

Center
for
Economic Research

No. 2000-89

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GOODS**

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September 2000

ISSN 0924-7815

On the Core of Semi-Infinite Transportation Games with Divisible Goods

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Abstract

The core of games arising from semi-infinite transportation situations with infinitely divisible goods is studied. In these situations one aims at maximizing the profit from transporting a good from a finite number of suppliers to an infinite number of demanders. This good is infinitely divisible. It is shown that the underlying primal and dual infinite programs have no duality gap. Further, there always exist so-called ε -core elements in the corresponding cooperative game. Conditions are provided under which the core of the corresponding games is nonempty.

Keywords: Infinite programs, transportation situations, cooperative games, core.

Journal of Economic Literature Classification Number: C71, C61.

1991 Mathematics Subject Classification Number: 90D12, 90C05, 90C34.

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1 Introduction

In this paper we study semi-infinite transportation situations with infinitely divisible goods and their corresponding (cooperative) transportation games. Special attention is paid to conditions (if any) under which there exist core elements or so-called ε -core elements of the transportation games.

In a (finite) transportation situation, units of a single *indivisible* good are transported from a finite number of suppliers to a finite number of demanders. If one unit of the good is transported from supplier i to demander j then this gives these agents a benefit of t_{ij} units. Of course, a supplier cannot supply more goods than he owns and a demander does not want to receive more units than she asked for. The goal of the suppliers and the demanders is to maximize their total benefits under these restrictions.

Sánchez-Soriano, López and García-Jurado ([8]) studied the core of transportation games, which are cooperative games arising from finite transportation situations. They show that there always exists an optimal solution of the dual linear program related to such a transportation situation. Any of these optimal solutions generates a core element of the transportation game but the set of core elements generated in this way need not exhaust the core.

Assignment situations are transportation situations where each supplier owns one unit of the good and each demander wants one unit of the good. In a sense, one is assigning a supplier to a demander and vice versa. Finite assignment games were first studied by Shapley and Shubik ([9]). Llorca, Tijs and Timmer ([5]) studied a semi-infinite extension of these games, namely games related to assignment situations with a countably infinite number of demanders. Other semi-infinite games arising from different linear programming situations are studied by Fragnelli, Patrone, Sideri and Tijs ([1]) and by Timmer, Llorca and Tijs ([10]).

A semi-infinite extension of transportation situations with indivisible goods is studied by Sánchez-Soriano, Llorca, Tijs and Timmer ([7]). Using the results of Llorca et al. ([5]), they show that the underlying infinite programs have no duality gap and that the core of the corresponding game is nonempty. Kortanek and Yamasaki ([3],[4]) also study semi-infinite transportation situations. The main difference with our analysis is that they assume that the total supply and the total demand for the good are equal (and consequently, finite) whereas we assume nothing about these quantities. Our analysis includes situations where the total supply is unequal to the total demand and situations with an infinite total demand. Further, we study semi-infinite transportation situations from a game-theoretic point of view, while they do not.

In this paper we study semi-infinite transportation situations with *infinitely divisible goods* and related games. These situations have a finite number of suppliers and a countably infinite number of demanders. The good that will be transported is assumed to be infinitely divisible. The goal of the suppliers and demanders is to maximize their total benefit such that no supplier supplies too much and any demander receives at most what she asked for. We start by showing that the underlying primal and dual infinite programs have no duality gap. In the proof of this result we need the corresponding result for semi-infinite transportation situations with indivisible goods by Sánchez-Soriano et al. ([7]). After this, the existence of so-called ε -core elements is shown. These are allocations of the total benefit over all suppliers and demanders in such a way that the gain of a subgroup of agents obtained by splitting off is at most ε , an arbitrarily small positive number. Finally we consider two types of situations: those with finite or infinite total demand. For both types of situations we show that there exist core elements of the corresponding transportation games. For the second type we need some conditions on the demands to achieve this result.

The organization of this paper is as follows. In the next section we present the main results for games arising from finite transportation situations and from semi-infinite transportation situations with indivisible goods. In section 3, games corresponding to semi-infinite transportation situations with infinitely divisible goods are introduced. We show that the underlying infinite programs have no duality gap. It is shown in section 4 that there always exist ε -core elements. Next, we consider two types of situations. In section 5 we study situations with a finite total demand. We prove that the core of the corresponding game is nonempty. Finally, in section 6 we consider situations with an infinite total demand. Conditions are provided under which the core of the corresponding game is nonempty.

2 Transportation problems and games

A (finite) transportation problem describes a situation in which demands at several locations for a certain indivisible good will be covered by supplies from other locations. The total demand for the good need not equal the total supply of the good. Transporting one unit of the good from a supply point to a demand point generates a certain profit. The goal of the suppliers and demanders is to find a transportation plan (how many of the good should be transported from any of the supply points to any of the demand points) that maximizes the total profit.

A *transportation problem* is a tuple (P, Q, T, s, d) where P and Q are respectively the finite sets of supply points and demand points. The transport of one unit of the good from supply point i to demand point j generates a profit of t_{ij} , a non-negative real number. The matrix $T = [t_{ij}]_{i \in P, j \in Q}$ contains all the profits per unit of the good. The supply at point $i \in P$ equals s_i units of the good and the demand at $j \in Q$ is d_j units where both s_i and d_j are natural numbers. The vectors $s = \{s_i\}_{i \in P}$ and $d = \{d_j\}_{j \in Q}$ contain respectively the supplies and demands of the good. For the sake of brevity we will use T to denote (P, Q, T, s, d) .

A *transportation plan* $X = [x_{ij}]_{i \in P, j \in Q}$ is a matrix where $x_{ij} \geq 0$ is the number of units of the good that will be transported from supply point i to demand point j . A supply point $i \in P$ can supply at most s_i units of the good, $\sum_{j \in Q} x_{ij} \leq s_i$, and a demand point $j \in Q$ wants to receive at most d_j units, $\sum_{i \in P} x_{ij} \leq d_j$. The maximal profit that can be obtained in this situation is

$$v_p(T) = \max \left\{ \sum_{(i,j) \in P \times Q} t_{ij} x_{ij} \mid X \text{ is a transportation plan} \right\}.$$

A transportation plan X is also called a feasible solution for T . Such a solution is *optimal* if it attains the maximal profit, $\sum_{(i,j) \in P \times Q} t_{ij} x_{ij} = v_p(T)$. It is well known that if we allow x_{ij} to be a nonnegative real number instead of an integer number then there exists an optimal integer solution.

Given a transportation problem T , the corresponding transportation game (N, w) is a cooperative transferable utility (TU) game with player set $N = P \cup Q$. Let $S \subset N$, $S \neq \emptyset$, be a coalition of players and define $P_S = P \cap S$ and $Q_S = Q \cap S$. If $S = P_S$ or $S = Q_S$, there are either only suppliers or demanders present, then no transport can take place and the worth $w(S)$ of coalition S equals zero. Otherwise, the worth $w(S)$ depends upon the transportation plans for this coalition. A transportation plan $X(S)$ for coalition S is a transportation plan for the transportation problem $T_S = (P_S, Q_S, [t_{ij}]_{i \in P_S, j \in Q_S}, \{s_i\}_{i \in P_S}, \{d_j\}_{j \in Q_S})$. In this case

$$\begin{aligned} w(S) &= \max \left\{ \sum_{(i,j) \in P_S \times Q_S} t_{ij} x_{ij} \mid X(S) \text{ is a transportation plan for } S \right\} \\ &= v_p(T_S) \end{aligned}$$

is the worth of coalition S . By convention $w(\emptyset) = 0$.

In Sánchez-Soriano et al. ([7]) these finite transportation problems are extended to *semi-infinite transportation problems* in which the set of demanders is the

countably infinite set $Q = \mathbb{N}$ (where \mathbb{N} is the set of natural numbers). Further s_i and d_j are natural numbers for all $i \in P$ and $j \in Q$, and the matrix $T = [t_{ij}]_{i \in P, j \in Q}$ is bounded, $\|T\|_\infty := \sup_{(i,j) \in P \times Q} |t_{ij}| < \infty$. We call these *discrete* situations since all the supplies and demands are integer numbers; we consider indivisible goods. This kind of situation can appear, for example, if a good has to be transported in containers from a finite set of warehouses to an infinite set of potential consumers.

Associated to such a semi-infinite transportation problem is a semi-infinite transportation game (N, w) with player set $N = P \cup Q$. As before, the worth of coalition S equals zero, $w(S) = 0$, if $S = P_S$ or $S = Q_S$ and

$$\begin{aligned} w(S) &= \sup \left\{ \sum_{(i,j) \in P_S \times Q_S} t_{ij} x_{ij} \mid X(S) \text{ is a transportation plan for } S \right\} \\ &= v_p(T_S) \end{aligned}$$

otherwise. If we allow x_{ij} to be a nonnegative real number then the dual problem D corresponding to the problem above for $S = N$ is

$$\inf \left\{ \sum_{i \in P} s_i u_i + \sum_{j \in Q} d_j v_j \mid u_i + v_j \geq t_{ij}, u_i, v_j \geq 0 \text{ for all } i \in P, j \in Q \right\}$$

with value $v_d(T)$. Notice that the primal and dual programs have an infinite number of variables and an infinite number of constraints. Sánchez-Soriano et al. ([7]) show that the primal and dual problem have the same value and that there exist optimal solutions for D .

When thinking about how to share the profit among the suppliers and the demanders, one can consider to share the profit according to an element in the core of the game. The *core* of a semi-infinite transportation game (N, w) is the set

$$C(w) = \left\{ x \in \mathbb{R}^N \mid \begin{array}{l} \sum_{i \in N} x_i = w(N) \text{ and} \\ \sum_{i \in S} x_i \geq w(S) \text{ for all } S \subset N, S \neq \emptyset \end{array} \right\}.$$

If a core-element x is proposed as a distribution of the total profit $w(N)$, then each coalition S will get at least as much as it can obtain on its own because $\sum_{i \in S} x_i \geq w(S)$. This implies that no coalition has an incentive to disagree with this proposal. A concept related to the core is the so-called *Owen set*¹, which is

¹Owen ([6]) presents a method to find a nonempty subset of the core of a linear production game. Gellekom et al. ([2]) names this set the ‘Owen set’.

defined by

$$\text{Owen}(T) = \left\{ x \in \mathbb{R}^N \mid \begin{array}{l} \exists (u, v) \text{ optimal for } D \text{ such that } x_k = s_k u_k \\ \text{if } k \in P \text{ and } x_k = d_k v_k \text{ if } k \in Q \end{array} \right\}.$$

An element of the Owen set is easy to find and it turns out to be an element of the core of the corresponding transportation game as well, that is, $\text{Owen}(T) \subset C(w)$.

From now on, we will consider a more general type of semi-infinite transportation problems in which the supplies s_i and the demands d_j are positive *real* numbers. We call these *continuous* situations. The underlying idea is to consider infinitely divisible goods like, e.g., gas or electricity.

3 A no-duality-gap theorem

In this section we show that (semi-infinite) continuous transportation situations have no duality gap, that is, $v_p(T) = v_d(T)$.

Let T be a continuous transportation problem and let $\delta > 0$ be a positive number. We scale the problem T in units of δ and we use overestimates and underestimates of the scaled supplies and demands to obtain two discrete transportation problems $T^{\delta+}$ and $T^{\delta-}$ that are related to T . With the help of these discrete problems we can show the absence of a duality gap between the primal and dual problems for T .

For a nonnegative real number a , the notations $\lceil a \rceil$ and $\lfloor a \rfloor$ stand for respectively the upper and lower integer part of a . So, $a - 1 \leq \lfloor a \rfloor \leq a \leq \lceil a \rceil \leq a + 1$ and $\lfloor a \rfloor, \lceil a \rceil \in \{0, 1, 2, 3, \dots\}$. The discrete problem $T^{\delta+}$, which refers to $(P, Q, T^{\delta+}, s^{\delta+}, d^{\delta+})$, is defined by

$$s_i^{\delta+} = \left\lceil \frac{s_i}{\delta} \right\rceil, \quad d_j^{\delta+} = \left\lceil \frac{d_j}{\delta} \right\rceil, \quad t_{ij}^{\delta+} = \delta t_{ij}$$

and for $T^{\delta-}$, which stands for $(P, Q, T^{\delta-}, s^{\delta-}, d^{\delta-})$, we define

$$s_i^{\delta-} = \left\lfloor \frac{s_i}{\delta} \right\rfloor, \quad d_j^{\delta-} = \left\lfloor \frac{d_j}{\delta} \right\rfloor, \quad t_{ij}^{\delta-} = \delta t_{ij}$$

for all $i \in P$ and $j \in Q$. The values $v_p(T^{\delta+})$ and $v_p(T^{\delta-})$ are closely related to the value $v_p(T)$ as the following lemma shows.

Lemma 3.1 *Let T be a continuous transportation problem and $T^{\delta+}$, $T^{\delta-}$ the associated discrete problems for $\delta > 0$. Then the following inequalities hold.*

1. $v_p(T) \leq v_p(T^{\delta+}) \leq v_p(T) + m\delta \|T\|_\infty$,
2. $v_p(T) \geq v_p(T^{\delta-}) \geq v_p(T) - m\delta \|T\|_\infty$.

Proof. Consider the transportation problem (P, Q, T', s', d') , where

$$s'_i = \frac{s_i}{\delta}, \quad d'_j = \frac{d_j}{\delta}, \quad t'_{ij} = \delta t_{ij} \quad \text{for all } i \in P, j \in Q.$$

This problem is obtained from the original one by scaling into units of δ . So, both T and T' have the same value, $v_p(T) = v_p(T')$.

Let X' be a feasible transportation plan for T' , then it is also feasible for $T^{\delta+}$. Therefore $v_p(T') \leq v_p(T^{\delta+})$. On the other hand, in $T^{\delta+}$ we have at most m units of the good more than in T' because of the overestimations, so $v_p(T^{\delta+})$ will exceed $v_p(T')$ with at most $m \|\delta T\|_\infty$. Hence $v_p(T^{\delta+}) \leq v_p(T') + m \|\delta T\|_\infty = v_p(T) + m\delta \|T\|_\infty$.

To show the second item, let X be a feasible transportation plan for $T^{\delta-}$. Such a plan is also feasible for T' and therefore $v_p(T') \geq v_p(T^{\delta-})$. On the other hand, in T' we have at most m units of the good more than in $T^{\delta-}$, so $v_p(T')$ will exceed $v_p(T^{\delta-})$ with at most in $m \|\delta T\|_\infty$. Hence $v_p(T) = v_p(T') \leq v_p(T^{\delta-}) + m \|\delta T\|_\infty$. \square

The next lemma shows that the dual value of the original problem is bounded by the dual values of the related discrete problems.

Lemma 3.2 *Let T be a continuous transportation problem and $T^{\delta+}$, $T^{\delta-}$ the associated discrete problems for $\delta > 0$. Then*

$$v_d(T^{\delta-}) \leq v_d(T) \leq v_d(T^{\delta+}).$$

Proof. Let T' be the continuous transportation problem as defined in the proof of lemma 3.1. Notice that the dual programs related to $T^{\delta+}$, T' and $T^{\delta-}$ have the same sets of feasible solutions. For the coefficients of the objective functions it holds that $s_i^{\delta-} \leq s'_i \leq s_i^{\delta+}$ and $d_j^{\delta-} \leq d'_j \leq d_j^{\delta+}$. Also, we are dealing with infimum problems, so $v_d(T^{\delta-}) \leq v_d(T') \leq v_d(T^{\delta+})$.

We continue by showing $v_d(T) = v_d(T')$. Let (u, v) be a feasible solution for D . Define the pair (u', v') by $u'_i = \delta u_i$ and $v'_j = \delta v_j$. Then (u', v') is a feasible solution of the dual program related to T' and these solutions have the same value:

$$\sum_{i \in P} s'_i u'_i + \sum_{j \in Q} d'_j v'_j = \sum_{i \in P} s_i u_i + \sum_{j \in Q} d_j v_j.$$

From this we conclude that $v_d(T) = v_d(T')$, which finishes the proof. \square

Now we present the main result in this section, which tell us that there is no duality gap for continuous transportation problems.

Theorem 3.3 *Let T be a continuous transportation problem. Then,*

$$v_p(T) = v_d(T).$$

Proof. From part 2 in lemma 3.1 and weak duality, we know that

$$v_p(T) - m\delta \|T\|_\infty \leq v_p(T^{\delta-}) \leq v_p(T) \leq v_d(T). \quad (3.1)$$

On the other hand, from the previous lemma, the absence of duality gaps for discrete problems (Sánchez-Soriano et al. [7]) and part 1 in lemma 3.1, we obtain

$$v_d(T) \leq v_d(T^{\delta+}) = v_p(T^{\delta+}) \leq v_p(T) + m\delta \|T\|_\infty. \quad (3.2)$$

The desired result is obtained from the observation that (3.1) and (3.2) hold for all $\delta > 0$. \square

4 Transportation games possess ε -core elements

Now we know that the primal and the dual programs of a continuous transportation problem have the same value, we can look at the problem of the division of $v_p(T) = v_d(T)$ among the involved agents.

For an arbitrary TU game (N, w) , where N is a finite or a countably infinite player set and $w : 2^N \rightarrow \mathbb{R}$ is a map assigning to each coalition $S \subset N$ a real number and $w(\emptyset) = 0$, interesting reward allocations are ε -core elements, where $\varepsilon \geq 0$. We say that $z \in \mathbb{R}^N$ is an ε -core element of the game (N, w) if the following two conditions hold.

$$\text{(Efficiency)} \quad \sum_{i \in N} z_i = w(N) \quad (4.1)$$

$$\text{(\varepsilon-stability)} \quad \sum_{i \in S} z_i \geq w(S) - \varepsilon \text{ for all } S \subset N \quad (4.2)$$

So, an ε -core element divides $w(N)$ among the players, and a coalition S can gain at most ε by splitting off. An ε -core element with $\varepsilon = 0$ is called a *core element*. Interesting questions for arbitrary games are: do there exist core elements and do

there exist ε -core elements for each $\varepsilon > 0$? For finite transportation games the answer to both questions is yes. For semi-infinite transportation games we will give an affirmative answer to the second question in the theorem below. We do not know the answer to the first question for general semi-infinite transportation games. Partial affirmative answers are given in the sections 5 and 6.

To find an ε -core element, $\varepsilon > 0$, of the transportation game we will use almost optimal solutions of the dual problem. This is done in the proof of the next theorem.

Theorem 4.1 *Let $\varepsilon > 0$ and let (N, w) be the cooperative game corresponding to the continuous transportation problem T as defined earlier. Then there exists a vector $z \in \mathbb{R}^N$ which is an ε -core element.*

Proof. If $w(N) = 0$ then $0 \in \mathbb{R}^N$ is an ε -core element for each $\varepsilon \geq 0$. So, suppose that $w(N) \neq 0$. Take $\delta > 0$ such that $\varepsilon = (1 + \delta)^{-1}\delta w(N)$. Let $(u, v) \in \mathbb{R}^P \times \mathbb{R}^Q$ be a feasible solution of D that satisfies

$$\sum_{i \in P} s_i u_i + \sum_{j \in Q} d_j v_j \leq (1 + \delta)v_d(T).$$

We call (u, v) a $\delta v_d(T)$ -optimal solution of the dual transportation problem. Let $0 \leq \delta' \leq \delta$ be such that

$$\sum_{i \in P} s_i u_i + \sum_{j \in Q} d_j v_j = (1 + \delta')v_d(T).$$

Define $z \in \mathbb{R}^N$ by $z_i = (1 + \delta')^{-1}s_i u_i$ for all $i \in P$ and $z_j = (1 + \delta')^{-1}d_j v_j$ for all $j \in Q$. We claim that z is an ε -core element of the game (N, w) . Clearly, the efficiency condition (4.1) holds:

$$\sum_{i \in N} z_i = v_d(T) = v_p(T) = w(N)$$

where we use the no-gap result of theorem 3.3. To show condition (4.2), note first that for $S \subset P$ or $S \subset Q$ we have $\sum_{i \in S} z_i \geq 0 = w(S) \geq w(S) - \varepsilon$. Take S such that $P_S \neq \emptyset$ and $Q_S \neq \emptyset$. Then

$$\begin{aligned} \sum_{i \in S} z_i &= \sum_{i \in P_S} z_i + \sum_{j \in Q_S} z_j = (1 + \delta')^{-1} \left(\sum_{i \in P_S} s_i u_i + \sum_{j \in Q_S} d_j v_j \right) \\ &\geq (1 + \delta')^{-1} w(S) = w(S) - (1 + \delta')^{-1} \delta' w(S) \\ &\geq w(S) - (1 + \delta')^{-1} \delta' w(N) \\ &\geq w(S) - (1 + \delta)^{-1} \delta w(N) = w(S) - \varepsilon \end{aligned}$$

where the first inequality follows from $w(S) = v_p(T_S) = v_d(T_S) \leq \sum_{i \in P_S} s_i u_i + \sum_{i \in Q_S} d_i v_i$. The second inequality follows from the monotonicity of w : for $S \subset U$ we have $w(S) = v_p(T_S) \leq v_p(T_U) = w(U)$. The last inequality holds because the function $f(x) = (1+x)^{-1}x$ is increasing in x for $x \geq 0$ and $\delta' \leq \delta$. We conclude that (4.2) is satisfied for z , which finishes the proof. \square

5 Transportation problems with finite total demand

In this section we consider continuous transportation problems (P, Q, T, s, d) with $\|T\|_\infty < \infty$ and also, without loss of generality, with $s_i > 0$ for all $i \in P$. Further we put the following condition on the demands.

$$\text{(Finite demand)} \quad \|d\|_1 := \sum_{j \in Q} d_j < \infty \quad (5.1)$$

For this type of transportation problems we show that the core of the corresponding game is nonempty. Useful will be the (fit) map $f : [0, \infty)^P \rightarrow [0, \infty)^Q$ defined by

$$f_j(x) = \max \left\{ (t_{ij} - x_i)_+ \mid i \in P \right\}$$

where a_+ is shorthand for $\max\{a, 0\}$, $a \in \mathbb{R}$. This fit map has the following properties for all $x, y \in \mathbb{R}_+^P$, $i \in P$ and $j \in Q$.

$$0 \leq f_j(x) \leq \|T\|_\infty \quad (5.2)$$

$$x_i + f_j(x) \geq t_{ij} \quad (5.3)$$

$$|f_j(x) - f_j(y)| \leq \|y - x\|_\infty := \max \{ |y_i - x_i| \mid i \in P \} \quad (5.4)$$

Condition (5.4) follows from the observation

$$-|y_i - x_i| \leq (t_{ij} - x_i)_+ - (t_{ij} - y_i)_+ \leq |y_i - x_i|$$

for all $j \in Q$. From (5.4) we can deduce for all $x, y \in \mathbb{R}_+^P$

$$\|f(x) - f(y)\|_\infty \leq \|y - x\|_\infty \quad (5.5)$$

where $\|a\|_\infty := \sup \{ |a_j| \mid j \in Q \}$ for all $a \in \mathbb{R}^Q$. From (5.2) and (5.3) follows that the vector $(x, f(x)) \in \mathbb{R}_+^P \times \mathbb{R}_+^Q$ is a feasible solution of the dual program D for all $x \in \mathbb{R}_+^P$. Furthermore,

$$v_j \geq f_j(u) \text{ for all feasible solutions } (u, v) \text{ for } D \text{ and all } j \in Q. \quad (5.6)$$

The theorem below shows the nonemptiness of the core of a transportation game in this setting.

Theorem 5.1 *Let T be a continuous transportation problem with $\|T\|_\infty < \infty$, $s > 0$, and $\|d\|_1 < \infty$. Then the corresponding transportation game (N, w) has $C(w) \neq \emptyset$.*

Proof. For each $\varepsilon \in (0, 1]$ take an ε -optimal solution $(u(\varepsilon), v(\varepsilon))$ of D (see the proof of theorem 4.1). So,

$$v_d(T) \leq \sum_{i \in P} s_i u_i(\varepsilon) + \sum_{j \in Q} d_j v_j(\varepsilon) \leq v_d(T) + \varepsilon.$$

In view of (5.6) we can assume that $v(\varepsilon) = f(u(\varepsilon))$. Take a decreasing sequence $\varepsilon(1), \varepsilon(2), \dots$ in $(0, 1]$ that converges to 0 such that the sequence $u(\varepsilon(1)), u(\varepsilon(2)), \dots$ converges to, say, \bar{u} . This is possible because the set $\{u(\varepsilon) \mid \varepsilon \in (0, 1]\}$ is a subset of $\{a \in \mathbb{R}_+^P \mid \sum_{i \in P} s_i a_i \leq v_d(T) + 1\}$, which is a compact² set. Then $\lim_{k \rightarrow \infty} u(\varepsilon(k)) = \bar{u}$ holds as a limit with respect to the $\|\cdot\|_\infty$ -norm. Let $z \in \mathbb{R}^N$ be defined by $z_i = s_i \bar{u}_i$ for all $i \in P$ and $z_j = d_j f_j(\bar{u})$ for all $j \in Q$. We show that z is a core element of (N, w) .

First note that by (5.5)

$$\lim_{k \rightarrow \infty} \|(u(\varepsilon(k)), f(u(\varepsilon(k)))) - (\bar{u}, f(\bar{u}))\|_\infty = 0$$

where $\|(a, b)\|_\infty = \max\{\|a\|_\infty, \|b\|_\infty\}$ for $a \in \mathbb{R}^P, b \in \mathbb{R}^Q$. Since the set of feasible solutions of D is closed with respect to $\|\cdot\|_\infty$ -limits we conclude that $(\bar{u}, f(\bar{u}))$ is a feasible solution of D . This implies immediately the stability of z :

$$\sum_{k \in S} z_k = \sum_{i \in P_S} s_i \bar{u}_i + \sum_{j \in Q_S} d_j f_j(\bar{u}) \geq w(S).$$

It remains to show that the efficiency condition (4.1) holds for z , that is, $\sum_{k \in N} z_k = w(N) = v_p(T)$ or

$$\sum_{i \in P} s_i \bar{u}_i + \sum_{j \in Q} d_j f_j(\bar{u}) = v_p(T).$$

We know that for all $k \in \mathbb{N}$

$$v_p(T) \leq \sum_{i \in P} s_i u_i(\varepsilon(k)) + \sum_{j \in Q} d_j f_j(u(\varepsilon(k))) \leq v_p(T) + \varepsilon.$$

²This follows from $s > 0$.

So, we are done if

$$\lim_{k \rightarrow \infty} \left(\sum_{i \in P} s_i (u_i(\varepsilon(k)) - \bar{u}_i) + \sum_{j \in Q} d_j (f_j(u(\varepsilon(k))) - f_j(\bar{u})) \right) = 0. \quad (5.7)$$

The absolute value of the expression behind the limit is at most equal to

$$\begin{aligned} & (\sum_{i \in P} s_i) \|u(\varepsilon(k)) - \bar{u}\|_\infty + (\sum_{j \in Q} d_j) \|f(u(\varepsilon(k))) - f(\bar{u})\|_\infty \\ & \leq (\|s\|_1 + \|d\|_1) \|u(\varepsilon(k)) - \bar{u}\|_\infty \end{aligned}$$

where the inequality follows from (5.5). Now (5.7) follows from (5.1) and from $\lim_{k \rightarrow \infty} \|u(\varepsilon(k)) - \bar{u}\|_\infty = 0$. \square

6 Transportation problems with infinite total demand

In this section we consider semi-infinite continuous transportation problems T with infinite total demand, $\sum_{j \in Q} d_j = \infty$, such that

$$d_j \geq \eta > 0 \text{ for all } j \in Q. \quad (6.1)$$

We assume that all demanders in Q want to receive at least the positive amount η . We can interpret this amount η as the minimal useful amount of the good. Our goal is to prove that the core of the corresponding transportation game is nonempty. We achieve this result in various steps, starting with a procedure to look for a finite transportation problem that is closely related to T .

Given a problem T as above, a demander j_0 is called a *good isolated demander* for a supply point i if $|\{j | t_{ij} \geq t_{ij_0}\}| < \infty$. We denote the set of good isolated demanders for i by $GID(i)$. For each supplier i we introduce a number t_i using the set $GID(i)$. Two cases can arise:

- a. If $GID(i) = \emptyset$, then $t_i = 0$.
- b. If $GID(i) \neq \emptyset$, then

$$t_i = \begin{cases} \max\{j | j \in GID(i)\}, & \sum_{h \in P} s_h > \sum_{j \in GID(i)} d_j, \\ \min\{k \in Q | v_p(T_i) = v_p(T_i^k)\}, & \sum_{h \in P} s_h \leq \sum_{j \in GID(i)} d_j, \end{cases}$$

where T_i refers to $(\{i\}, Q, [t_{ij}]_{j \in Q}, \sum_{h \in P} s_h, d)$ and T_i^k to the problem $(\{i\}, \{1, 2, \dots, k\}, [t_{ij}]_{j=1}^k, \sum_{h \in P} s_h, (d_1, \dots, d_k))$.

Now, define $t^* = \max_{i \in P} t_i$. Closely related to T is the finite transportation problem $(P, \{1, 2, \dots, t^*, t^* + 1\}, \bar{T}, s, \bar{d})$, in short: \bar{T} . Here, \bar{T} is the $m \times (t^* + 1)$ -matrix with entries

$$\bar{t}_{ij} = \begin{cases} t_{ij}, & j \leq t^*, \\ \sup \{t_{ik} | k \geq t^* + 1\}, & j = t^* + 1. \end{cases}$$

The demand vector \bar{d} , which has $t^* + 1$ entries, is defined by $\bar{d}_j = d_j$ if $j \leq t^*$ and $\bar{d}_{t^*+1} = \sum_{h \in P} s_h + 1$. The example below illustrates this method.

Example 6.1 Consider the semi-infinite transportation situation (P, Q, T, s, d) with $P = \{1, 2, 3, 4\}$, $Q = \{1, 2, \dots\}$, $d = (2.3, 5, 5, 5, 1.7, 3, 2, 1, 1, 1, 1, \dots)$, $s = (2, 1.1, 4.4, 3)$ and with T as below.

$$\begin{array}{c} \begin{array}{cccccccccccc} 2.3 & 5 & 5 & 5 & 1.7 & 3 & 2 & 1 & 1 & 1 & 1 & \dots & d_j \\ 2 & 5 & 5 & 5 & 5 & 5 & 4 & 5 & 5 & 5 & 5 & \dots & \\ 1.1 & 8 & 1 & 3\frac{1}{2} & 2 & 6 & 7 & 3\frac{3}{4} & 3\frac{4}{5} & 3\frac{5}{6} & 3\frac{6}{7} & 3\frac{7}{8} & \dots & \\ 4.4 & 1 & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{3}{4} & 1 & 1\frac{5}{6} & 1\frac{6}{7} & 1\frac{7}{8} & 1\frac{8}{9} & 1\frac{9}{10} & 1\frac{10}{11} & \dots & \\ 3 & 1 & 6 & 7 & 6 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & \dots & \end{array} \\ s_i \end{array} = T$$

In this situation we have $GID(1) = \emptyset$, $GID(2) = \{1, 5, 6\}$, $GID(3) = \emptyset$ and $GID(4) = \{2, 3, 4\}$. Consequently, $t_1 = t_3 = 0$. To determine t_2 we notice that $\sum_{h \in P} s_h = 10.5 > d_1 + d_5 + d_6 = 7$. Thus $t_2 = \max\{j | j \in GID(2)\} = 6$. Finally, for t_4 we have $\sum_{h \in P} s_h = 10.5 < d_2 + d_3 + d_4 = 15$. Hence, $t_4 = \min\{k \in Q | v_p(T_4^k) = v_p(T_4)\}$ with

$$T_4 : \quad \begin{array}{c} \begin{array}{cccccccccccc} 2.3 & 5 & 5 & 5 & 1.7 & 3 & 2 & 1 & 1 & 1 & 1 & \dots & d_j \\ 10.5 & 1 & 6 & 7 & 6 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & \dots & \end{array} \\ \sum_{h \in P} s_h \end{array},$$

so, $t_4 = 4$. Now $t^* = \max\{0, 6, 0, 4\} = 6$ and

$$\bar{T} : \quad \begin{array}{c} \begin{array}{cccccccc} 2.3 & 5 & 5 & 5 & 1.7 & 3 & 11.5 & \bar{d}_j \\ 2 & 5 & 5 & 5 & 5 & 5 & 5 & \\ 1.1 & 8 & 1 & 3\frac{1}{2} & 2 & 6 & 7 & 4 \\ 4.4 & 1 & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{3}{4} & 1 & 1\frac{5}{6} & 2 \\ 3 & 1 & 6 & 7 & 6 & 2 & 1 & 2 \end{array} \\ s_i \end{array}$$

is the finite transportation problem related to T .

Lemma 6.1 *Let T be a transportation problem as defined above, and let \bar{T} be the corresponding finite transportation problem. Then,*

$$v_p(T) = v_p(\bar{T}).$$

Proof. Let X be a feasible plan for T . Define the plan \bar{X} for \bar{T} as follows:

$$\bar{x}_{ij} = \begin{cases} x_{ij}, & j \leq t^*, \\ \sum_{k=t^*+1}^{\infty} x_{ik}, & j = t^* + 1. \end{cases}$$

The plan \bar{X} is feasible for \bar{T} and

$$\begin{aligned} \sum_{i \in P} \sum_{j \in Q} x_{ij} t_{ij} &= \sum_{i \in P} \sum_{j=1}^{t^*} x_{ij} t_{ij} + \sum_{i \in P} \sum_{j=t^*+1}^{\infty} x_{ij} t_{ij} \\ &\leq \sum_{i \in P} \sum_{j=1}^{t^*} \bar{x}_{ij} \bar{t}_{ij} + \sum_{i \in P} \bar{x}_{i(t^*+1)} \bar{t}_{i(t^*+1)} \\ &= \sum_{i \in P} \sum_{j=1}^{t^*+1} \bar{x}_{ij} \bar{t}_{ij}. \end{aligned}$$

For any feasible plan X for T there exists a feasible plan \bar{X} for \bar{T} with a value that is equal or larger. Hence, $v_p(\bar{T}) \geq v_p(T)$.

Next, we show that $v_p(T) \geq v_p(\bar{T}) - \varepsilon$ for all $\varepsilon > 0$. Without loss of generality assume that $\sum_{i \in P} s_i > 0$. Let $\varepsilon > 0$, let $i \in P$ and let \bar{X} be an optimal solution for \bar{T} . Define the plan X for T by $x_{ij} = \bar{x}_{ij}$, $j \leq t^*$. If $\bar{x}_{i(t^*+1)} = 0$ then define $x_{ij} = 0$ for all $j > t^*$. Otherwise there exists a *finite* set of demanders $K \subset Q$ such that $\sum_{j \in K} d_j \geq \bar{x}_{i(t^*+1)}$ and for all $j \in K$ it holds that $j > t^*$ and $t_{ij} \geq \bar{t}_{i(t^*+1)} - \varepsilon / \sum_{i \in P} s_i$. This finite set K exists because of condition (6.1). Now we can divide the amount $\bar{x}_{i(t^*+1)}$ over the demanders in K in a feasible way. The plan X is feasible for T and

$$\begin{aligned} v_p(T) &\geq \sum_{i \in P} \sum_{j \in Q} x_{ij} t_{ij} = \sum_{i \in P} \sum_{j=1}^{t^*} x_{ij} t_{ij} + \sum_{i \in P} \sum_{j=t^*+1}^{\infty} x_{ij} t_{ij} \\ &\geq \sum_{i \in P} \sum_{j=1}^{t^*} \bar{x}_{ij} \bar{t}_{ij} + \sum_{i \in P} \bar{x}_{i(t^*+1)} \left(\bar{t}_{i(t^*+1)} - \varepsilon / \sum_{i \in P} s_i \right) \\ &\geq \sum_{i \in P} \sum_{j=1}^{t^*+1} \bar{x}_{ij} \bar{t}_{ij} - \varepsilon. \end{aligned}$$

We conclude that $v_p(T) \geq v_p(\bar{T}) - \varepsilon$ for all $\varepsilon > 0$. □

Example 6.2 For the situation in example 6.1 we have $v_p(T) = s_1 t_{12} + s_2 t_{21} + s_3 \sup_{j \in Q} t_{3j} + s_4 t_{43} = 48.6$ and $v_p(\bar{T}) = s_1 \bar{t}_{12} + s_2 \bar{t}_{21} + s_3 \bar{t}_{37} + s_4 \bar{t}_{43} = 48.6$.

The following example indicates that condition (6.1) is necessary in lemma 6.1.

Example 6.3 Let T be the transportation problem with $P = \{1\}$, $Q = N$ and

$$\begin{array}{c}
 2 \\
 s_i
 \end{array}
 \begin{array}{cccccccccccc}
 1 & 1 & 1 & \frac{1}{2} & 1 & \frac{1}{4} & 1 & \frac{1}{8} & 1 & \frac{1}{16} & 1 & \cdots \\
 \hline
 1 & 1 & 1 & \frac{1}{2} & 1 & \frac{1}{3} & 1 & \frac{1}{4} & 1 & \frac{1}{5} & 1 & \cdots
 \end{array}
 \begin{array}{c}
 d_j \\
 = T
 \end{array}$$

In this situation the total demand is infinite and $\inf_{j \in Q} d_j = 0$. Hence, condition (6.1), $d_j \geq \eta > 0$, $j \in Q$, does not hold. The value of T is

$$v_p(T) = 1 + \sum_{j=1}^{\infty} \frac{1}{2^j} \left(2 - \frac{1}{j+1} \right),$$

which lies in-between 2 and 3. The core of the corresponding one-person game is $C(w) = \{v_p(T)\}$. From $GID(1) = \emptyset$ we obtain $t^* = t_1 = 0$. The finite transportation problem \bar{T} corresponding to T is

$$\begin{array}{c}
 2 \\
 s_1
 \end{array}
 \begin{array}{c}
 3 \\
 \hline
 2
 \end{array}
 \begin{array}{c}
 d_1 \\
 = \bar{T}
 \end{array}$$

with value $v_p(\bar{T}) = 4$. We conclude that $v_p(T) \neq v_p(\bar{T})$.

The lemma below shows that any optimal solution for $D(\bar{T})$, the dual problem related to \bar{T} , has a special property.

Lemma 6.2 *Let T be a transportation problem as above, and \bar{T} the corresponding finite problem. If (\bar{u}, \bar{v}) is optimal for $D(\bar{T})$ then $\bar{v}_{t^*+1} = 0$.*

Proof. Since \bar{T} is a finite problem we know that there exist optimal solutions for $D(\bar{T})$ and for \bar{T} itself. Let (\bar{u}, \bar{v}) be optimal for $D(\bar{T})$ and let \bar{X} be an optimal plan for \bar{T} . According to the complementary slackness conditions

$$\sum_{j=1}^{t^*+1} \bar{v}_j \left(\bar{d}_j - \sum_{i \in P} \bar{x}_{ij} \right) = 0.$$

Since all quantities are nonnegative, this reduces to

$$\bar{v}_j \left(\bar{d}_j - \sum_{i \in P} \bar{x}_{ij} \right) = 0 \quad (6.2)$$

for $j = 1, \dots, t^* + 1$. By definition of \bar{d}_{t^*+1} and by \bar{X}

$$\bar{d}_{t^*+1} > \sum_{i \in P} s_i \geq \sum_{i \in P} \sum_{j=1}^{t^*+1} \bar{x}_{ij} \geq \sum_{i \in P} \bar{x}_{i(t^*+1)}.$$

Together with (6.2) we conclude that $\bar{v}_{t^*+1} = 0$. \square

With this result we can show that there exists an optimal solution for the dual problem related to T .

Theorem 6.3 *Let T be a transportation problem as above. Then there exists an optimal solution for $D(T)$.*

Proof. Let T be a transportation problem with infinite demand and \bar{T} the corresponding finite problem. Let (\bar{u}, \bar{v}) be optimal for $D(\bar{T})$. So,

$$\begin{cases} \bar{u}_i + \bar{v}_j \geq \bar{t}_{ij} = t_{ij}, & i \in P, j \leq t^*, \\ \bar{u}_i + \bar{v}_{t^*+1} \geq \bar{t}_{i(t^*+1)} = \sup \{t_{ij} | j \geq t^* + 1\}, & i \in P. \end{cases}$$

From lemma 6.2 we know $\bar{v}_{t^*+1} = 0$. Define $\alpha(\bar{v}) = (\bar{v}_1, \dots, \bar{v}_{t^*}, 0, 0, \dots)$. Then $\bar{u}_i \geq \sup \{t_{ij} : j \geq t^* + 1\}$ which implies $\bar{u}_i + \alpha(\bar{v})_j \geq t_{ij}$ for all $i \in P$ and $j \geq t^* + 1$. Hence, $(\bar{u}, \alpha(\bar{v}))$ is a feasible solution for $D(T)$.

According to lemma 6.1

$$\begin{aligned} v_p(T) &= v_p(\bar{T}) = v_d(\bar{T}) = \sum_{i \in P} s_i \bar{u}_i + \sum_{j=1}^{t^*+1} \bar{d}_j \bar{v}_j \\ &= \sum_{i \in P} s_i \bar{u}_i + \sum_{j \in Q} d_j \alpha(\bar{v})_j \geq v_d(T) = v_p(T) \end{aligned}$$

where the inequality follows from $(\bar{u}, \alpha(\bar{v}))$ being a feasible solution for $D(T)$. We conclude that

$$\sum_{i \in P} s_i \bar{u}_i + \sum_{j \in Q} d_j \alpha(\bar{v})_j = v_d(T),$$

the solution $(\bar{u}, \alpha(\bar{v}))$ is optimal for $D(T)$. \square

Finally, we are able to show that the corresponding cooperative game has a nonempty core.

Theorem 6.4 Let T be a transportation problem as above and (N, w) its corresponding game. Then the core of this game, $C(w)$, is a nonempty set.

Proof. From theorem 6.3 we know that there exists an optimal solution for $D(T)$. Let (u, v) be such an optimal solution. Define the vector x by $x_k = s_k u_k$ for $k \in P$ and $x_k = d_k v_k$ for $k \in Q$. Then $x \in \text{Owen}(T) \subset C(w)$. \square

Example 6.4 Consider once again the semi-infinite transportation situation in example 6.1. An optimal solution for $D(\bar{T})$ is (\bar{u}, \bar{v}) with $\bar{u} = (5, 8, 2, 7)$ and $\bar{v} = (0, \dots, 0)$. Then $(\bar{u}, \alpha(\bar{v}))$ is optimal for $D(T)$ with $\alpha(\bar{v}) = (0, 0, \dots)$. According to the previous theorem, the vector $x = ((10, 8.8, 8.8, 21), (0, 0, \dots))$ is an element of the core $C(w)$.

As example 6.3 shows the procedure used in this section to find core elements cannot be used to prove that the core of a more general semi-infinite continuous transportation game with infinite total demand is nonempty. An alternative approach could be the one described in section 5, but it fails because the total demand is infinite. So, it remains an open question whether the core is nonempty for games corresponding to continuous transportation situations with infinite total demand and without a positive lower bound on the demands.

References

- [1] Fragnelli, V., F. Patrone E. Sideri and S. Tijs: “Balanced Games Arising from Infinite Linear Models,” *Mathematical Methods of Operations Research* 50 (1999), 385–397.
- [2] Gellekom, J.R.G. van, J.A.M. Potters, J.H. Reijnierse, M.C. Engel, and S.H. Tijs: “Characterization of the Owen Set of Linear Production Processes,” *Games and Economic Behavior* 32 (2000), 139–156.
- [3] Kortanek, K.O. and M. Yamasaki: “Semi-Infinite Transportation Problems”, *Journal of Mathematical Analysis and Applications* 88 (1982), 555–565.
- [4] Kortanek, K.O. and M. Yamasaki: “Discrete Infinite Transportation Problems”, *Discrete Applied Mathematics* 58 (1995), 19–33.

- [5] Llorca, N., S. Tijs and J. Timmer: “Semi-Infinite Assignment Problems and Related Games,” *CentER Discussion Paper 9974*, Tilburg University, The Netherlands (1999).
- [6] Owen, G.: “On the Core of Linear Production Games,” *Mathematical Programming* 9 (1975), 358–370.
- [7] Sánchez-Soriano, J., N. Llorca, S. Tijs and J. Timmer: “Semi-Infinite Assignment and Transportation Games,” *CentER Discussion Paper 2000-43*, Tilburg University, The Netherlands (2000).
- [8] Sánchez-Soriano, J., M.A. López and I. García-Jurado: “On the Core of Transportation Games,” to appear in *Mathematical Social Sciences* (2000).
- [9] Shapley, L.S. and S. Shubik: “The Assignment Game I: The Core,” *International Journal of Game Theory* 1 (1972), 111–130.
- [10] Timmer, J., N. Llorca and S. Tijs: “Games Arising from Infinite Production Situations,” to appear in *International Game Theory Review* (2000).