Center
for
Economic Research

No. 2000-38

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March 2000

ISSN 0924-7815

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Abstract

The aim of the present study is to construct a state feedback controller for a given linear system that minimizes the worst-case effect of an L_2-bounded disturbance. Our setting is different from the usual framework of H_∞-theory in that we consider nonzero initial conditions. The situation is modeled in a game theoretical framework, in which the controller designer acts as a minimizing player, and the uncertainty as a maximizing player. We show that a saddle-point equilibrium exists and find an optimal controller.

Keywords: Linear Uncertain Systems, Game Theory, Algebraic Riccati Equations.

Journal of Economic Literature Classification Numbers: C70, C72 and C73.

1 Introduction

Consider a linear time invariant system, described by the equations

$$\dot{x} = Ax + Bu + Ew, \quad x(0) = x_0, \quad z = Cx + Du,$$

where x denotes the state variable with initial value x_0, u the control variable, w the unknown disturbance, and z the output variable. The central problem of the present paper is to determine a linear stabilizing state feedback law that minimizes the supremum of the L_2-norm of the output, where the supremum is taken over all L_2-disturbances with norm bounded by ε, for a given positive number ε. For $x_0 = 0$, this problem reduces to the state feedback H_∞ control problem, which has been extensively studied in the literature, see for instance [1], [5], or [10]. For $x_0 \neq 0$, the corresponding maximin problem has been studied by Chen [4]. Following the lines of

H_∞ theory, it seems more natural to study the minimax problem. This is the main motivation for the present paper.

It is not surprising that a state feedback law which solves the problem will depend on x_0. Especially in applications where initial states may deviate substantially from zero, such feedback controllers could be interesting, since they specifically take this initial state into account. One could also replace the parameter x_0 in the feedback law by the current state $x(t)$, which gives a nonlinear state feedback controler with certain robustness properties.

The problem allows for a convenient description in terms of a two-person zero-sum differential game. The two players involved are the controller designer and the uncertainty. The designers goal is to minimize the L_2-norm of the output by choosing a suitable stabilizing linear state feedback law, whereas the uncertainty tries to maximize this norm by choosing a suitable disturbance, both taking each others' action into account. One usually looks for saddle-point equilibria in these games. It is well-known that if such a game has a saddle-point equilibrium, its lower and upper value are equal (see for instance [9]). Hence a saddle point equilibrium immediately gives a solution of both the minimax and the maximin problem. We refer to [2] for a general treatment of dynamic game theory, and more specifically to [1], where the H_∞ problem is treated using a game theoretic approach.

The organization of the paper is as follows. In Section 2 some notations are introduced, and a couple of standing assumptions are made. Furthermore, the central problem will be rephrased in a mathematical framework. In Section 3, the "inner optimization problem" of the minimax problem will be studied. This is the problem of maximizing the norm of the output over all ε-bounded disturbances for a fixed control law. The main part of the paper is Section 4, in which it will be shown that there exists a saddle-point equilibrium.

2 Assumptions, Notations, and Preliminaries

Concerning the matrix-quintuple (A, B, C, D, E), we assume that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, and $E \in \mathbb{R}^{n\times q}$, with (A, B) stabilizable, and D injective. We also assume that the quadruple (A, B, C, D) has no invariant zeros on the imaginary axis. Next, define the set

$$\mathcal{F} := \{F | A + BF \text{ is stable}\}.$$

Note that the stabilizability of (A, B) ensures that $\mathcal{F} \neq \emptyset$. For each $F \in \mathcal{F}$, we shall write $A_F = A + BF$, and $C_F = C + DF$.

The Hilbert space consisting of L_2-functions on $(0, \infty)$ with k components will be denoted by L_2^k. This space is supplied with the inner product $\langle f, g\rangle := \int_0^\infty f(t)^T g(t)dt$, and corresponding norm $\|f\| := \langle f, f\rangle^{\frac{1}{2}}$. For each $\delta > 0$, we introduce the sets

$$B_\delta^k := \left\{ f \in L_2^k \middle| \, \|f\| \leq \delta \right\}, \quad \partial B_\delta^k := \left\{ f \in L_2^k \middle| \, \|f\| = \delta \right\}.$$

If $F \in \mathcal{F}$, then $z \in L_2^p$, and we have $z(t) = C_F e^{tA_F} x_0 + C_F \int_0^t e^{(t-\tau)A_F} E w(\tau) d\tau$, with $w \in L_2^q$. The output can be interpreted as the image of an affine operator from L_2^q to L_2^p. Indeed, define for $F \in \mathcal{F}$, and for $x_0 \in I\!R^n$, the function $z_{F,x_0} \in L_2^p$ by $z_{F,x_0}(t) := C_F e^{tA_F} x_0$ and the linear operator $\mathcal{G}_F : L_2^q \to L_2^p$ by $(\mathcal{G}_F w)(t) = \int_0^t T_F(t-\tau) w(\tau) d\tau$, where $T_F(t) = C_F e^{tA_F} E$ is the closed-loop impulse response between disturbance and output, then we have $z = z_{F,x_0} + \mathcal{G}_F w$. The performance criterion is now defined as the map $\varphi : \mathcal{F} \times L_2^q \times I\!R^n \to I\!R$ given by

$$\varphi(F, w, x_0) = \|z_{F,x_0} + \mathcal{G}_F w\|^2. \tag{1}$$

We shall also be concerned with the L_2-induced operator norm of $\mathcal{G}_F$, usually denoted by $\|\mathcal{G}_F\|_\infty$ (and often called the H_∞-norm), which is defined by

$$\|\mathcal{G}_F\|_\infty = \sup_{w \in L_2^q,\ w \neq 0} \frac{\|\mathcal{G}_F w\|}{\|w\|}. \tag{2}$$

The central problem of the paper can now be formulated as follows.

Problem 2.1 Let $\varepsilon > 0$, and $x_0 \in I\!R^n$. Find

$$\hat{F}(x_0, \varepsilon) := \arg \min_{F \in \mathcal{F}} \sup_{w \in B_\varepsilon^q} \varphi(F, w, x_0).$$

As explained in the introduction, the game theoretical approach suggests looking for saddle-point equilibria. Such an equilibrium is formally defined as a pair $(\hat{F}, \hat{w}) \in \mathcal{F} \times B_\varepsilon^q$ such that

$$\varphi(\hat{F}, w, x_0) \leq \varphi(\hat{F}, \hat{w}, x_0) \leq \varphi(F, \hat{w}, x_0)$$

for each F and w, and for given ε and x_0. One can easily verify that an $\hat{F}$ corresponding to a saddle-point equilibrium indeed solves Problem 2.1.

If there exists an $F \in \mathcal{F}$ such that $C_F = 0$, the problem is solved immediately, since $F = -(D^T D)^{-1} D^T C$ yields $\varphi = 0$ in that case. In order to exclude this trivial case, we shall use the nonminimum phase assumption, i.e. we assume that $A - B(D^T D)^{-1} D^T C$ has at least one eigenvalue in the open right-half plane.

We shall use the following two well-known results from H_∞ theory. The first theorem is sometimes called the bounded real lemma. The second theorem gives a complete solution of the infinite horizon state feedback H_∞ control problem.

Theorem 2.2 (See for instance [10], pp. 360-361) *Let $\gamma > 0$, and $F \in \mathcal{F}$. Then $\|\mathcal{G}_F\|_\infty < \gamma$ if and only if there exists a solution X of the algebraic Riccati equation*

$$C_F^T C_F + A_F^T X + X A_F + \gamma^{-2} X E E^T X = 0 \tag{3}$$

such that $A_F + \gamma^{-2} E E^T X$ is stable.

Theorem 2.3 (See for instance [3], pp. 239-240) *Let $\gamma > 0$. The following statements are equivalent:*

(i) *There exists an $F \in \mathcal{F}$ such that $\|\mathcal{G}_F\|_\infty < \gamma$.*

(ii) *There exists a positive semi-definite solution P of the algebraic Riccati equation*

$$C^TC + A^TP + PA + \\ + \gamma^{-2}PEE^TP - (PB + C^TD)(D^TD)^{-1}(B^TP + D^TC) = 0 \qquad (4)$$

such that $A - B(D^TD)^{-1}(B^TP + D^TC) + \gamma^{-2}EE^TP$ is stable.

If the latter condition holds, a suitable F is given by $F = -(D^TD)^{-1}(B^TP + D^TC)$.

3 Maximization of the Criterion with a Fixed Control Variable

In the present section we study Problem 2.1 for a fixed $F \in \mathcal{F}$. This problem reduces then to maximizing the map $w \mapsto \varphi(F, w, x_0)$, restricted to the set B^q_ε, for given x_0 and ε. The first part of the analysis in this section (until Remark 3.3) is meant to develop some intuition concerning this problem, but will not be used to derive a solution in terms of the system data. This will be done in the second part.

Consider the system $\dot{x} = A_Fx + Ew$, $x(0) = x_0$, and $z = C_Fx$, and recall that we have introduced the notation $\varphi(F, w, x_0) = \|z\|^2$. Hence, the supremum from Problem 2.1 can be written as

$$\overline{\varphi}(F, x_0) := \sup_{w \in B^q_\varepsilon} \varphi(F, w, x_0). \qquad (5)$$

If $\varepsilon = 1$, and $x_0 = 0$, so that the map $w \mapsto z$ is linear, this supremum is actually the H_∞-norm of the system under consideration, and is, by definition, equal to $\|\mathcal{G}_F\|_\infty$. In order to find this norm, one usually looks for the infimum of all γ's, for which there exists a solution X of the algebraic Riccati equation

$$C_F^TC_F + A_F^TX + XA_F + \gamma^{-2}XEE^TX = 0 \qquad (6)$$

such that $A_F + \gamma^{-2}EE^TX$ is stable. It is well-known that this infimum is equal to $\|\mathcal{G}_F\|_\infty$. See also Theorem 2.2.

Now, let $x_0 \neq 0$. Obviously, the situation becomes different, although, as we will see, the same algebraic Riccati equation will appear. The dependence of z on w becomes affine, instead of linear. Indeed, we have $z = z_{F,x_0} + \mathcal{G}_Fw$, with $z_{F,x_0} \neq 0$. Since we are maximizing over a compact set, the supremum in (5) is actually a maximum. It is also clear that this maximum is attained at the boundary. So, we do in fact have $\overline{\varphi}(F, x_0) = \max_{w \in \partial B^q_\varepsilon} \varphi(F, w, x_0)$, which transforms the problem into a maximization problem with an equality constraint. It is clear that this problem is equivalent to maximizing the criterion

$$\varphi_\kappa(F, w, x_0) := \|z_{F,x_0} + \mathcal{G}_F w\|^2 - \kappa\|w\|^2,$$

where the parameter κ is chosen such as to make the norm of unconstrained maximizers of this criterion equal to ε. Now, if $\kappa > \|\mathcal{G}_F^* \mathcal{G}_F\| = \|\mathcal{G}_F\|_\infty^2 =: \kappa_F$, the operator $\kappa - \mathcal{G}_F^* \mathcal{G}_F$ is positive definite, and φ_κ can then by a standard completion of the squares be rewritten as

$$\begin{aligned}\varphi_\kappa(F, w, x_0) = &- \langle(\kappa - \mathcal{G}_F^* \mathcal{G}_F)(w - w_{F,\kappa}), w - w_{F,\kappa}\rangle + \\ &+ \langle w_{F,\kappa}, \mathcal{G}_F^* z_{F,x_0}\rangle + \langle z_{F,x_0}, z_{F,x_0}\rangle,\end{aligned}$$

where we introduced the notation

$$w_{F,\kappa} := (\kappa - \mathcal{G}_F^* \mathcal{G}_F)^{-1} \mathcal{G}_F^* z_{F,x_0}. \tag{7}$$

Clearly, if $\kappa > \kappa_F$, then φ_κ is maximized at $w_{F,\kappa}$. Hence, if we can find a $\kappa > \kappa_F$ such that $\|w_{F,\kappa}\| = \varepsilon$, the maximum of $\varphi(F, w, x_0)$ is attained by $w_{F,\kappa}$. The existence of such a κ is partly answered by the following lemma.

Lemma 3.1 *The function $f : \kappa \mapsto \|w_{F,\kappa}\|^2$, for $\kappa > \kappa_F$, is strictly decreasing and approaches zero for $\kappa \to \infty$.*

Proof: Since $\kappa > \kappa_F$, we can write

$$(\kappa - \mathcal{G}_F^* \mathcal{G}_F)^{-1} = \kappa^{-1}\left(\mathcal{I} - \kappa^{-1} \mathcal{G}_F^* \mathcal{G}_F\right)^{-1} = \kappa^{-1} \sum_{i=0}^{\infty} \left(\kappa^{-1} \mathcal{G}_F^* \mathcal{G}_F\right)^i.$$

Consequently,

$$\begin{aligned} f(\kappa) &= \left\langle (\kappa - \mathcal{G}_F^* \mathcal{G}_F)^{-1} v, (\kappa - \mathcal{G}_F^* \mathcal{G}_F)^{-1} v \right\rangle = \\ &= \kappa^{-2} \left\langle \sum_{i=0}^{\infty} \left(\kappa^{-1} \mathcal{G}_F^* \mathcal{G}_F\right)^i v, \sum_{i=0}^{\infty} \left(\kappa^{-1} \mathcal{G}_F^* \mathcal{G}_F\right)^i v \right\rangle = \\ &= \kappa^{-2}\|v\|^2 + 2\kappa^{-3}\|\mathcal{G}_F v\|^2 + 3\kappa^{-4}\|\mathcal{G}_F^* \mathcal{G}_F v\|^2 + \cdots. \end{aligned}$$

with $v = \mathcal{G}_F^* z_{F,x_0}$. We see that f is a positive linear combination of decreasing functions. Hence, f is strictly decreasing. Furthermore, it is obvious that $f(\kappa) \downarrow 0$, for $\kappa \to \infty$. □

Remark 3.2 If $f(\kappa)$ is unbounded for $\kappa \downarrow \kappa_F$, there clearly exists a unique $\kappa > \kappa_F$ such that $\|w_{F,\kappa}\| = \varepsilon$, for each $\varepsilon > 0$. However, if $f(\kappa)$ is bounded, for $\kappa \downarrow \kappa_F$, we have to analyse the situation further. This case occurs when

$$\mathcal{G}_F^* z_{F,x_0} \in \operatorname{Im}(\kappa_F - \mathcal{G}_F^* \mathcal{G}_F),$$

or, since $\operatorname{Im} T = (\operatorname{Ker} T)^{\perp}$ for any self-adjoint operator T,

$$\mathcal{G}_F^* z_{F,x_0} \perp \operatorname{Ker}(\kappa_F - \mathcal{G}_F^* \mathcal{G}_F). \tag{8}$$

Hence, if x_0 is such that (8) holds, then $f(\kappa)$ is bounded for $\kappa \downarrow \kappa_F$, and if ε is larger than this bound, the equation $\|w_{F,\kappa}\| = \varepsilon$ has no solution. In the rest of the paper we shall assume that we are not in this special situation; i.e. we assume that x_0 is such that one can find a $\kappa > \kappa_F$ such that $\|w_{F,\kappa}\| = \varepsilon$. The case in which (8) holds is left for further research.

Remark 3.3 At this point in the analysis, one could proceed from (7) with computing a maximizer by determining an expression for $w_{F,\kappa}$ in terms of the system data. For this, one has to invert the operator $\kappa - \mathcal{G}_F^* \mathcal{G}_F$, which boils down to inverting a Wiener-Hopf operator. The theory from Gohberg et al. [6], is particularly useful for this. We shall however proceed in a different way.

Replace γ in (6) by $\sqrt{\kappa}$. Denote the infimum of all the κ's for which there exists a solution X of (6) such that $A_F + \kappa^{-1} E E^T X$ is stable by κ_F. Of course, this is the same κ_F that we introduced earlier in the present section. It will be convenient to display the dependence of X on κ explicitly. Therefore, we shall write X_κ. An important conclusion from the analysis so far is that we are looking for a specific $\kappa > \kappa_F$, that corresponds uniquely to a disturbance that maximizes the criterion, unlike H_∞ theory, where one usually looks for κ_F. There is a reasonably simple method to find an expression for this maximizer. This method is based on a completion of the squares. The next lemma states the result.

Lemma 3.4 *For each $\kappa > \kappa_F$, and $w \in L_2^q$ with $\|w\| = \varepsilon$, we have*

$$\varphi(F, w, x_0) = x_0^T X_\kappa x_0 + \kappa \varepsilon^2 - \kappa \int_0^\infty \left| w(t) - \kappa^{-1} E^T X_\kappa x(t) \right|^2 dt, \tag{9}$$

where x is generated by the system $\dot{x} = A_F x + E w$, $x(0) = x_0$.

Proof: By definition, we have $\varphi(F, w, x_0) = \int_0^\infty x^T C_F^T C_F x dt$. Adding and substracting the expression $\int_0^\infty \frac{d}{dt} x^T X_\kappa x dt$ from the right hand side yields

$$\begin{aligned} \varphi(F, w, x_0) = x_0^T X_\kappa x_0 + & \\ & + \int_0^\infty \left(x^T (C_F^T C_F + A_F^T X_\kappa + X_\kappa A_F) x + 2 w^T E^T X_\kappa x \right) dt = \\ = x_0^T X_\kappa x_0 + \kappa \varepsilon^2 - \kappa \int_0^\infty \left| w - \kappa^{-1} E^T X_\kappa x \right|^2 dt. & \end{aligned}$$

In the latter equality we used the algebraic Riccati equation (6), and the fact that $w \in \partial B_\varepsilon^q$. □

Let $w_{F,\kappa} := \kappa^{-1} E^T X_\kappa x$. It can be shown that this agrees with the expression by which $w_{F,\kappa}$ is defined in (7). Formula (9) clearly reveals that if we can find a $\kappa > \kappa_F$ such that $\|w_{F,\kappa}\| = \varepsilon$, then $w_{F,\kappa}$ is a maximizer. Generically, one can find such a κ, at least if $x_0 \neq 0$ (see Remark 3.2). Now, in order to compute the right κ, note that $\|w_{F,\kappa}\|^2 = \kappa^{-2} x_0^T S x_0$, where S is the solution of the Lyapunov equation $(A + \kappa^{-1} E E^T X_\kappa)^T S + S(A + \kappa^{-1} E E^T X_\kappa) = -X_\kappa E E^T X_\kappa$. Hence, the κ that makes the norm of $w_{F,\kappa}$ equal to ε is a solution of $\kappa^{-2} x_0^T S x_0 = \varepsilon^2$. We arrive at the following result.

Theorem 3.5 *Let $F \in \mathcal{F}$, and $\varepsilon > 0$. Let X_κ be the solution of the algebraic Riccati equation*

$$C_F^T C_F + A_F^T X + X A_F + \kappa^{-1} X E E^T X = 0. \tag{10}$$

such that $A_F + \kappa^{-1} E E^T X$ is stable. Denote the infimum of all κ's for which there exists such an X by κ_F. Let S_κ be the solution of the Lyapunov equation

$$(A_F + \kappa^{-1} E E^T X)^T S + S(A_F + \kappa^{-1} E E^T X) = -X E E^T X \tag{11}$$

Furthermore, let x_0 be such that there exists a $\overline{\kappa} > \kappa_F$ that satisfies $\overline{\kappa}^{-2} x_0^T S_{\overline{\kappa}} x_0 = \varepsilon^2$. Then we have

$$\sup_{w \in B_\varepsilon^q} \varphi(F, w, x_0) = \max_{w \in \partial B_\varepsilon^q} \varphi(F, w, x_0) = x_0^T X_{\overline{\kappa}} x_0 + \overline{\kappa} \varepsilon^2. \tag{12}$$

This maximum is attained by $\overline{w} := \overline{\kappa}^{-1} E^T X_{\overline{\kappa}} x$, where x is generated by the system $\dot{x} = (A_F + \overline{\kappa}^{-1} E E^T X_{\overline{\kappa}}) x$, $x(0) = x_0$.

Remark 3.6 The matrix $\kappa^{-1} S_\kappa$ is the observability grammian of the system $\dot{x} = (A_F + \kappa^{-1} E E^T X_\kappa) x$, $x(0) = x_0$, $y = \kappa^{-1} E^T X_\kappa x$, which is exactly the system that generates $w_{F,\kappa}$, i.e. $y = w_{F,\kappa}$. It is well-known that the quantity $\kappa^{-2} x_0^T S_\kappa x_0$ equals the observation energy of the same system, and this energy is equal to ε^2 if $\kappa = \overline{\kappa}$.

4 The Saddle-Point Equilibrium

The analysis of the present section starts by defining a pair $(\hat{F}, \hat{w}) \in \mathcal{F} \times \partial B_\varepsilon^q$, which will turn out to be a saddle-point equilibrium.

In order to define the pair $(\hat{F}, \hat{w})$, consider the algebraic Riccati equation

$$\begin{aligned} &C^T C + A^T P + PA + \\ &\quad + \kappa^{-1} P E E^T P - (PB + C^T D)(D^T D)^{-1}(B^T P + D^T C) = 0 \end{aligned} \tag{13}$$

which also appeared in Theorem 2.3 (with γ replaced by $\sqrt{\kappa}$). From H_∞ theory, it is well-known that for κ large enough, there exists a unique positive semi-definite solution of this equation, such that $A_{F_P} + \kappa^{-1}EE^TP$ is stable, where $F_P := -(D^TD)^{-1}(B^TP + D^TC)$, which is in $\mathcal{F}$, according to Theorem 2.3. Denote the infimum of all such κ's by κ^*. It is obvious that $\kappa^* > 0$. For each $\kappa > \kappa^*$, we shall denote the solution of (13) by P_κ, and define $w_\kappa \in L_2^q$ by $w_\kappa = \kappa^{-1}E^TP_\kappa x_\kappa$, where x_κ is generated by the system $\dot{x}_\kappa = (A_{F_P} + \kappa^{-1}EE^TP_\kappa)x_\kappa$, $x_\kappa(0) = x_0$. It is easily seen and in fact well-known from the theory of algebraic Riccati equations that P_κ is equal to X_κ, as defined in Theorem 3.5, with $F = F_{P_\kappa}$. Hence, generically (typically depending on x_0, see Remark 3.2) there exists a $\kappa > \kappa^*$, such that $\|w_\kappa\| = \varepsilon$, for each $\varepsilon > 0$. Denote it by $\hat{\kappa}$. Now, the pair that will turn out to be a saddle-point equilibrium is defined by

$$(\hat{F}, \hat{w}) := (F_{\hat{P}}, w_{\hat{\kappa}}), \text{ where } \hat{P} := P_{\hat{\kappa}}. \tag{14}$$

We want to show that $\varphi(\hat{F}, \hat{w}, x_0) < \varphi(F, \hat{w}, x_0)$, for all $F \neq \hat{F}$. Note that this statement implies that $F \mapsto \varphi(F, \hat{w}, x_0)$ is *uniquely* minimized at $F = \hat{F}$. The next lemma forms the basis for this property and can easily be obtained, again by a completion of the squares. We omit the proof.

Lemma 4.1 *For each $F \in \mathcal{F}$, $w \in L_2^q$ with $\|w\| = \varepsilon$, and $x_0 \neq 0$, we have*

$$\varphi(F, w, x_0) = x_0^T\hat{P}x_0 + \hat{\kappa}\varepsilon^2 + \\ + \int_0^\infty \left|D(F - \hat{F})x(t)\right|^2 dt - \hat{\kappa}\int_0^\infty \left|w(t) - \hat{\kappa}^{-1}E^T\hat{P}x(t)\right|^2 dt, \tag{15}$$

where x is generated by the system $\dot{x} = A_Fx + Ew$, $x(0) = x_0$.

Motivated by formula (15), we define for each $F \in \mathcal{F}$, $w \in B_\varepsilon^q$, and $x_0 \neq 0$, the expression $\psi(F, w, x_0)$ by

$$\psi(F, w, x_0) := \int_0^\infty \left|D(F - \hat{F})x(t)\right|^2 dt - \hat{\kappa}\int_0^\infty \left|w(t) - \hat{\kappa}^{-1}E^T\hat{P}x(t)\right|^2 dt, \tag{16}$$

where x is generated by the system $\dot{x} = A_Fx + Ew$, $x(0) = x_0$. Our next goal is to prove that if $w = \hat{w}$, this expression is positive for each $F \neq \hat{F}$. This will complete the proof of the minimizing property.

Lemma 4.2 *For each $F \in \mathcal{F}$, and $x_0 \neq 0$, we have*

$$\psi(F, \hat{w}, x_0) > 0, \text{ if } F \neq \hat{F}, \text{ and } \psi(\hat{F}, \hat{w}, x_0) = 0.$$

Proof: Define $v := D(\hat{F} - F)x$, and $\zeta := \hat{w} - \hat{\kappa}^{-1}E^T\hat{P}x$, where x is generated by the system $\dot{x} = A_Fx + E\hat{w}$, $x(0) = x_0$. We then have $\psi(F, \hat{w}, x_0) = \int_0^\infty |v|^2dt -$

$\hat{\kappa}\int_0^\infty |\zeta|^2 dt$. Define moreover $\xi := x_{\hat{\kappa}} - x$. It can easily be seen that $\dot{\xi} = A_{\hat{F}}\xi + B(D^TD)^{-1}D^Tv$, $\xi(0) = 0$, and $\zeta = \hat{\kappa}^{-1}E^T\hat{P}\xi$. Now, since $\int_0^\infty \frac{d}{dt}\xi^T\hat{P}\xi dt = 0$, it follows that

$$\begin{aligned}\psi(F,\hat{w},x_0) &= \int_0^\infty \left(v^Tv - \hat{\kappa}^{-1}\xi^T\hat{P}EE^T\hat{P}\xi - \frac{d}{dt}\xi^T\hat{P}\xi\right)dt = \\ &= \int_0^\infty \Big(|v - D(D^TD)^{-1}B^T\hat{P}\xi|^2 + \\ &\qquad -\xi^T(A_{\hat{F}}^T\hat{P} + \hat{P}A_{\hat{F}} + \hat{P}(B(D^TD)^{-1}B^T + \hat{\kappa}^{-1}EE^T)\hat{P})\xi\Big)\,dt.\end{aligned}$$

The Riccati equation (13), with $\kappa = \hat{\kappa}$, can be rewritten as

$$\begin{aligned}&C^T(I - D(D^TD)^{-1}D^T)C + \\ &\qquad + A_{\hat{F}}^T\hat{P} + \hat{P}A_{\hat{F}} + \hat{P}\left(B(D^TD)^{-1}B^T + \hat{\kappa}^{-1}EE^T\right)\hat{P} = 0.\end{aligned}$$

Hence,

$$\psi(F,\hat{w},x_0) = \int_0^\infty \Big(|v - D(D^TD)^{-1}B^T\hat{P}\xi|^2 + \xi^TC^T(I - D(D^TD)^{-1}D^T)C\xi\Big)\,dt.$$

Since $I - D(D^TD)^{-1}D^T$ is positive semi-definite, we conclude that $\psi(F,\hat{w},x_0) \geq 0$ for all F. Moreover, if $\psi(F,\hat{w},x_0) = 0$, it follows that $v = D(D^TD)^{-1}B^T\hat{P}\xi$, which implies that ξ is generated by $\dot{\xi} = (A_{\hat{F}} + B(D^TD)^{-1}B^T\hat{P})\xi$, $\xi(0) = 0$. Obviously, this gives us $\xi = 0$, and thus $v = 0$, or, equivalently, $F = \hat{F}$. This completes the proof. □

This lemma completes the proof of the minimizing property of the saddle-point equilibrium. Formula (15) displays the minimal value, i.e.

$$\varphi(\hat{F},\hat{w},x_0) = x_0^T\hat{P}x_0 + \hat{\kappa}\varepsilon^2. \tag{17}$$

We have also shown that this minimal value is uniquely attained by $F = \hat{F}$. We proceed by proving the maximizing part of the saddle-point equilibrium. This is basically a consequence of the previous section. Indeed, by applying Theorem 3.5 to $F = \hat{F}$, one gets $\hat{\kappa} = \overline{\kappa}$, immediately implying that $\varphi(\hat{F},w,x_0) < \varphi(\hat{F},\hat{w},x_0)$, for each $w \in B_\varepsilon^q$. Again, we conclude that the maximal value of $w \mapsto \varphi(\hat{F},w,x_0)$, i.e. $x_0^T\hat{P}x_0 + \hat{\kappa}\varepsilon^2$, is *uniquely* attained by $\hat{w}$. We summarize the results of the present section in the following theorem, which is the main result of our study.

Theorem 4.3 *Let $\varepsilon > 0$. Let P_κ be the solution of the algebraic Riccati equation*

$$\begin{aligned}&C^TC + A^TP + PA + \\ &\qquad + \kappa^{-1}PEE^TP - (PB + C^TD)(D^TD)^{-1}(B^TP + D^TC) = 0\end{aligned}$$

such that $A_{F_\kappa}+\kappa^{-1}EE^TP_\kappa$ *is stable, where* $F_\kappa := -(D^TD)^{-1}(B^TP_\kappa+D^TC)$. *Denote the infimum of all* κ*'s for which there exists such a* P *by* κ^*. *Let* S_κ *be the solution of the Lyapunov equation*

$$\left(A_{F_\kappa}+\kappa^{-1}EE^TP_\kappa\right)^T S + S\left(A_{F_\kappa}+\kappa^{-1}EE^TP_\kappa\right) = -PEE^TP.$$

Furthermore, let x_0 *be such that there exists a* $\hat\kappa > \kappa^*$ *that satisfies* $\hat\kappa^{-2}x_0^TS_{\hat\kappa}x_0 = \varepsilon^2$. *Let* $\hat F = F_{\hat\kappa}$, *and* $\hat w = \hat\kappa^{-1}E^TP_{\hat\kappa}x$, *where* x *is generated by the system* $\dot x = (A_{F_{\hat\kappa}} + \hat\kappa^{-1}EE^TP_{\hat\kappa})x$, $x(0) = x_0$. *Then we have*

$$\varphi(F_{\hat\kappa}, w, x_0) < \varphi(F_{\hat\kappa}, \hat w, x_0) < \varphi(F, \hat w, x_0),$$

and a solution of Problem 2.1 is given by $\hat F$.

5 Concluding Remarks

The most important conclusion of our research is that there exists a saddle-point equilibrium of the two person zero-sum game with the controller designer as minimizing player and the uncertainty as maximizing player. The controller corresponding to the saddle point equilibrium also solves the minimax problem, and the disturbance corresponding to this equilibrium solves the maximin problem, in agreement with the result of Chen [4].

We have solved the problem here under a certain (generic) assumption on the initial state. The situation in which this assumption is not satisfied calls for further investigation. Another topic for further research could be the analysis of the properties of the nonlinear feedback controller $\hat F(x(t), \varepsilon)$ for given ε.

Acknowledgement

We would like to thank S. Weiland for an inspiring discussion about the topic. We are also grateful to A. Sasane for his help with some functional analytic aspects.

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