Non-linear Asset Valuation on Markets with Frictions

ANJA DE WAEGENAERE[*]   ROBERT KAST[†]

ANDRE LAPIED[‡]

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Abstract: This paper provides a non-linear pricing rule for the valuation of assets on financial markets with intermediaries. The non-linearity arises from the fact that dealers charge a price for their intermediation between buyer and seller. The pricing rule we propose is an alternative for the well-known no-arbitrage pricing on markets without frictions. The price of an asset equals the signed Choquet integral of its discounted payoff with respect to a concave signed capacity. We show that this pricing rule is consistent with equilibrium. Furthermore, equilibria are shown to satisfy a notion of constrained Pareto optimality.

Key words: asset pricing, bid-ask spreads, set function, Choquet integral, General Equilibrium, constrained Pareto optimality.

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1 Introduction

On financial markets without frictions, no-arbitrage pricing allows to price non-marketed redundant assets using the equilibrium prices of the marketed assets (see for instance Magill and Shafer (1991)). The equilibrium prices of the marketed assets determine a (set of) *risk neutral* probability distribution(s) such that the equilibrium price of a redundant asset equals the mathematical expectation of its discounted payoff with respect to this (these) probability distribution(s). This pricing rule is consistent with equilibrium in the sense that, introducing a redundant asset at its no-arbitrage price does not affect the equilibrium allocations (see for example Harrison and Kreps (1979)). On markets with frictions however, a pricing rule will in general be non-linear. Consider for example bid-ask spreads or transaction costs. Then clearly prices (as a function of asset payoffs) are non-linear, since the price an agent has to pay for buying an asset is strictly larger than the price an agent receives for selling it. Therefore equilibrium asset prices cannot be represented by the mathematical expectation of their discounted payoff with respect to a probability measure.

In this paper, we propose a non-linear pricing rule that allows for bid-ask spreads. This pricing rule essentially amounts to replacing the risk-neutral probability measure appearing in no-arbitrage pricing by a concave signed capacity[1] ν. The price of an asset then equals the signed Choquet integral[2], which in general is non-linear, of its discounted payoff with respect to ν.

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[1]Consider a measure space $(\Omega, \mathcal{A})$. A set function $\mu : \mathcal{A} \to I\!R$ is concave if for all $A, B \in \mathcal{A}$ one has $\mu(A \cup B) \leq \mu(A) + \mu(B) - \mu(A \cap B)$. A set function μ is a signed capacity if it satisfies $\mu(\Omega) = 1$ and $\mu(\emptyset) = 0$. Signed capacities generalize capacities since they need not satisfy monotonicity with respect to set inclusion.

[2]See for instance Schmeidler (1986) for details on the Choquet integral, and De Waegenaere and Wakker (1996) for details on the signed Choquet integral.

We define a market model with bid-ask spreads and show that the above described pricing rule is consistent with General Equilibrium pricing in the sense that: i) the equilibrium price of a marketed asset equals the signed Choquet integral of its discounted payoffs with respect to an equilibrium concave signed capacity, and ii) introducing a redundant asset at the price equal to the signed Choquet integral of its discounted payoffs with respect to this concave signed capacity does not affect the equilibrium allocations.

2   Choquet equilibrium prices

We consider a two period asset market model with dealers charging bid-ask spreads, show that equilibrium exists, and give a characterization of the equilibrium prices of the assets. There are J nominal assets, indexed by $j \in \mathcal{J} := \{1, 2, \ldots, J\}$. The assets can be traded in the first period, and yield payoff in the second period. There are S possible states of the world at the second period, indexed by $s \in \Omega := \{1, 2, \ldots, S\}$. For simplicity of notation, we assume that there are no spot markets, i.e. there is only one good at each state of the world, and assets yield payoff in quantities of this good. A consumption bundle is a vector $x = (x_0, x_1, \ldots, x_S)^t \in I\!R_+^{S+1}$, consisting of x_0 units of the good in the first period, and x_s units of the good in the second period if state s occurs, for $s \in \Omega$. The payoff of asset $j \in \mathcal{J}$ is denoted by a vector $A_j \in I\!R^S$. The matrix of asset payoffs is denoted $A \in I\!R^{S \times J}$. There are I agents, indexed by $i \in \mathcal{I} = \{1, 2, \ldots, I\}$ with utility functions $u^i : I\!R_+^{S+1} \to I\!R_+$. They have initial endowments $w^i = (w_0^i, w_1^i, \ldots, w_S^i)^t, i \in \mathcal{I}$, and maximize utility by trading asset portfolios $z = (z_1, \ldots, z_J)^t \in I\!R^J$.

The model differs from the standard incomplete markets model in the sense that assets can only be traded through the intermediation of dealers. Again for simplicity of notation, we assume that there is only one dealer. The

presence of a dealer is formalized by the fact that for each asset $A_j, j \in \mathcal{J}$, there is a buying price $q(A_j)$ and a selling price $-q(-A_j)$. Typically, one will have that $q(A_j) > -q(-A_j)$, i.e. the dealer can make a profit equal to the bid-ask spread $\gamma_j := q(A_j) + q(-A_j)$, by buying the asset from an agent for the price $-q(-A_j)$ and selling it to an agent for the price $q(A_j)$. Furthermore, when a portfolio consisting of more than one asset is traded, the dealer takes into account that hedging effects can reduce the risk of the portfolio. Consequently, he might allow a price discount in this case. More precisely, for a portfolio $z \in I\!\!R^J$, in general one will have that $q(Az) + q(-Az) \leq \sum_{j=1}^{J} |z_j|(q(A_j) + q(-A_j))$, i.e. the spread on a portfolio is less than or equal to the sum of the individual spreads. When however the payoff vectors X and Y of two portfolios are comonotonic[3], then $Var(X+Y) \geq Var(X) + Var(Y)$, so there is no hedging effect when combining the two portfolios. Consequently, the dealer does not allow a price discount in this case. In short, we assume that the price functional $q : I\!\!R^S \to I\!\!R$ used by the dealer has the following properties:

Properties P_1:

P_{11}) *Continuity.*

P_{12}) *Sub-additivity, i.e. $q(X + Y) \leq q(X) + q(Y)$ for all payoff vectors $X, Y \in I\!\!R^S$.*

P_{13}) *Comonotonic additivity, i.e. if X and Y are comonotonic vectors in $I\!\!R^S$, then $q(X + Y) = q(X) + q(Y)$.*

When the dealer charges spreads, the inequality in P_{12} will in general be strict for non-comonotonic assets, and consequently, there is no probability

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[3]Vectors $X, Y \in I\!\!R^S$ are comonotonic if $(X_s - X_t)(Y_s - Y_t) \geq 0$ for all $s, t \in \Omega$. When X and Y denote payoff vectors of portfolios, then comonotonicity means that they increase each others risk, since they move in the same direction.

measure such that the price of an arbitrary portfolio equals the expected value of its discounted payoff with respect to that probability measure. Consequently, traditional no-arbitrage pricing cannot be used in this case. In the sequel, we show how the signed Choquet integral can be used to value portfolios on markets with frictions. Let us therefore first recall the definition of the signed Choquet integral.

Definition 2.1 *For any set function μ on $(\Omega, 2^\Omega)$, and any random variable X on $(\Omega, 2^\Omega)$, the signed Choquet integral of X w.r.t. μ, denoted by $\int X d\mu$, is defined as follows.*

(i) *Take a permutation $\rho(.)$ on Ω that is compatible with X, i.e.*
$$X(\rho(1)) \geq \cdots \geq X(\rho(S)).$$

(ii) *Define $\pi_{\rho(s)} := \mu(\{\rho(1), \ldots, \rho(s)\}) - \mu(\{\rho(1), \ldots, \rho(s-1)\})$, for all $s \geq 2$, and $\pi_{\rho(1)} := \mu(\{\rho(1)\}) - \mu(\emptyset)$.*

Then the signed Choquet integral of X with respect to μ is given by:

$$\int X d\mu = \sum_{s=1}^{S} \pi_s X(s). \tag{1}$$

Notice that, if $X(i) = X(j)$ for some $i \neq j$, then the rank-ordering $\rho(.)$ above is not uniquely defined. It is elementarily verified that then any rank-ordering $\rho(.)$ that is compatible with X gives the same result, so that the signed Choquet integral is well-defined.

Notice furthermore that, when μ is a signed measure (i.e. μ is additive and $\mu(\emptyset) = 0$), then $\pi_{\rho(s)} = \mu(\{\rho(s)\})$ for all $s \in \Omega$, and consequently, one has $\int X d\mu = \sum_{s=1}^{S} \mu(\{s\}) X(s)$, i.e. the signed Choquet integral equals the Lebesque integral of X with respect to μ. Hence, when μ is a probability measure, then $\int X d\mu$ equals the expected value of X with respect to μ.

Since in our case, $\Omega = \{1, 2, \ldots, S\}$, and $\mathcal{A} = 2^\Omega$, we can represent a stochastic variable on $(\Omega, \mathcal{A})$ by a vector $X \in I\!\!R^S$. Therefore in the sequel, with slight abuse of notation, for any vector $X \in I\!\!R^S$, and any set function μ, $\int X d\mu$ denotes the signed Choquet integral of the stochastic variable $\tilde{X}$ on $(\Omega, \mathcal{A})$ defined by $\tilde{X}(s) := X_s$, for all $s \in \Omega$.

As stated above, with traditional no-arbitrage pricing, there exists a probability measure such that the price of an asset equals the Lebesque integral of its discounted payoff with respect to this measure. In the sequel, we show that, by replacing the Lebesque integral by the more general signed Choquet integral, one obtains a pricing rule that is applicable to markets with frictions as described above. The following lemma is crucial.

Lemma 2.1 *A functional $q : I\!\!R^S \to I\!\!R$ satisfies properties P_1 if and only if there exists a concave set function μ satisfying $\mu(\emptyset) = 0$ such that $q(X) = \int X d\mu$, for all $X \in I\!\!R^S$.*

Proof: From theorem 1 in De Waegenaere and Wakker (1996), we know that a functional $q(.)$ can be represented as a signed Choquet integral with respect to a set function μ iff $q(.)$ satisfies P_{11} and P_{13}. Clearly, without loss of generality, one can take $\mu(\emptyset) = 0$. Given that $q(X) = \int X d\mu$ for all $X \in I\!\!R^S$, it follows from theorem 3 in De Waegenaere and Wakker (1996) that μ is concave iff the functional $q(.)$ satisfies property P_{12}. This concludes the proof.

For this reason, asset valuation by means of a price functional $q(.)$ that satisfies P_1 will in the sequel be called *Choquet valuation*, and the corresponding functional will be called a *Choquet functional*.

Corollary 2.1 *For every Choquet functional $q(.)$ that assigns a positive price to the riskless bond, i.e. $q((1,1,\ldots,1)^t) > 0$, there exists a concave signed capacity ν such that $q(X) = q((1,1,\ldots,1)^t) \int X d\nu$ for all $X \in I\!R^S$.*

Proof: From lemma 2.1 it follows that there exists a concave set function μ satisfying $\mu(\emptyset) = 0$ such that $q(X) = \int X d\mu$, for all $X \in I\!R^S$. By taking $X = (1,1,\ldots,1)^t$, it follows that $\mu(\Omega) > 0$. Every concave set function μ satisfying $\mu(\Omega) > 0$ and $\mu(\emptyset) = 0$ can be normalized to a concave signed capacity ν as follows $\nu(.) := \mu(.)/\mu(\Omega)$. By definition 2.1 one has $\int X d(t\nu) = t \int X d\nu$ for all $X \in I\!R^S$, $t \in I\!R$, and any set function ν. This yields the desired result.

The above corollary shows that a dealer who uses Choquet valuation and assigns a positive price to the riskless bond performs asset valuation in a way similar to no-arbitrage pricing in the frictionless case, but with the probability measure replaced by a concave signed capacity. The non-linearity of this pricing rule then clearly generates profit for the dealer. We suppose that each agent $i \in \mathcal{I}$ has a share ξ^i in the dealer's firm, with $\sum_{i=1}^I \xi^i = 1$. After trade, the dealer's profit is redistributed amongst the agents proportional to their shares. Keeping in mind lemma 2.1 this gives rise to the following definition.

Definition 2.2 *A set function $\mu : 2^\Omega \to I\!R$ is an equilibrium set function if there exist consumption bundles $\{\overline{x}^i, i \in \mathcal{I}\}$, asset portfolios $\{\overline{z}^i, i \in \mathcal{I}\}$, and dealer's profit $\pi^d \in I\!R_+$ satisfying:*

i) $(\overline{x}^i, \overline{z}^i) \in argmax_{(x,z) \in B^i(\mu, \pi^d)} u^i(x), \quad i \in \mathcal{I},$

ii) $\sum_{i=1}^I \overline{x}^i = \sum_{i=1}^I w^i,$

iii) $\sum_{i=1}^I \overline{z}^i = 0,$

iv) $\pi^d = \sum_{i=1}^I \int A\overline{z}^i d\mu$

where, the budget set of agent $i \in \mathcal{I}$ is given by:

$$B^i(\mu, \pi^d) := \left\{ (x, z) \in I\!\!R_+^{S+1} \times I\!\!R^J \;\middle|\; \begin{array}{l} x_0 \leq w_0^i - \int Az d\mu + \xi^i \pi^d \\ x_s \leq w_s^i + (Az)_s, \quad s \in \Omega \end{array} \right\}.$$

Before going to the main theorem, it is interesting to notice the following. From theorem 5 in De Waegenaere and Wakker (1996) it follows that[4], for each price functional $q(.)$ with properties P_1, and therefore by lemma 2.1 for every signed Choquet integral with respect to a concave set function satisfying $\mu(\emptyset) = 0$, there exists a vector $\pi \in I\!\!R^S$, which can be interpreted as a vector of state prices, and a positive, continuous functional $\Psi(\pi, .) : I\!\!R^S \rightarrow I\!\!R_+$, such that $q(X) = \pi X + \Psi(\pi, X)$, for all $X \in I\!\!R^S$. Therefore, a dealer using Choquet valuation applies a pricing rule that consists of a linear part, πX, the "price" of X, to which he adds a positive, sub-additive part, $\Psi(\pi, X)$, which represents the "spread" he charges for his intermediation. For this reason, $\Psi(., .)$ will be called the *spread functional*. Now the following definition follows naturally.

Definition 2.3 *A spread functional $\Psi(., .) : I\!\!R^S \times I\!\!R^S \rightarrow I\!\!R_+$ is compatible with properties P_1 if for every $\pi \in I\!\!R^S$, the functional $q(.) := \pi. + \Psi(\pi, .)$ satisfies properties P_1. Let $\mathcal{C}$ denote the set of all set functions on $(\Omega, 2^\Omega)$. The set functions that correspond to a given spread functional $\Psi(., .)$ that is compatible with P_1 are given by $\mathcal{C}(\Psi) := \{\mu \in \mathcal{C} | \exists \pi \in I\!\!R^S : \pi X + \Psi(\pi, X) = \int X d\mu$, for all $X \in I\!\!R^S\}$.*

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[4]It is shown in theorem 5 in De Waegenaere and Wakker (1996) that for a concave set function μ satisfying $\mu(\emptyset) = 0$, the signed Choquet integral of a stochastic variable X with respect to μ is given by $\int X d\mu = \max \left\{ \int X dP | P \in \mathcal{P}(\mu) \right\}$, where $\mathcal{P}(\mu) = \{P | P$ is an additive set function such that $P(A) \leq \mu(A)$ for all $A \in \mathcal{A}$, and $P(\Omega) = \mu(\Omega)\}$.

We will consider spread functionals that satisfy the following regularity properties.

Properties P_2:

$P_{21)}$ $\Psi(.,.)$ *is continuous.*

$P_{22)}$ *When $\{\pi^n : n \to \infty\}$ is such that $\lim_{n\to\infty}(\pi^n A_j) = +\infty(resp. \ -\infty)$ for some $j \in \mathcal{J}$, then $\lim_{n\to\infty}(-\pi^n A_j + \Psi(\pi^n, -A_j)) = -\infty$ (resp. $\lim_{n\to\infty}(\pi^n A_j + \Psi(\pi^n, A_j)) = -\infty$).*

The intuition behind P_{22} is as follows. Suppose that there would be no dealer, i.e. $\Psi(.,.) = 0$. Then it is well known that potential equilibrium values for the asset prices $q_j := \pi A_j$ are bounded. Indeed, a sequence such that $\lim_{n\to\infty} |q_j^n| = \infty$ leads to unbounded aggregate demand for the good in at least one state. Now P_{22} essentially says that the spread functional is such that when the price of an asset becomes very high (resp. low), the net amount received when selling it (price - dealer's charge) gets very high (resp. the total cost for buying it (price + dealer's charge) gets very low). Consequently, spreads do not prevent that aggregate demand becomes unbounded.

Notice that properties P_2 are satisfied for all spread functionals that are compatible with P_1 and do not depend on π. A concrete example of a spread functional that is compatible with P_1 and satisfies P_2 is given in section 4. More general examples of spread functionals being compatible with P_1 and satisfying P_2 can be constructed using theorem 5 in De Waegenaere and Wakker (1996).

We can now go to the main theorem, which can be shown under the following regularity assumptions.

Assumptions $\mathcal{A}$:

A_1) *The utility functions are continuous, strictly increasing, and quasi-concave.*

A_2) *The initial endowments are strictly positive, i.e. $w_s^i > 0$, for all $i \in \mathcal{I}$ and $s \in \{0, 1, 2, \ldots, S\}$.*

A_3) *There is no redundancy in the asset's payoffs, i.e. rank $(A)=J$.*

Now the main theorem reads as follows.

Theorem 2.1 *Under assumptions $\mathcal{A}$, for every spread functional $\Psi(.,.)$ that is compatible with properties P_1, and satisfies properties P_2, an equilibrium concave set function $\mu^* \in \mathcal{C}(\Psi)$ exists. When the riskless bond in redundant, i.e. $(1,1,\ldots,1)^t \in <A>$, then there exists a concave signed capacity ν^* such that the equilibrium price of a portfolio $z \in I\!\!R^J$, equals $q_{rb}^* \int Azd\nu^*$, where q_{rb}^* denotes the equilibrium price of the riskless bond.*

Proof: Definition 2.3 combined with lemma 2.1 implies that, for a given $\Psi(.,.)$ that is compatible with P_1, budget sets can be rewritten as follows,

$$B^i(\pi, \pi^d) := \left\{ (x,z) \in I\!\!R_+^{S+1} \times I\!\!R^J \;\middle|\; \begin{array}{l} x_0 \leq w_0^i - \pi(Az) - \Psi(\pi, Az) + \xi^i \pi^d \\ x_s \leq w_s^i + (Az)_s, \quad s \in \Omega \end{array} \right\},$$

where the vector $\pi \in I\!\!R^S$ now becomes the equilibrium variable. Since $\Psi(.,.)$ is compatible with P_1, it follows from theorem 3 in De Waegenaere and Wakker (1996) that $\Psi(\pi,.)$ is a convex function for all $\pi \in I\!\!R^S$. Assumptions $\mathcal{A}$ and the fact that $\Psi(.,.)$ is compatible with properties P_1 therefore imply that budget sets are non-empty, closed and convex for every $\pi \in I\!\!R^S$ and every $\pi^d \geq 0$. Furthermore, property P_{22} implies that potential equilibrium values for πA are bounded. Existence of an equilibrium π^* can therefore be shown following the lines of Werner (1985). An equilibrium μ^* is then given

by $\mu^*(A) := \pi^* 1_A + \Psi(\pi^*, 1_A)$ for all $A \in 2^\Omega$, where 1_A satisfies $(1_A)_s = 1$ for $s \in A$ and $(1_A)_s = 0$ for $s \in \Omega \setminus A$.

Now let $\mu^* \in \mathcal{C}(\Psi)$ be an equilibrium set function. It then follows that the equilibrium price of the riskless bond is given by $q_{rb}^* = \int (1, 1, \ldots, 1)^t d\mu^* = \mu^*(\Omega)$. By no-arbitrage arguments, clearly $q_{rb}^* > 0$. Then corollary 2.1 says that there exists a concave signed capacity ν^* such that $\int X d\mu^* = q_{rb}^* \int X d\nu^*$ for all $X \in \mathbb{R}^S$. This yields the desired result.

3   Constrained Pareto optimality

In this section, we address the issue of Pareto optimality. The example in the next section shows that, even with complete markets, one cannot expect to have Pareto optimality. In this model, there are three sources of Pareto inefficiency. The first one is the incompleteness of the market. The second one is the redistribution of the dealer's profit according to the shares in the dealer's firm. The last one is the fact that the presence of frictions indirectly limits the level of transactions. This leads to the following definition.

Definition 3.1 *At a set function $\mu : 2^\Omega \to \mathbb{R}$, and dealer's profit $\pi^d \in \mathbb{R}_+$, the set of feasible allocations is defined as follows:*

$$
\mathcal{F}(\mu, \pi^d) := \left\{ x \in \mathbb{R}_+^{I \times (S+1)} \middle| \exists z \in \mathbb{R}^{I \times J} : \begin{array}{c} x_s^i = w_s^i + (Az^i)_s, \quad s \in \Omega \\ \sum_{i=1}^I z^i = 0, \quad \sum_{i=1}^I x_0^i = \sum_{i=1}^I w_0^i \\ \sum_{i=1}^I \int Az^i d\mu \leq \pi^d \end{array} \right\}.
$$

We can show the following.

Theorem 3.1 *Suppose that μ^* is an equilibrium set function, and let $(x^*, z^*, \mu^*, \pi^{d*})$ satisfy $i), ii), iii)$ and $iv)$ in definition 2.2. Then there does not exist an allocation $\tilde{x} \in \mathcal{F}(\mu^*, \pi^{d*})$ such that $u^i(\tilde{x}^i) \geq u^i(x^{*i})$ for all $i \in \mathcal{I}$ with at least one strict inequality.*

Proof: Goes along the usual lines.

Notice that this concept of constrained Pareto optimality is similar to notions of second order optimality in the context of markets with taxation (see for instance Guesnerie (1994)).

4   Example

In the following example, we compute all equilibrium concave set functions for a specified economy, and show that the only Pareto optimal equilibrium is the one that corresponds to no bid-ask spreads.

Consider a market with two agents, two assets, one good and two states of the world at date one. There is no consumption at date zero. The date one endowment of agent 1 is given by $(0,2)^t$. The date one endowment of agent 2 is given by $(1,0)^t$.

Agent 1 has utility function $u^1(x_1, x_2) := 2\ln(x_1) + \ln(x_2)$. Agent 2 has utility function $u^2(x_1, x_2) := \ln(x_1) + 2\ln(x_2)$. The dealer's firm is owned by agent two, i.e. $\xi^2 = 1$, and $\xi^1 = 0$. The asset payoffs are given by: $A^1 = (1,0)^t$ and $A^2 = (0,1)^t$. We consider price functionals $q(.)$ given by $q(X) = \int X d\mu$ for all $X \in I\!R^S$, where the concave set function μ is the equilibrium variable. Notice that, for a given concave set function μ, the price functional equals $q(X) = \int X d\mu = \pi X + \gamma|X_2 - X_1|$, with $\pi_1 = \mu(\{1,2\}) - \mu(\{2\})$, $\pi_2 = \mu(\{1,2\}) - \mu(\{1\})$, and $\gamma = \mu(\{1\}) + \mu(\{2\}) - \mu(\{1,2\})$. On the other hand, for any $\gamma \geq 0$, the above used spread functional is compatible with a concave set function. Since there is no consumption at date zero, we can without loss of generality normalize the price of the riskless bond $(1,1)^t$ to 1, i.e. $\int(1,1)^t d\mu = \mu(\{1,2\}) = 1$. In the sequel, we denote $\mu_1 = \mu(\{1\})$, and $\mu_2 = \mu(\{2\})$. It is straightforward to see that the equilibria in this economy

are then given by:

$$x^1 = \left(\frac{4}{3}\frac{(1-\mu_1)}{\mu_1}, \frac{2}{3}\right)^t, \quad x^2 = \left(\frac{1}{3} + \frac{\pi^d}{3(1-\mu_2)}, \frac{2}{3}\frac{(1-\mu_2)}{\mu_2} + \frac{2}{3}\frac{\pi^d}{\mu_2}\right)^t,$$

for all μ_1, μ_2 and π^d satisfying:

$$\frac{1}{3} \le \mu_2 \le \frac{3}{5}, \quad \mu_1 = \frac{4(1-\mu_2)}{7-9\mu_2}, \quad \pi^d = 3\mu_2 - 1.$$

For Pareto optimality however, it is necessary that $x_1^1 = 4x_2^1/(2+3x_2^1)$.

Consequently, the only Pareto optimal equilibrium is:

$$x^1 = (\frac{2}{3}, \frac{2}{3})^t, \quad x^2 = (\frac{1}{3}, \frac{4}{3})^t$$
$$\mu_1 = \frac{2}{3}, \quad \mu_2 = \frac{1}{3}, \quad \pi^d = 0,$$

i.e., it is the unique equilibrium where there is no bid-ask spread ($\pi^d = 0$), and for this equilibrium the set function mu is a probability measure, since then $\mu_1 + \mu_2 = 1 = \mu(\{1,2\})$.

Remark: The multiplicity of equilibria is a result of the fact that we allow for $\mu_1 + \mu_2 > 1$, i.e. we allow for spreads, and consequently μ is allowed to be a concave signed capacity instead of a probability measure. When restricting to $\mu_1 + \mu_2 = 1$, one gets a unique equilibrium (without spreads).

5   Conclusion

Asset pricing with transaction costs or bid-ask spreads has been widely developed recently (see for example Boyle and Vorst (1992) and Bensaid et al. (1992)). However, the link between asset pricing and equilibrium on markets with frictions has not been given much attention. This paper provides such a link. It is shown that a non-linear pricing rule, Choquet pricing, can

be used to price redundant assets in a way consistent with equilibrium. The rule essentially amounts to replacing the risk-neutral probability distribution appearing in no-arbitrage pricing by a concave signed capacity. It is shown that "equilibrium" signed capacities exist, and that the equilibrium allocations satisfy a notion of Pareto optimality that is very similar to notions of Pareto optimality appearing in the taxation literature.

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