i,s} \cdot q_h^{i,s} \geq \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} \\ 0 & \text{if } \sum_{i=1}^{L} \sum_{s=1}^{K_i} p^{i,s} \cdot q_h^{i,s} < \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} \end{cases}$$

where it is postulated that all divisions by zero equal zero[2].

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[1] i.e., if u represents $\succeq$ then for all $x \in \Re^L_{++}$, $\partial u_h / \partial x^i > 0$ for each $i = 1, 2, \ldots, L$

[2] It should be emphasized that this is a benchmark model. The restrictions on offer strategies could be written differently. For example one might allow individuals to make offers up to their whole endowment in each trading post. Other restrictions with different implications are also conceivable.

Consumers are viewed as solving the problem:

$$\max \left\{ u_h \left(\left(x_h^i \left(\left(b_h^{i,s}, q_h^{i,s}, B_{-h}^{i,s}, Q_{-h}^{i,s} \right)_{s=1}^{K_i} \right) \right)_{i=1}^{L} \right) \; : \; (b_h, q_h) \in S_h^{\mathbf{k}} \right\} \tag{1}$$

An equilibrium is defined as a profile $\left\{ (b_h, q_h) \in S_h^{\mathbf{k}} \, : \, h \in H \right\}$ that forms a Nash equilibrium.

Notice that, due to the bankruptcy rule above, in an equilibrium with positive bids and offers individuals can be viewed as solving the following problem:

$$\max \left\{ u_h \left(\left(x_h^i \left(\left(b_h^{i,s}, q_h^{i,s}, B_{-h}^{i,s}, Q_{-h}^{i,s} \right)_{s=1}^{K_i} \right) \right)_{i=1}^{L} \right) \; : \; \sum_{i=1}^{L} \sum_{s=1}^{K_i} p^{i,s} \cdot q_h^{i,s} \geq \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} \right\} \tag{2}$$

Given an economy $\mathcal{E}$ it will be useful to denote by $\mathcal{E}^{\mathbf{k}}$ the market game which is characterized by $\mathbf{k} = \{K_i\}_{i=1}^{L}$ trading posts. According to this notation $\mathcal{E}^{\mathbf{1}}$, where $\mathbf{1}$ is the L dimensional vector with coordinates equal to 1, is the standard market game with a single trading post per commodity. Furthermore, we denote by $\mathbf{NE}(\mathcal{E}^{\mathbf{k}})$ the set of Nash equilibrium strategy profiles of the game $\mathcal{E}^{\mathbf{k}}$ and by $\mathbf{E}(\mathcal{E}^{\mathbf{k}})$ the set of consumption allocations that correspond to the elements of $\mathbf{NE}(\mathcal{E}^{\mathbf{k}})$.

We report here two elementary facts that will be useful in the development. In what follows we consider an equilibrium of our economy, where at least two trading posts are active, i.e., prices are positive and there is trade. Let $z_h^{i,s} = b_h^{i,s}/p^{i,s} - q_h^{i,s}$ denote the net trade in commodity i from trading post s for an individual $h \in H$.

Fact 2.1 *Consider a budget feasible profile* (b, q)*. If* $b_h^{i,s} \cdot q_h^{i,s} > 0$*, there is a budget feasible* $(\hat{b}_h, \hat{q}_h)$ *with* $\hat{b}_h^{i,s} \cdot \hat{q}_h^{i,s} = 0$ *such that net trades and clearing price remain unchanged. Conversely, if* $b_h^{i,s} \cdot q_h^{i,s} = 0$ *and* $b_h^{i,s} + q_h^{i,s} > 0$ *there is a budget feasible* $(\hat{b}_h, \hat{q}_h)$ *with* $\hat{b}_h^{i,s} \cdot \hat{q}_h^{i,s} > 0$*, which results in the same net trades and clearing price.*

Proof:

See [5].

Fact 2.2 *Consider a budget feasible profile* (b, q)*, where* $b_h^{i,s} = q_h^{i,s} = 0$ *and* $p^{i,s} > 0$*. Then there is a budget feasible* $(\hat{b}_h, \hat{q}_h)$ *with* $\hat{b}_h^{i,s} \cdot \hat{q}_h^{i,s} > 0$*, which results in the same net trades and clearing price.*

Proof:

This is a simple consequence of the fact that every individual can be viewed as making a 'wash' sale in the trading posts where he is non-active. Suppose that for some $h \in H$ we have $b_h^{i,s} = q_h^{i,s} = 0$. Given $p^{i,s} = \frac{B_{-h}^{i,s}}{Q_{-h}^{i,s}} > 0$, this agent can be viewed as using a strategy $\hat{b}_h^{i,s} \cdot \hat{q}_h^{i,s} > 0$ such that $\frac{\hat{b}_h^{i,s}}{\hat{q}_h^{i,s}} = \frac{B_{-h}^{i,s}}{Q_{-h}^{i,s}}$. Notice that in this way $z_h^{i,s} = 0$ and furthermore $p^{i,s} = \frac{B_{-h}^{i,s}}{Q_{-h}^{i,s}} = \frac{B_{-h}^{i,s}+\hat{b}_h^{i,s}}{Q_{-h}^{i,s}+\hat{q}_h^{i,s}}$. In other words the allocation and price remain the same with this transformation in the strategy of individual h. It is easy to verify that the new strategy is also budget feasible $\square$

3 Characterization of equilibria

In this section we present some elementary facts about equilibria with many trading posts. We begin with the following proposition that characterizes equilibrium prices in two trading posts for a commodity:

Proposition 3.1 *In equilibrium, the prices between every pair of trading posts (s, r) of a commodity i, must satisfy the following (non-arbitrage) condition:*

$$\left(p^{i,s}\right)^2 = \frac{B_{-h}^{i,s}}{Q_{-h}^{i,s}} \cdot \frac{Q_{-h}^{i,r}}{B_{-h}^{i,r}} \cdot \left(p^{i,r}\right)^2, \ \forall h \in H$$

Proof:
Notice that the statement is trivially true if one or both prices are zero, so it remains to prove it for the case where $p^{i,s} \cdot p^{i,r} > 0$.
Consider any individual $h \in H$. By facts 2.2 and 2.1 respectively, this agent can be considered active on both trading posts r and s for commodity i and in particular, active on both sides in each post. Fix one such strategy $\left(\bar{b}_h, \bar{q}_h\right)$ that is best response to (B_{-h}, Q_{-h}) and denote by $\bar{B}$ and $\bar{Q}$ the corresponding aggregates. Taking the total differential of the distribution rule we obtain:

$$dx_h^i = \sum_{s=1}^{K_i} \frac{B_{-h}^{i,s} \cdot \bar{Q}^{i,s}}{\left(\bar{B}^{i,s}\right)^2} \cdot db_h^{i,s} \Leftrightarrow \sum_{s=1}^{K_i} \frac{B_{-h}^{i,s}}{\bar{B}^{i,s}} \cdot dq_h^{i,s} \tag{3}$$

Also by totally differentiating the budget constraint we obtain:

$$\sum_{s=1}^{K_i} \frac{\bar{B}^{i,s} \cdot Q_{-h}^{i,s}}{\left(\bar{Q}^{i,s}\right)^2} \cdot dq_h^{i,s} \Leftrightarrow \sum_{s=1}^{K_i} \frac{Q_{-h}^{i,s}}{\bar{Q}^{i,s}} \cdot db_h^{i,s} = 0 \tag{4}$$

Solving (4) for $db_h^{i,r}$ yields:

$$db_h^{i,r} = \sum_{s=1}^{K_i} \frac{\bar{B}^{i,s} \cdot Q_{-h}^{i,s} \cdot \bar{Q}^{i,r}}{\left(\bar{Q}^{i,s}\right)^2 \cdot Q_{-h}^{i,r}} \cdot dq_h^{i,s} \Leftrightarrow \sum_{s \neq r} \frac{Q_{-h}^{i,s} \cdot \bar{Q}^{i,r}}{\bar{Q}^{i,s} \cdot Q_{-h}^{i,r}} \cdot db_h^{i,s} \tag{5}$$

Substituting (5) into (3) we further obtain:

$$dx_h^i = \sum_{s \neq r} \left[\frac{B_{-h}^{i,s} \bar{Q}^{i,s}}{\left(\bar{B}^{i,s}\right)^2} \Leftrightarrow \frac{Q_{-h}^{i,s} B_{-h}^{i,r} \left(\bar{Q}^{i,r}\right)^2}{\bar{Q}^{i,s} Q_{-h}^{i,r} \left(\bar{B}^{i,r}\right)^2} \right] db_h^{i,s} + \sum_{s \neq r} \left[\frac{\bar{B}^{i,s} Q_{-h}^{i,s} \left(\bar{Q}^{i,r}\right)^2}{\left(\bar{Q}^{i,s}\right)^2 Q_{-h}^{i,r} \bar{B}^{i,r}} \Leftrightarrow \frac{B_{-h}^{i,s}}{\bar{B}^{i,s}} \right] dq_h^{i,s}$$

The equation above describes the changes in the final holdings of commodity i, for feasible 'shifts' in the bids and offers on the K_i trading posts. Now, in equilibrium it must be the case that $dx^i \leq 0$ for all $db_h^{i,s}$ and $dq_h^{i,s}$ where $s \neq r$. Hence, it must be:

$$\frac{B_{-h}^{i,s} \bar{Q}^{i,s}}{\left(\bar{B}^{i,s}\right)^2} \Leftrightarrow \frac{Q_{-h}^{i,s} B_{-h}^{i,r} \left(\bar{Q}^{i,r}\right)^2}{\bar{Q}^{i,s} Q_{-h}^{i,r} \left(\bar{B}^{i,r}\right)^2} = 0, \quad \forall s \neq r$$

which is equivalent to:

$$\left(p^{i,s}\right)^2 = \frac{B_{-h}^{i,s}}{Q_{-h}^{i,s}} \cdot \frac{Q_{-h}^{i,r}}{B_{-h}^{i,r}} \cdot \left(p^{i,r}\right)^2 \tag{6}$$

Furthermore, notice that the conclusion is independent of the best response chosen for individual h. Since the same must be true for all individuals our claim is proved $\square$

Corollary 3.1 *In equilibrium, for each pair of individuals $h \neq k$ and each pair of trading posts s, r of commodity i,* $\frac{B_{-h}^{i,s}}{Q_{-h}^{i,s}} \cdot \frac{Q_{-h}^{i,r}}{B_{-h}^{i,r}} = \frac{B_{-k}^{i,s}}{Q_{-k}^{i,s}} \cdot \frac{Q_{-k}^{i,r}}{B_{-k}^{i,r}}$

Remark 3.1 Notice that the above condition implies that there are at most $H \Leftrightarrow 1$ independent equations per pair of trading posts for each commodity, that must be satisfied in equilibrium.

Finally, we have the following conclusion regarding the relationship between the prices in two trading posts for a commodity.

Corollary 3.2 *In equilibrium,* $p^{i,s} = p^{i,r} \Leftrightarrow \frac{B_{-h}^{i,s}}{Q_{-h}^{i,s}} = \frac{B_{-h}^{i,r}}{Q_{-h}^{i,r}}$, *for some $h \in H$.*

Remark 3.2 It should be clear from the above corollary that in an atomless economy prices across all active posts would be uniform. However, if we consider sequences of economies with an increasing number of agents, the *asymptotic convergence* of price distributions across posts needs some careful consideration.

Corollary 3.3 *If in equilibrium $z_h^{i,r} = z_h^{i,s} = 0$ for some $h \in H$ then $p^{i,s} = p^{i,r}$.*

Notice that the above results do not guarantee the equality of prices in two posts in general. In fact, we have the following result.

Proposition 3.2 *If in equilibrium we have $z_h^{i,s} \cdot z_h^{i,r} \leq 0$, where $\left|z_h^{i,s}\right| + \left|z_h^{i,r}\right| \neq 0$ for some $h \in H$, then $p^{i,r} \neq p^{i,s}$.*

Proof:
Suppose that $z_h^{i,s} \leq 0$ and $z_h^{i,r} > 0$. By fact (2.1) it can be assumed that $b_h^{i,r} > 0$, $q_h^{i,s} \geq 0$ and $b_h^{i,s} = q_h^{i,r} = 0$. It follows that $p^{i,s} \leq \frac{B_{-h}^{i,s}}{Q_{-h}^{i,s}}$ and $p^{i,r} > \frac{B_{-h}^{i,r}}{Q_{-h}^{i,r}}$. Using these two inequalities along with proposition (3.1) we obtain the following:

$$\left(p^{i,r}\right)^2 = \frac{B_{-h}^{i,r}}{Q_{-h}^{i,r}} \cdot \frac{Q_{-h}^{i,s}}{B_{-h}^{i,s}} \cdot \left(p^{i,s}\right)^2 \leq \frac{1}{p^{i,s}} \cdot \frac{B_{-h}^{i,r}}{Q_{-h}^{i,r}} \cdot \left(p^{i,s}\right)^2 < p^{i,r} \cdot p^{i,s}$$

Thus, we conclude that $p^{i,r} < p^{i,s}$ $\square$

Corollary 3.4 *There is no equilibrium where for some pair of agents $h, k \in H$ and some pair of posts r, s for a commodity i, we have that $z_h^{i,r} \cdot z_k^{i,r} < 0$ and $z_h^{i,s} = z_k^{i,s} = 0$.*

Proof:
Suppose that such an equilibrium exists and let $p^{i,r}, p^{i,s}$ be the clearing prices in the two posts. Without loss of generality suppose that $z_h^{i,r} > 0$ and $z_k^{i,r} < 0$. An application of proposition 3.2 to agent h implies $p^{i,r} < p^{i,s}$. A similar application to agent k implies $p^{i,r} > p^{i,s}$, a contradiction $\square$

Remark 3.3 The last corollary excludes the possibility of equilibria where two disjoint subsets of individuals trade a commodity in different posts.

The proposition above gives a sufficient condition that guarantees the inequality of prices in two trading posts. It is not clear however that indeed there exist equilibria where this condition holds. Therefore, in order to highlight the significance of multiple trading posts, we provide an example in the next section, which demonstrates the possibility of distinct equilibrium prices for some commodities.

4 Equilibria with non-uniform prices

4.1 An example

The example that follows features an equilibrium with *two distinct positive* prices for each commodity. In this way the idea of multiple trading posts becomes meaningful.

Our example consists of three agents $H = 1, 2, 3$ and three goods $L = 1, 2, 3$. We postulate two trading posts for each commodity. The consumption set of each agent is thus $\Re_+^3$. For reasons that will become clear shortly we do not specify the utility functions of agents at this point. Finally, endowments are given as follows:

$$\mathbf{e}_1 = (55/9, 123/9, 92/9)\,, \mathbf{e}_2 = (123/9, 92/9, 55/9)\,, \mathbf{e}_3 = (92/9, 55/9, 123/9)$$

First consider the market for commodity 1. It can be verified that the following strategy profile satisfies equation (6), so given the strategies of any two agents it is impossible for the remaining agent to increase his net trade in commodity 1 by bids and offers across posts:

$$\begin{aligned}
\left(b_1^{11}, b_2^{11}, b_3^{11}\right) &= \left(\left(\frac{56}{54}\right)^2 \cdot \frac{5}{9}, \left(\frac{56}{54}\right)^2 \cdot \frac{3}{9}, \left(\frac{56}{54}\right)^2 \cdot \frac{1}{9}\right) \\
\left(b_1^{12}, b_2^{12}, b_3^{12}\right) &= \left(\frac{4}{9}, 0, \frac{4}{9}\right) \\
\left(q_1^{11}, q_2^{11}, q_3^{11}\right) &= (2, 4, 1) \\
\left(q_1^{12}, q_2^{12}, q_3^{12}\right) &= (1, 2, 3)
\end{aligned}$$

For this profile of orders in the two trading posts for commodity 1 we have:

$$p^{11} = \left(\frac{56}{54}\right)^2 \cdot \frac{1}{7} \ \text{ and } \ p^{12} = \frac{8}{54}$$

Thus, the net trades of each agent in each trading post are as follows:

$$\left(z_1^{11}, z_1^{12}\right) = \left(\frac{17}{9}, 2\right), \left(z_2^{11}, z_2^{12}\right) = \left(-\frac{15}{9}, -2\right), \left(z_3^{11}, z_3^{12}\right) = \left(-\frac{2}{9}, 0\right)$$

Hence, with this profile of bids and offers, the consumption allocation of commodity 1 accruing to each agent is as follows:

$$\left(x_1^1, x_2^1, x_3^1\right) = (10, 10, 10)$$

By rotating the indices in the strategy profile above we can do the same in the markets for commodities 2 and 3. Thus, we have:

-For commodity 2:

$$\begin{aligned}
\left(b_1^{21}, b_2^{21}, b_3^{21}\right) &= \left(\left(\frac{56}{54}\right)^2 \cdot \frac{3}{9}, \left(\frac{56}{54}\right)^2 \cdot \frac{1}{9}, \left(\frac{56}{54}\right)^2 \cdot \frac{5}{9}\right) \\
\left(b_1^{22}, b_2^{22}, b_3^{22}\right) &= \left(0, \frac{4}{9}, \frac{4}{9}\right) \\
\left(q_1^{21}, q_2^{21}, q_3^{21}\right) &= (4, 1, 2) \\
\left(q_1^{22}, q_2^{22}, q_3^{22}\right) &= (2, 3, 1)
\end{aligned}$$

The corresponding prices in the two trading posts for commodity 2 are:

$$p^{21} = \left(\frac{56}{54}\right)^2 \cdot \frac{1}{7} \ \text{ and } \ p^{22} = \frac{8}{54}$$

-For commodity 3:

$$\begin{aligned}
\left(b_1^{31}, b_2^{31}, b_3^{31}\right) &= \left(\left(\frac{56}{54}\right)^2 \cdot \frac{1}{9}, \left(\frac{56}{54}\right)^2 \cdot \frac{5}{9}, \left(\frac{56}{54}\right)^2 \cdot \frac{3}{9}\right) \\
\left(b_1^{32}, b_2^{32}, b_3^{32}\right) &= \left(\frac{4}{9}, \frac{4}{9}, 0\right) \\
\left(q_1^{31}, q_2^{31}, q_3^{31}\right) &= (1, 2, 4) \\
\left(q_1^{32}, q_2^{32}, q_3^{32}\right) &= (3, 1, 2)
\end{aligned}$$

As before prices in the two trading posts for commodity 3 are:

$$p^{31} = \left(\frac{56}{54}\right)^2 \cdot \frac{1}{7} \text{ and } p^{32} = \frac{8}{54}$$

Thus, each agent ends up with consumption:

$$\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = (10, 10, 10)$$

Furthermore, from the way the strategies have been constructed, the proposed strategy profile is budget feasible for each agent $h = 1, 2, 3$:

$$\begin{aligned}
\sum_{s=1}^{2}\sum_{i=1}^{3} p^{is} q_h^{is} &= p^{i1}\sum_{i=1}^{3} q_h^{i1} + p^{i2}\sum_{i=1}^{3} q_h^{i2} \\
&= p^{i1}\sum_{h=1}^{3} q_h^{i1} + p^{i2}\sum_{h=1}^{3} q_h^{i2} \\
&= p^{i1}Q^{i1} + p^{i2}Q^{i2} \\
&= \sum_{h=1}^{3} b_h^{i1} + \sum_{h=1}^{3} b_h^{i2} \\
&= \sum_{i=1}^{3} b_h^{i1} + \sum_{i=1}^{3} b_h^{i2} \\
&= \sum_{s=1}^{2}\sum_{i=1}^{3} b_h^{is}
\end{aligned}$$

It remains to find utility functions for each agent, so that the above profile of strategies is a Nash equilibrium. To this end recall that the first order conditions of (2) require that for each pair of commodities i and j, we must have[3]:

$$\frac{\partial u_h/\partial x_h^i}{\partial u_h/\partial x_h^j} = \frac{Q_{-h}^{i1}(B^{i1})^2 B_{-h}^{j1}(Q^{j1})^2}{B_{-h}^{i1}(Q^{i1})^2 Q_{-h}^{j1}(B^{j1})^2} \tag{7}$$

$$\frac{\partial u_h/\partial x_h^i}{\partial u_h/\partial x_h^j} = \frac{Q_{-h}^{i2}(B^{i2})^2 B_{-h}^{j1}(Q^{j1})^2}{B_{-h}^{i2}(Q^{i2})^2 Q_{-h}^{j1}(B^{j1})^2} \tag{8}$$

[3]It can be shown (see [4] proposition 2.4) that those conditions are also sufficient.

$$\frac{\partial u_h/\partial x_h^i}{\partial u_h/\partial x_h^j} = \frac{Q_{-h}^{i1}(B^{i1})^2 B_{-h}^{j2}(Q^{j2})^2}{B_{-h}^{i1}(Q^{i1})^2 Q_{-h}^{j2}(B^{j2})^2} \tag{9}$$

$$\frac{\partial u_h/\partial x_h^i}{\partial u_h/\partial x_h^j} = \frac{Q_{-h}^{i2}(B^{i2})^2 B_{-h}^{j2}(Q^{j2})^2}{B_{-h}^{i2}(Q^{i2})^2 Q_{-h}^{j2}(B^{j2})^2} \tag{10}$$

$$\left(\frac{B^{i1}}{Q^{i1}}\right)^2 = \frac{B_{-h}^{i1}}{Q_{-h}^{i1}} \cdot \frac{Q_{-h}^{i2}}{B_{-h}^{i2}} \cdot \left(\frac{B^{i2}}{Q^{i2}}\right)^2 \tag{11}$$

$$\left(\frac{B^{j1}}{Q^{j1}}\right)^2 = \frac{B_{-h}^{j1}}{Q_{-h}^{j1}} \cdot \frac{Q_{-h}^{j2}}{B_{-h}^{j2}} \cdot \left(\frac{B^{j2}}{Q^{j2}}\right)^2 \tag{12}$$

It easy to show that in view of (11) and (12), which the proposed profile of strategies satisfies by construction, the marginal rate of substitution between commodities i and j that satisfies any one of the above equations, will automatically satisfy the rest also. Hence, in order to determine the marginal utilities we need only solve the following system:

$$\frac{\partial u_h/\partial x_h^i}{\partial u_h/\partial x_h^j} = \frac{Q_{-h}^{i1}(B^{i1})^2 B_{-h}^{j1}(Q^{j1})^2}{B_{-h}^{i1}(Q^{i1})^2 Q_{-h}^{j1}(B^{j1})^2}, \quad \text{where } i \neq j.$$

However, only two of the three equations of this system are independent. Thus, we can fix the marginal utility of, say, commodity 3 at an arbitrary level and calculate the corresponding marginal utilities of commodities 1 and 2 that solve the system. Certainly, the same can be done for each agent. In this way any utility function which is concave and its gradient takes the values calculated above at the point $(10, 10, 10)$, will be a solution to our problem. Upon substitution of the values for bids and offers we have:

$$\text{Agent 1}: \frac{\partial u_1}{\partial x_1^1} = \frac{5}{3} \cdot \frac{\partial u_1}{\partial x_1^3}, \; \frac{\partial u_1}{\partial x_1^2} = \frac{2}{3} \cdot \frac{\partial u_1}{\partial x_1^3}$$

$$\text{Agent 2}: \frac{\partial u_2}{\partial x_2^1} = \frac{2}{5} \cdot \frac{\partial u_2}{\partial x_2^3}, \; \frac{\partial u_2}{\partial x_2^2} = \frac{3}{5} \cdot \frac{\partial u_2}{\partial x_2^3}$$

$$\text{Agent 3}: \frac{\partial u_3}{\partial x_3^1} = \frac{3}{2} \cdot \frac{\partial u_3}{\partial x_3^3}, \; \frac{\partial u_3}{\partial x_3^2} = \frac{5}{2} \cdot \frac{\partial u_3}{\partial x_3^3}$$

Now, by assigning values to the marginal utility of commodity 3 as well as the initial and final utility levels -taking care so that the final utility is higher than the initial one- we have, for each agent, a system of three partial differential equations with three conditions, which characterizes the set of functions that solves our problem. Thus, we have a characterization of the family of utility functions for which the proposed allocation is a Nash equilibrium. This fact also demonstrates in an informal way the robustness of our example. For concreteness we report one solution:

$$
\begin{aligned}
u_1\left(x_1^1, x_1^2, x_1^3\right) &= 5log(x_1^1) + 2log(x_1^2) + 3log(x_1^3) \\
u_2\left(x_2^1, x_2^2, x_2^3\right) &= 2log(x_2^1) + 3log(x_2^2) + 5log(x_2^3) \\
u_1\left(x_3^1, x_3^2, x_3^3\right) &= 3log(x_3^1) + 5log(x_3^2) + 2log(x_3^3)
\end{aligned}
$$

4.2 Asymptotic behavior of price disparities

Clearly price disparities are due to the finite number of traders and the presence of multiple trading posts. It is natural then to wonder in which way price disparities are related to the number of agents, the number of trading posts or any other primitives of the economy. The following results are devoted to this question.

Let $(b, q) \in \mathbf{NE}(\mathcal{E}^{\mathbf{k}})$ be any non-uniform equilibrium. It can be assumed without loss of generality (see proposition 5.2 in the next section) that trading posts with the same price have been consolidated. Thus, $\mathcal{E}^{\mathbf{k}}$, where $\mathbf{k}$ is such that $K_i > 2$ for some $i = 1, 2, \ldots, L$, is the game with the minimum number of trading posts for which $x(b, q) \in \mathbf{E}(\mathcal{E}^{\mathbf{k}})$. We may further assume, by renumbering posts if necessary, that $p^{i,1} = min\left\{p^{i,r} : \;\; r = 1, 2, \ldots, K_i\right\}$. Define:

$$
g^i(b, q) = sup\left\{\frac{p^{i,r}}{p^{i,1}} \Leftrightarrow 1 : \;\; r = 1, 2, \ldots, K_i\right\}
$$

We have the following lemma:

Lemma 4.1 $g^i(b, q) \leq sup\left\{\frac{b_h^{i,1}}{B_{-h}^{i,1}} + \frac{q_h^{i,r}}{Q_{-h}^{i,r}} + \frac{b_h^{i,1}}{B_{-h}^{i,1}} \cdot \frac{q_h^{i,r}}{Q_{-h}^{i,r}} : r = 1, 2, \ldots, K_i\right\}$, *for all* $h \in H$.

Proof: Fix one $h \in H$. By simple manipulation of 3.1 we have that:

$$
\begin{aligned}
\frac{p^{i,r}}{p^{i,1}} &= \frac{Q^{i,r}}{B^{i,r}} \cdot \frac{B_{-h}^{i,r}}{Q_{-h}^{i,r}} \cdot \frac{Q_{-h}^{i,1}}{B_{-h}^{i,1}} \cdot \frac{B^{i,1}}{Q^{i,1}} \\
&= \frac{Q_{-h}^{i,1}}{Q^{i,1}} \cdot \frac{B_{-h}^{i,r}}{B^{i,r}} \cdot \left[1 + \frac{b_h^{i,1}}{B_{-h}^{i,1}} + \frac{q_h^{i,r}}{Q_{-h}^{i,r}} + \frac{b_h^{i,1}}{B_{-h}^{i,1}} \cdot \frac{q_h^{i,r}}{Q_{-h}^{i,r}}\right] \\
&\leq 1 + \frac{b_h^{i,1}}{B_{-h}^{i,1}} + \frac{q_h^{i,r}}{Q_{-h}^{i,r}} + \frac{b_h^{i,1}}{B_{-h}^{i,1}} \cdot \frac{q_h^{i,r}}{Q_{-h}^{i,r}}
\end{aligned}
$$

Since the same inequality is true for each h, we conclude that

$$
\forall h \in H, \;\; \frac{p^{i,r}}{p^{i,1}} \Leftrightarrow 1 \leq sup\left\{\frac{b_h^{i,1}}{B_{-h}^{i,1}} + \frac{q_h^{i,r}}{Q_{-h}^{i,r}} + \frac{b_h^{i,1}}{B_{-h}^{i,1}} \cdot \frac{q_h^{i,r}}{Q_{-h}^{i,r}} : r = 1, 2, \ldots, K_i\right\}.
$$

Finally, the conclusion of the lemma follows by taking the supremum of the lefthand side of the last inequality $\square$

Theorem 4.1 *Let $(b,q) \in \mathbf{NE}(\mathcal{E}^{\mathbf{k}})$, where $\mathbf{k}$ is such that $K_i \geq 2$ for some $i = 1, 2, \ldots, L$. We have:*

$$(i) \qquad \text{Given } \epsilon > 0, \text{ we have: } \#H \geq (K_i + 1)\left(\frac{\sqrt{1+\epsilon}}{\sqrt{1+\epsilon} - 1}\right) \Rightarrow g^i(b,q) \leq \epsilon$$

$$(ii) \qquad \text{If } \#H > K_i + 1 \quad \text{then} \quad g^i(b,q) \leq \frac{(K_i+1)(2 \cdot \#H - K_i - 1)}{(\#H - K_i - 1)^2}$$

Proof: We show the contrapositive of (i). Suppose that $g^i(b,q) > \epsilon$.
For each agent $h \in H$ let $\theta_h = \max \left\{ \frac{b_h^{i,1}}{B_{-h}^{i,1}}, \left(\frac{q_h^{i,r}}{Q_{-h}^{i,r}}\right)_{r=1}^{K_i} \right\}$
From the lemma above we have that $g^i(b,q) \leq 2 \cdot \theta_h + \theta_h^2$ for all $h \in H$, from which it follows that $\theta_h > -1 + \sqrt{1+\epsilon} = \eta(\epsilon)$, $\forall h \in H$. From the definition of θ_h it follows that for each $h \in H$ either $\frac{b_h^{i,1}}{B_{-h}^{i,1}} > \eta(\epsilon)$ or $\frac{q_h^{i,r}}{Q_{-h}^{i,r}} > \eta(\epsilon)$ for some $r = 1, 2, \ldots, K_i$. Thus, we have:

$$\forall h \in H, \text{ either } \frac{b_h^{i,1}}{B^{i,1}} > \frac{\eta(\epsilon)}{1+\eta(\epsilon)} \text{ or } \frac{q_h^{i,r}}{Q^{i,r}} > \frac{\eta(\epsilon)}{1+\eta(\epsilon)} \text{ for some } r = 1, 2, \ldots, K_i.$$

Let $V = \left\{ h \in H : \frac{b_h^{i,1}}{B^{i,1}} > \frac{\eta(\epsilon)}{1+\eta(\epsilon)} \right\}$ and $V_r = \left\{ h \in H : \frac{q_h^{i,r}}{Q^{i,r}} > \frac{\eta(\epsilon)}{1+\eta(\epsilon)} \right\}$.
We have: $\#V \frac{\eta(\epsilon)}{1+\eta(\epsilon)} < \sum_{h \in V} \frac{b_h^{i,1}}{B^{i,1}} \leq 1$, so $\#V < \frac{1+\eta(\epsilon)}{\eta(\epsilon)}$. Similarly $\#V_r < \frac{1+\eta(\epsilon)}{\eta(\epsilon)}$.
Recall now that $H = \left(\bigcup_{r=1}^{K_i} V_r\right) \cup V$, so $\#H \leq \sum_{r=1}^{K_i} \#V_r + \#V < (K_i+1)\left(\frac{1+\eta(\epsilon)}{\eta(\epsilon)}\right)$.
Thus, we have shown that: $g^i(b,q) > \epsilon \Rightarrow \#H < (K_i+1)\left(\frac{1+\eta(\epsilon)}{\eta(\epsilon)}\right)$.

Finally, the second claim of the theorem follows now directly by solving the last inequality for ϵ, taking into account the hypothesis $\#H > K_i + 1$. In this way we have for any $\epsilon > 0$:

$$g^i(b,q) > \epsilon \Rightarrow \frac{(K_i+1)(2 \cdot \#H - K_i - 1)}{(\#H - K_i - 1)^2} > \epsilon,$$

which implies the statement of claim (ii) $\square$

The implication of the last theorem for the asymptotic behavior of price disparities is crystalized in the corollary that follows[4].

Consider a sequence of economies and associated market games $\mathcal{E}_n^{\mathbf{k}_n}$ where $\#H^n \to \infty$ and a sequence $(b_n, q_n) \in \mathbf{NE}(\mathcal{E}_n^{\mathbf{k}_n})$. Define $z_n = \#H^n / K^n + 1$, where $K^n = max\{K_i^n : i = 1, 2, \ldots, L\}$.

Corollary 4.1 *$z_n \to \infty \Rightarrow g^i(b_n, q_n) \to 0$ for all $i = 1, 2, \ldots, L$.*

Proof: Since $z_n \to \infty$ we have that eventually $z_n > 1$, so according to the above theorem

$$\forall i, \; g^i(b_n, q_n) \leq \frac{2z_n - 1}{(z_n - 1)^2}.$$

[4] The statement of this corollary has been suggested to me by J-F Mertens.

Thus, as $z_n \to \infty$ we have $g^i(b_n, q_n) \to 0$ for each $i = 1, 2, \ldots, L$ $\square$

We conclude that 'large' price disparities across posts of a commodity are compatible only with a limited number of agents. As the number of agents becomes large then price disparities across posts of a commodity tend to zero, at a rate that depends only on the relative number of agents and trading posts, provided that along the sequence there are more agents than trading posts. In particular, the distribution of characteristics along the sequence is irrelevant! Although this conclusion might seem odd, the intuition behind it is clear: 'small' agents can take advantage of price disparities just as well as 'large' agents. Thus, the distribution of endowments and preferences should not be relevant to the degree of price variability. This result seems encouraging for the study of asymptotic convergence to the competitive model. Although from a formal point of view the above theorem is not quite an equivalence result, it does hint that the common limit of prices across all posts of a commodity will be competitive.

5 Equilibria with uniform prices

It can be shown that every equilibrium allocation of the standard model with a single post per commodity can be obtained as a uniform equilibrium of the model with multiple trading posts[5]. This fact ensures that we do not 'miss' any equilibria by augmenting the number of trading posts. We provide here a proof for a game with two trading posts per commodity. The proof can be extended to the general case via an inductive argument.

Consider the market games $\mathcal{E}^{\mathbf{1}}$ and $\mathcal{E}^{\mathbf{2}}$ (where $\mathbf{2}$ is the L dimensional vector with all coordinates equal to 2). Let $x \in \mathbf{E}(\mathcal{E}^{\mathbf{1}})$ be an equilibrium allocation corresponding to the strategy profile $\{(b_h, q_h) \in S_h^{\mathbf{1}} : h \in H\}$. For each index $i = 1, 2, \ldots, L$ choose $0 < t_i < 1$ and consider the profile $\{(\hat{b}_h, \hat{q}_h) \in S_h^{\mathbf{2}} : h \in H\}$ for the game $\mathcal{E}^{\mathbf{2}}$, where for each $h \in H$:

$$(\hat{b}_h^{i,s}, \hat{q}_h^{i,s}) = \begin{cases} t_i \cdot (b_h^i, q_h^i) & s = 1 \\ (1 \Leftrightarrow t_i) \cdot (b_h^i, q_h^i) & s = 2 \end{cases}$$

Notice that this strategy profile results in a uniform distribution of prices across posts for each commodity and the same commodity allocation x as in the game $\mathcal{E}^{\mathbf{1}}$.

Claim 5.1 *The profile of strategies $(\hat{b}_h, \hat{q}_h)_{h \in H}$ constructed above is an equilibrium for the game $\mathcal{E}^{\mathbf{2}}$. In particular $x \in \mathbf{E}(\mathcal{E}^{\mathbf{2}})$.*

Proof: See appendix.

Proceeding in the same way as above one can establish the following result:

[5]This fact is also an indirect existence proof for our model.

Proposition 5.1 *Given two vectors with positive integer coordinates* $\mathbf{k}, \mathbf{t} \in \Re^L$ *where* $\mathbf{t} \geq \mathbf{k}$*, we have* $\mathbf{E}(\mathcal{E}^{\mathbf{k}}) \subseteq \mathbf{E}(\mathcal{E}^{\mathbf{t}})$*.*

Remark 5.1 This result gives an alternative view to the appearance of trivial equilibria in this class of market games: If, according to our notation, we denote by $\mathcal{E}^{\mathbf{0}}$ the market game with no trading posts (hence autarky is the only equilibrium) then $\mathbf{E}(\mathcal{E}^{\mathbf{0}}) \subseteq \mathbf{E}(\mathcal{E}^{\mathbf{1}}) \subseteq \mathbf{E}(\mathcal{E}^{\mathbf{2}}) \subseteq \ldots$ In this way the no trade equilibrium is 'implanted' into all subsequent models.

The next result establishes that the converse inclusion holds if one confines attention the set of uniform equilibria.

Proposition 5.2 *Let* $x \in \mathbf{E}(\mathcal{E}^{\mathbf{t}})$ *be uniform,i.e., there is a unique price that clears all the trading posts where each commodity is traded. Then* $x \in \mathbf{E}(\mathcal{E}^{\mathbf{k}})$ *for all* $\mathbf{k} \leq \mathbf{t}$*. In particular,* $x \in \mathbf{E}(\mathcal{E}^{\mathbf{1}})$*.*

Proof: See appendix.

6 Conclusion

In summary we conclude that this much is true: Every equilibrium of the market game with a single post can be obtained as an equilibrium of the market game with multiple posts. Furthermore, given an equilibrium of a game with multiple posts, the consolidation of strategies over all posts with equal prices is an equilibrium for a model with fewer trading posts. In particular, the set of uniform price equilibria is invariant with respect to the number of posts. On the other hand there are equilibria of the game with multiple trading posts which cannot be captured by the market game with one trading post. These are equilibria where the 'law of one price' fails. The appearance of such equilibria is entirely due to the lack of perfectly competitive conditions in trade. As the number of agents increases the price variation across posts for a commodity becomes smaller.

Thus, imposing on the general exchange context any number of trading posts seems to be the source of two problems: First, the appearance of trivial equilibria which can be viewed as being inherited from the game with no trading posts. This observation hints that trivial equilibria appear because trading posts are imposed rather than evolve in the model. Second, that the set of equilibria of the model depends on the structure of trading posts that one imposes. In particular, the single trading post model excludes non-uniform equilibria.

The central message of this paper is that there is a need to elaborate on the concept of trading posts. The structure of trading posts determines the dimensionality of the strategy sets of agents in the underlying game. Thus, alternative structures of posts give rise to a variety of games that differ in their strategy spaces, so from an abstract game theoretic point of view it seems natural that the set of equilibria depends on the structure of posts. However, from an economic point of view this fact is extremely important because it affects the outcomes predicted by the economic model in intriguing ways, as the appearance of equilibria with non-uniform prices shows. In lack of a particular structure of posts suggested by the economic model, it is not clear which is the game underlying the economy. Thus, it is necessary to develop a foundation of the structure of trading posts, which in turn will determine the underlying game.

7 Appendix

Proof of Claim 5.1 Suppose the claim is not true. Then for some agent $h \in H$ there exists a strategy $(\beta_h, \theta_h) \in S_h$ so that:

$$u_h\left(\left(x_h^i\left(\beta_h^{i,s}, \theta_h^{i,s}, \hat{B}_{-h}^{i,s}, \hat{Q}_{-h}^{i,s}\right)_{s=1}^2\right)_{i=1}^L\right) > u_h\left(\left(x_h\left(\hat{b}_h^{i,s}, \hat{q}_h^{i,s}, \hat{B}_{-h}^{i,s}, \hat{Q}_{-h}^{i,s}\right)_{s=1}^2\right)_{i=1}^L\right)$$

Without loss of generality we may assume that for each commodity $i = 1, 2, \ldots, L$, $\beta_h^{i,1} \cdot \theta_h^{i,1} = 0$ and $\beta_h^{i,2} \cdot \theta_h^{i,2} = 0$ (otherwise we can find another feasible strategy for which this is true while, at the same time, prices, allocations and budget remain unchanged). Note that, given our assumptions on preferences and endowments, the bankruptcy rule ensures that:

$$\sum_{i=1}^{L}\sum_{s=1}^{2}\frac{\hat{B}_{-h}^{i,s}}{\hat{Q}_{-h}^{i,s}+\theta_h^{i,s}}\cdot\theta_h^{i,s} \geq \sum_{i=1}^{L}\sum_{s=1}^{2}\beta_h^{i,s} \tag{13}$$

Given (β_h, θ_h), consider the following sets of commodities: $L_1 = \left\{i : \beta_h^{i,1} > 0, \beta_h^{i,2} > 0\right\}$, $L_2 = \left\{i : \theta_h^{i,1} > 0, \theta_h^{i,2} > 0\right\}$, and $L_3 = \left\{i : \beta_h^{i,s} > 0, \theta_h^{i,r} > 0, s \neq r\right\}$.

- Step I Consider the commodities $i \in L_3$ (i.e., the commodities where the consumer is a net buyer in one trading post and a net seller in the other).
In this case form a new strategy $(\hat{\beta}_h, \hat{\theta}_h) \in S_h$ where for $i \notin L_3$ we have $\hat{\beta}_h^i = \beta_h^i$, $\hat{\theta}_h^i = \theta_h^i$ while, for $i \in L_3$, $\hat{\beta}_h^{i,s} = t_i\beta_h^{i,s}$, $\hat{\beta}_h^{i,r} = (1-t_i)\beta_h^{i,s}$, $\hat{\theta}_h^{i,s} = t_i\theta_h^{i,r}$, $\hat{\theta}_h^{i,r} = (1-t_i)\theta_h^{i,r}$. With this strategy the prices in the two posts of the ith commodity would become equal:

$$p^{i,s} = \frac{\hat{B}_{-h}^{i,s} + t_i\cdot\beta_h^{i,s}}{\hat{Q}_{-h}^{i,s} + t_i\cdot\theta_h^{i,r}} = \frac{t_i\cdot B_{-h}^{i} + t_i\cdot\beta_h^{i,s}}{t_i\cdot Q_{-h}^{i} + t_i\cdot\theta_h^{i,r}} = \frac{(1-t_i)\cdot B_{-h}^{i} + (1-t_i)\cdot\beta_h^{i,s}}{(1-t_i)\cdot Q_{-h}^{i} + (1-t_i)\cdot\theta_h^{i,r}} = p^{i,r}$$

Thus, with this strategy the net trade of consumer h is: $\beta_h^{i,s}\cdot\left(\frac{Q_{-h}^{i,s}+\theta_h^{i,r}}{B_{-h}^{i,s}+\beta_h^{i,s}}\right) - \theta_h^{i,r}$. Notice that this net trade is at least as big as the net trade resulting from the initial strategy, i.e.,

$$\beta_h^{i,s}\cdot\left(\frac{Q_{-h}^{i,s}+\theta_h^{i,r}}{B_{-h}^{i,s}+\beta_h^{i,s}}\right) - \theta_h^{i,r} \geq \beta_h^{i,s}\cdot\left(\frac{t_i\cdot Q_{-h}^{i,s}}{t_i\cdot B_{-h}^{i,s}+\beta_h^{i,s}}\right) - \theta_h^{i,r}$$

Therefore, by shifting a proportion of the bid (offer) from the first trading post to the second and at the same time shifting the same proportion of the offer (bid) from the second to the first, the consumer can achieve an allocation which is at least as good as the original (keeping the strategy fixed in the other commodities). Notice that doing so is budget feasible because:

$$\theta_h^{i,r}\cdot\left(\frac{B_{-h}^{i,s}+\beta_h^{i,s}}{Q_{-h}^{i,s}+\theta_h^{i,r}}\right) \geq \theta_h^{i,r}\cdot\left(\frac{(1-t_i)\cdot B_{-h}^{i,s}}{(1-t_i)\cdot Q_{-h}^{i,s}+\theta_h^{i,r}}\right)$$

We conclude then that if the consumer can improve over the first allocation by a selling-buying strategy, then he can do so by either selling or buying in both trading posts. Hence, we can assume without loss of generality that $L_3 = \emptyset$.

- Step II Consider now $i \in L_1$ (i.e., the commodities where the consumer is a net purchaser in both trading posts).
In this case the total net trade of commodity i is given by:

$$\beta_h^{i,1} \cdot \frac{\hat{Q}_{-h}^{i,1}}{\hat{B}_{-h}^{i,1} + \beta_h^{i,1}} + \beta_h^{i,2} \cdot \frac{\hat{Q}_{-h}^{i,2}}{\hat{B}_{-h}^{i,2} + \beta_h^{i,2}} = \beta_h^{i,1} \cdot \frac{t_i \cdot Q_{-h}^i}{t_i \cdot B_{-h}^i + \beta_h^{i,1}} + \beta_h^{i,2} \cdot \frac{(1-t_i) \cdot Q_{-h}^i}{(1-t_i) \cdot B_{-h}^i + \beta_h^{i,2}}$$

In the game $\mathcal{E}^1$ the same net trade could be achieved via a bid $\hat{\beta}_h^i$ that solves the equation:

$$\hat{\beta}_h^i \cdot \frac{Q_{-h}^i}{B_{-h}^i + \hat{\beta}_h^i} = \beta_h^{i,1} \cdot \frac{t_i \cdot Q_{-h}^i}{t_i \cdot B_{-h}^i + \beta_h^{i,1}} + \beta_h^{i,2} \cdot \frac{(1-t_i) \cdot Q_{-h}^i}{(1-t_i) \cdot B_{-h}^i + \beta_h^{i,2}}$$

which can be solved to yield:

$$\hat{\beta}_h^i = \frac{t_i \cdot (1-t_i) \cdot B_{-h}^i \cdot (\beta_h^{i,1} + \beta_h^{i,2}) + \beta_h^{i,1} \cdot \beta_h^{i,2}}{t_i \cdot (1-t_i) \cdot B_{-h}^i + (1-t_i)^2 \cdot \beta_h^{i,1} + t_i^2 \cdot \beta_h^{i,2}}$$

In other words the agent h could receive the same net trade of commodity i in the game $\mathcal{E}^1$ by adopting the above strategy. It can be easily verified that:

$$\beta_h^{i,1} + \beta_h^{i,2} \geq \hat{\beta}_h^i \tag{14}$$

- Step III Now consider the commodities $i \in L_2$ (i.e., those commodities for which the consumer is a net seller in both trading posts).

In this case the consumer's net trade is : $-(\theta_h^{i,1} + \theta_h^{i,2})$ and the receipts from those sales are:

$$\theta_h^{i,1} \cdot \frac{\hat{B}_{-h}^{i,1}}{\hat{Q}_{-h}^{i,1} + \theta_h^{i,1}} + \theta_h^{i,2} \cdot \frac{\hat{B}_{-h}^{i,2}}{\hat{Q}_{-h}^{i,2} + \theta_h^{i,2}} = \theta_h^{i,1} \cdot \frac{t_i \cdot B_{-h}^i}{t_i \cdot Q_{-h}^i + \theta_h^{i,1}} + \theta_h^{i,2} \cdot \frac{(1-t_i) \cdot B_{-h}^i}{(1-t_i) \cdot Q_{-h}^i + \theta_h^{i,2}}$$

In the game $\mathcal{E}^1$ the same revenue could be raised via the following offer:

$$\hat{\theta}_h^i \cdot \frac{B_{-h}^i}{Q_{-h}^i + \hat{\theta}_h^i} = \theta_h^{i,1} \cdot \frac{t_i \cdot B_{-h}^i}{t_i \cdot Q_{-h}^i + \theta_h^{i,1}} + \theta_h^{i,2} \cdot \frac{(1-t_i) \cdot B_{-h}^i}{(1-t_i) \cdot Q_{-h}^i + \theta_h^{i,2}}$$

This equation can be readily solved to obtain:

$$\hat{\theta}_h^i = \frac{t_i \cdot (1-t_i) \cdot Q_{-h}^i \cdot (\theta_h^{i,1} + \theta_h^{i,2}) + \theta_h^{i,1} \cdot \theta_h^{i,2}}{t_i \cdot (1-t_i) \cdot Q_{-h}^i + (1-t_i)^2 \cdot \theta_h^{i,1} + t_i^2 \cdot \theta_h^{i,2}}$$

It can be verified that:

$$\theta_h^{i,1} + \theta_h^{i,2} \geq \hat{\theta}_h^i \tag{15}$$

Therefore, this strategy would give the consumer a net trade in commodities $i \in L_2$ that would be at least as big as the one with offers in two trading posts.

-Step IV Consider now the strategy $\left(\hat{\beta}_h, \hat{\theta}_h\right) \in \Re_+^L$ which is defined as follows:

$$\left(\hat{\beta}_h, \hat{\theta}_h\right) = \begin{cases} \left(\frac{t_i \cdot (1-t_i) \cdot B^i_{-h} \cdot (\beta_h^{i,1} + \beta_h^{i,2}) + \beta_h^{i,1} \cdot \beta_h^{i,2}}{t_i \cdot (1-t_i) \cdot B^i_{-h} + (1-t_i)^2 \cdot \beta_h^{i,1} + t_i^2 \cdot \beta_h^{i,2}}, 0\right) & i \in L_1 \\ \left(0, \frac{t_i \cdot (1-t_i) \cdot Q^i_{-h} \cdot (\theta_h^{i,s} + \theta_h^{i,r}) + \theta_h^{i,s} \cdot \theta_h^{i,r}}{t_i \cdot (1-t_i) \cdot Q^i_{-h} + (1-t_i)^2 \cdot \theta_h^{i,s} + t_i^2 \cdot \theta_h^{i,r}}\right) & i \in L_2 \end{cases}$$

Using (13) and (14) from above, it turns out that this strategy is budget feasible in the game $\mathcal{E}^{\mathbf{1}}$:

$$\begin{aligned} \sum_{i=1}^{L} \left(\frac{B^i_{-h} + \hat{\beta}^i_h}{Q^i_{-h} + \hat{\theta}^i_h} \cdot \hat{\theta}^i_h \right) &= \sum_{i \in L_1} \left(\frac{B^i_{-h} + \hat{\beta}^i_h}{Q^i_{-h} + \hat{\theta}^i_h} \cdot \hat{\theta}^i_h \right) + \sum_{i \in L_2} \left(\frac{B^i_{-h} + \hat{\beta}^i_h}{Q^i_{-h} + \hat{\theta}^i_h} \cdot \hat{\theta}^i_h \right) \\ &= \sum_{i \in L_2} \left(\frac{B^i_{-h}}{Q^i_{-h} + \hat{\theta}^i_h} \right) \cdot \hat{\theta}^i_h \\ &= \sum_{i \in L_2} \left(\frac{t_i \cdot B^i_{-h}}{t_i \cdot Q^i_{-h} + \theta_h^{i,1}} \cdot \theta_h^{i,1} + \frac{(1-t_i) \cdot B^i_{-h}}{(1-t_i) \cdot Q^i_{-h} + \theta_h^{i,2}} \cdot \theta_h^{i,2} \right) \\ &\geq \sum_{i \in L_1} \left(\beta_h^{i,1} + \beta_h^{i,2} \right) \\ &\geq \sum_{i \in L_1} \hat{\beta}^i_h \\ &= \sum_{i=1}^{L} \hat{\beta}^i_h \end{aligned}$$

Recall that for each $i = 1, 2, \ldots, L$:

$$x_h^i \left(\hat{\beta}^i_h, \hat{\theta}^i_h, B^i_{-h}, Q^i_{-h} \right) \geq x_h^i \left(\left(\beta_h^{i,s}, \theta_h^{i,s}, \hat{B}^{i,s}_{-h}, \hat{Q}^{i,s}_{-h} \right)_{s=1}^{2} \right)$$

By the monotonicity of preferences we have that:

$$\begin{aligned} u_h \left(\left(x_h^i \left(\hat{\beta}^i_h, \hat{\theta}^i_h, B^i_{-h}, Q^i_{-h} \right) \right)_{i=1}^{L} \right) &\geq u_h \left(\left(x_h^i \left(\left(\beta_h^{i,s}, \theta_h^{i,s}, \hat{B}^{i,s}_{-h}, \hat{Q}^{i,s}_{-h} \right)_{s=1}^{2} \right) \right)_{i=1}^{L} \right) \\ &> u_h \left(\left(x_h^i \left(\left(\hat{b}_h^{i,s}, \hat{q}_h^{i,s}, \hat{B}^{i,s}_{-h}, \hat{Q}^{i,s}_{-h} \right)_{s=1}^{2} \right) \right)_{i=1}^{L} \right) \\ &= u_h \left(\left(x_h^i \left(b^i_h, q^i_h, B^i_{-h}, Q^i_{-h} \right) \right)_{i=1}^{L} \right) \end{aligned}$$

which contradicts the fact that $(b_h, q_h)_{h \in H}$ is a Nash equilibrium for the game $\mathcal{E}^{\mathbf{1}}$. So our original hypothesis is ruled out and our claim is proved $\square$

Proof of proposition 5.2:
We prove that $x \in \mathbf{E}(\mathcal{E}^{\mathbf{1}})$. The rest follows as a consequence of the previous result.

Let $(b,q) \in \mathbf{NE}(\mathcal{E}^{\mathbf{t}})$ be the profile of strategies which gives rise to the allocation x. By assumption: $p^{i,s} = p^{i,r} = p^i$ for $s \neq r$. Recall that by corollary 3.2 this implies that for each $h \in H$:

$$\frac{B_{-h}^{i,1}}{Q_{-h}^{i,1}} = \frac{B_{-h}^{i,2}}{Q_{-h}^{i,2}} = \ldots = \frac{B_{-h}^{i,T_i}}{Q_{-h}^{i,T_i}} \tag{16}$$

For each $h \in H$ consider the strategy $\left(\hat{b}_h, \hat{q}_h\right) \in \Re_+^L \times \Re_+^L$ defined as follows:

$$\hat{b}_h^i = \sum_{s=1}^{T_i} b_h^{i,s},\ \hat{q}_h^i = \sum_{s=1}^{T_i} q_h^{i,s} \tag{17}$$

Certainly, $\left(\hat{b}_h, \hat{q}_h\right) \in S^{\mathbf{1}}$ for all $h \in H$. Furthermore, it is easy to see that $(\hat{b}_h, \hat{q}_h)$ is budget feasible for each $h \in H$. With this profile agent h obtains the net trade:

$$\begin{aligned}
z_h^i &= \hat{b}_h^i \cdot \frac{\sum_{h\in H} \hat{q}_h^i}{\sum_{h\in H} \hat{b}_h^i} - \hat{q}_h^i \\
&= (\sum_{s=1}^{T_i} b_h^{i,s}) \cdot \frac{\sum_{h\in H}\sum_{s=1}^{T_i} q_h^{i,s}}{\sum_{h\in H}\sum_{s=1}^{T_i} b_h^{i,s}} - \sum_{s=1}^{T_i} q_h^{i,s} \\
&= (\sum_{s=1}^{T_i} b_h^{i,s}) \cdot \frac{\sum_{s=1}^{T_i} Q^{i,s}}{\sum_{s=1}^{T_i} B^{i,s}} - \sum_{s=1}^{T_i} q_h^{i,s} \\
&= \frac{\sum_{s=1}^{T_i} b_h^{i,s}}{p^i} - \sum_{s=1}^{T_i} q_h^{i,s} \\
&= \sum_{s=1}^{T_i} z_h^{i,s}
\end{aligned}$$

Thus, with the profile $(\hat{b}_h, \hat{q}_h)_{h\in H}$ in the game $\mathcal{E}^{\mathbf{1}}$ each individual obtains the same consumption allocation as in the game $\mathcal{E}^{\mathbf{t}}$. It follows that:

$$u_h\left(\left(x_h^i\left(\hat{b}_h^i, \hat{q}_h^i, \hat{B}_{-h}^i, \hat{Q}_{-h}^i\right)\right)_{i=1}^L\right) = u_h\left(\left(x_h^i\left(\left(b_h^{i,s}, q_h^{i,s}, B_{-h}^{i,s}, Q_{-h}^{i,s}\right)_{s=1}^{T_i}\right)\right)_{i=1}^L\right) \tag{18}$$

We claim that $(\hat{b}, \hat{q}) \in \mathbf{NE}(\mathcal{E}^{\mathbf{1}})$.
Suppose not. Then there would exist $h \in H$ and a budget feasible $(\beta_h, \theta_h) \in S_h^{\mathbf{1}}$ such that:

$$u_h\left(\left(x_h^i\left(\beta_h^i, \theta_h^i, \hat{B}_{-h}^i, \hat{Q}_{-h}^i\right)\right)_{i=1}^L\right) > u_h\left(\left(x_h^i\left(\hat{b}_h^i, \hat{q}_h^i, \hat{B}_{-h}^i, \hat{Q}_{-h}^i\right)\right)_{i=1}^L\right) \tag{19}$$

In this case consider the strategy, $\left(\hat{\beta}_h, \hat{\theta}_h\right) \in S_h^{\mathbf{t}}$ for agent h defined as follows: for each $i = 1, 2, \ldots, L$ and $s = 1, 2, \ldots, T_i$

$$\hat{\beta}_h^{i,s} = \frac{B_{-h}^{i,s}}{\sum_{i=1}^{T_i} B_{-h}^{i,s}} \cdot \beta_h^i,\quad \hat{\theta}_h^{i,s} = \frac{Q_{-h}^{i,s}}{\sum_{i=1}^{T_i} Q_{-h}^{i,s}} \cdot \theta_h^i \tag{20}$$

By (17) and the fact that (β_h, θ_h) has been assumed feasible, it follows that given $(b_a, q_a)_{a \neq h}$, the strategy defined in (20) is budget feasible for the individual h. Furthermore, with that strategy in the game $\mathcal{E}^{\mathbf{t}}$ this agent would obtain the net trade:

$$\hat{z}_h^i = \sum_{i=1}^{T_i} \hat{\beta}_h^{i,s} \frac{Q_{-h}^{i,s} + \hat{\theta}_h^{i,s}}{B_{-h}^{i,s} + \hat{\beta}_h^{i,s}} - \sum_{i=1}^{T_i} \hat{\theta}_h^{i,s} \tag{21}$$

Using (16) we have: $\frac{Q_{-h}^{i,s} + \hat{\theta}_h^{i,s}}{B_{-h}^{i,s} + \hat{\beta}_h^{i,s}} = \frac{Q_{-h}^{i,r} + \hat{\theta}_h^{i,r}}{B_{-h}^{i,r} + \hat{\beta}_h^{i,r}}$ for all $s \neq r$. It follows that:

$$\forall s = 1, 2, \ldots, T_i, \quad \frac{Q_{-h}^{i,s} + \hat{\theta}_h^{i,s}}{B_{-h}^{i,s} + \hat{\beta}_h^{i,s}} = \frac{\sum_{i=1}^{T_i} Q_{-h}^{i,s} + \hat{\theta}_h^{i,s}}{\sum_{i=1}^{T_i} B_{-h}^{i,s} + \hat{\beta}_h^{i,s}} = \frac{Q_{-h}^i + \theta_h^i}{B_{-h}^i + \beta_h^i}$$

Substituting this into (21) above we conclude that:

$$\hat{z}_h^i = \frac{Q_{-h}^i + \theta_h^i}{B_{-h}^i + \beta_h^i} \cdot \left(\sum_{i=1}^{T_i} \hat{\beta}_h^{i,s} \right) - \sum_{i=1}^{T_i} \hat{\theta}_h^{i,s} = \beta_h^i \cdot \frac{Q_{-h}^i + \theta_h^i}{B_{-h}^i + \beta_h^i} - \theta_h^i \tag{22}$$

But the last term is exactly the net trade that h receives with the strategy (β_h, θ_h) in the game $\mathcal{E}^{\mathbf{1}}$. Hence, we have:

$$u_h \left(\left(x_h^i \left(\left(\hat{\beta}_h^{i,s}, \hat{\theta}_h^{i,s}, B_{-h}^{i,s}, Q_{-h}^{i,s} \right)_{s=1}^{T_i} \right) \right)_{i=1}^{L} \right) = u_h \left(\left(x_h^i \left(\beta_h^i, \theta_h^i, \hat{B}_{-h}^i, \hat{Q}_{-h}^i \right) \right)_{i=1}^{L} \right)$$

Combining the last equation with (18) and (19) from above we conclude that:

$$u_h \left(\left(x_h^i \left(\left(\hat{\beta}_h^{i,s}, \hat{\theta}_h^{i,s}, B_{-h}^{i,s}, Q_{-h}^{i,s} \right)_{s=1}^{T_i} \right) \right)_{i=1}^{L} \right) > u_h \left(\left(x_h^i \left(\left(b_h^{i,s}, q_h^{i,s}, B_{-h}^{i,s}, Q_{-h}^{i,s} \right)_{s=1}^{T_i} \right) \right)_{i=1}^{L} \right)$$

which contradicts the hypothesis that $(b, q) \in \mathbf{NE}(\mathcal{E}^{\mathbf{t}})$ $\square$

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