CentER

Discussion Paper

No. 2003–94

COMBINATORIAL DESIGNS WITH TWO SINGULAR VALUES II. PARTIAL GEOMETRIC DESIGNS

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October 2003

ISSN 0924-7815

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Combinatorial designs with two singular values
II. Partial geometric designs

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2000 Mathematics Subject Classification: 05B20, 05B25, 05C50, 62K10

Abstract

In this and an earlier paper [17] we study combinatorial designs whose incidence matrix has two distinct singular values. These generalize (v, k, λ) designs, and include uniform multiplicative designs and partial geometric designs. Here we study the latter, which are precisely the designs with constant replication and block size. We collect most known results, give new characterization results, and we enumerate, partly by computer, all small ones.

1 Introduction

Combinatorial designs (a set of points, a set of blocks, and an incidence relation between those) are usually defined in terms of nice combinatorial properties, such as "each block has the same size", "every pair of points occurs in the same number of blocks", etc.. Many combinatorial designs defined in this way have the property that their $(0, 1)$-incidence matrix has nice algebraic properties. These algebraic properties are in turn relevant to the statistical properties of the designs.

Here we start from the point of view of such an algebraic property, i.e., the property that the incidence matrix N has two distinct singular values (the positive square roots of the (nonzero) eigenvalues of NN^T). Designs with zero or one singular value are trivial: they are empty or complete, respectively. Designs with two singular values include (v, k, λ) designs and transversal designs, but also some less familiar designs such as partial geometric designs and uniform multiplicative designs. The latter are precisely the nonsingular designs, and these are studied in an earlier paper [17]. Here we study partial geometric designs, that is, the designs with constant replication and constant block size. Partial geometric designs

*The research of E.R. van Dam has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences

were introduced (combinatorially) by Bose, Shrikhande, and Singhi [6], and were studied by Bose, Bridges, and Shrikhande [2, 3, 4, 5]. More recently Bagchi and Bagchi [1] studied the statistical properties of partial geometric designs. Here we shall collect most of earlier results, derive new theoretical characterizations, and enumerate, partly by computer, all small ones. By doing this we also give a partial answer to the question which designs have three eigenvalues, which was raised recently by Bayley (cf. [12]).

There is an important connection to algebraic graph theory in the sense that the incidence graphs of the studied designs are precisely the bipartite biregular graphs with four (in case of symmetric designs) or five eigenvalues. Graphs with few distinct eigenvalues have been studied before by the authors, cf. [10, 14, 15, 16], but so far not much attention has been paid to bipartite graphs.

2 Designs with two singular values

In order to eliminate some trivialities, we assume that the studied designs (and their bipartite incidence graphs) are connected, i.e., that there is no (nontrivial) subset of points and subset of blocks such that all incidences are between those subsets, or between their complements. Consequently the Perron-Frobenius theory (cf. [8, p. 80]) can be applied, and it follows that the largest singular value has multiplicity one and a positive eigenvector.

A design whose incidence matrix N has two singular values $\sigma_0 > \sigma_1$ has an incidence graph whose adjacency matrix

$$A = \begin{bmatrix} O & N \\ N^T & O \end{bmatrix}$$

has eigenvalues $\pm\sigma_0$, $\pm\sigma_1$, and possibly 0. Let α be the positive eigenvector of NN^T with eigenvalue σ_0^2, normalized (for future purposes) such that $\alpha^T\alpha = \sigma_0(\sigma_0^2 - \sigma_1^2)$. Let β be such that $N^T\alpha = \sigma_0\beta$, then $N\beta = \sigma_0\alpha$ and β is the positive eigenvector of N^TN with eigenvalue σ_0^2 such that $\beta^T\beta = \alpha^T\alpha$. Now the vectors $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $y = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$ are eigenvectors of A with the eigenvalues $\pm\sigma_0$, respectively. It moreover follows that $(A^2 - \sigma_1^2 I)A$, which has rank 2, equals $\frac{1}{2}(xx^T - yy^T)$. From this we derive that

$$NN^TN = \sigma_1^2 N + \alpha\beta^T.$$

Going back from this equation to the fact that N has two singular values is easy, so we have the following.

Proposition 1 *A non-complete design with incidence matrix N has two singular values $\sigma_0 > \sigma_1$ if and only if $NN^TN = \sigma_1^2 N + \alpha\beta^T$, where α and β are the respective positive eigenvectors of NN^T and N^TN with eigenvalue σ_0^2 such that $\alpha^T\alpha = \beta^T\beta = \sigma_0(\sigma_0^2 - \sigma_1^2)$.*

In [17] we studied the nonsingular designs with two singular values. From $NN^TN = \sigma_1^2 N + \alpha\beta^T = \sigma_1^2 N + \alpha\alpha^T N/\sigma_0$ we then obtain from the invertibility of N that $NN^T = \sigma_1^2 I + \alpha\alpha^T/\sigma_0$, which is the defining equation for uniform multiplicative designs (the vector α in [17] is normalized differently though).

Before arriving at the main part of this paper, we make a few more observations.

2.1 Designs with smallest rank

The designs with two singular values, and (smallest) rank two are easily determined. Such a design, or its dual, has incidence matrix N of the form

$$N = \begin{bmatrix} J_{v_1,b_0} & J_{v_1,b_1} & O_{v_1,b_2} \\ J_{v_2,b_0} & O_{v_2,b_1} & J_{v_2,b_2} \end{bmatrix}$$

where v_1, v_2, b_0, b_1 are positive, and $J_{m,n}$ and $O_{m,n}$ denote all-ones and all-zeroes matrices of size $m \times n$, respectively. The corresponding incidence graphs can be seen as paths of length 4 or 5 of which each vertex, and its incident edges, are multiplied.

2.2 Non-square uniform multiplicative designs

If N is a uniform multiplicative design, then $M = [N\ N\ ...\ N]$ is a non-square uniform multiplicative design, i.e., $MM^T = \sigma_1^2 I + \alpha\alpha^T$ for some σ_1 and α. The dual design is no longer uniform multiplicative, however; but it has two singular values.

If N is a symmetric design, then $[N\ J]$ (with J a, not necessarily square, all-ones matrix) is also a non-square uniform multiplicative design. Some methods of [17] can be applied in the general study of such, and other, designs with two singular values. We will not go deeper into this matter, however, and turn our attention to the designs with constant replication and constant block size.

3 Partial geometric designs

3.1 General observations

In the remainder of this paper we consider designs with two singular values with constant replication, say r, and constant block size, say k. Hence we have $N\mathbf{j} = r\mathbf{j}$ and $N^T\mathbf{j} = k\mathbf{j}$, where $\mathbf{j}$ is the all-ones vector. From this we find that $\sigma_0 = \sqrt{rk}$, $\alpha = \sqrt{\frac{\sqrt{rk}(rk-\sigma_1^2)}{v}}\mathbf{j}$, and $\beta = \sqrt{\frac{\sqrt{rk}(rk-\sigma_1^2)}{b}}\mathbf{j}$, where v and $b = vr/k$ are the numbers of points and blocks of the design, respectively. From Proposition 1 we now find that

$$NN^TN = \sigma_1^2 N + \frac{k(rk - \sigma_1^2)}{v} J.$$

This is more or less the combinatorial definition of partial geometric designs by Bose, Shrikhande, and Singhi [6]. They called a (non-complete) design partial geometric with parameters (r, k, t, c) if each point has replication r, each block has size k, and for each point-block pair (p, B), the number of incident point-block pairs (p', B'), $p' \neq p$, $B' \neq B$, with p' in B and p in B' equals c or t, depending on whether p is in B or not, respectively. In matrix form this definition is equivalent to the equations

$$NN^TN = (r + k - 1 + c - t)N + tJ, \quad NJ = rJ, \quad N^TJ = kJ.$$

It was first observed by Bose, Bridges, and Shrikhande [2] that this is indeed equivalent to N having two singular values, constant row sums, and constant column sums.

From the above equations we find that

$$\sigma_1 = \sqrt{r + k - 1 + c - t}$$

and

$$t = \frac{k(rk - \sigma_1^2)}{v}.$$

This implies that

$$v = k((r-1)(k-1) + t - c)/t,$$

an expression already observed by Bose, Shrikhande, and Singhi [6].

If m_1 is the multiplicity of σ_1^2 as an eigenvalue of NN^T, then by considering the trace of NN^T we find that $rk + m_1\sigma_1^2 = vr$, and thus

$$m_1 = (v - k)r/\sigma_1^2$$

(cf. [2]). The integrality of this expression and the above expression for t turn out to be important restrictions for the parameter sets that can occur. Another restriction on the parameters is that kc and rc are both even. Indeed, for a fixed point p, the number of triples (p', B, B') with $p' \neq p$, $B' \neq B$, and p, p' both incident with both B and B', is even (since we can interchange B and B') and equals rc. Similarly kc is even.

We note that one can easily check that the complement of a partial geometric design (i.e., the design with incidence matrix $J - N$) is also partial geometric. The complement of a connected design may be disconnected, however. This occurs precisely when we take the complement of the disjoint union of at least three copies of a complete design.

More interesting examples of partial geometric designs are given by partial geometries (cf. [9]); this is exactly the case $c = 0$. Other examples are (v, k, λ) designs. Bose, Shrikhande, and Singhi [6] already characterized these block designs among the partial geometric designs (by using Cauchy's inequality). Here we derive the following equivalent characterization.

Proposition 2 *A partial geometric design with parameters (r, k, t, c) and singular values $\sqrt{rk}$ and σ_1 is a (v, k, λ) design if and only if $t = k(r - \sigma_1^2)$. In this case $\lambda = r - \sigma_1^2$.*

Proof. The equation $t = k(r - \sigma_1^2)$ is equivalent to $m_1 = v - 1$. Hence this is equivalent to $NN^T - \sigma_1^2 I$ having rank 1 (with eigenvector $\mathbf{j}$), hence to $NN^T = \sigma_1^2 I + (r - \sigma_1^2)J$. □

There is an easy way to construct infinitely many partial geometric designs from a given partial geometric design. If D is a design, then we denote by $D \otimes J_{m,n}$ the design obtained from D by replacing each point by m "identical" points, and each block by n "identical" blocks, and preserving the incidence relation. If N is the incidence matrix of D, then $D \otimes J_{m,n}$ has incidence matrix $N \otimes J_{m,n}$, where $\otimes$ denotes the Kronecker product. If D is partial geometric with parameters (r, k, t, c), then $D \otimes J_{m,n}$ is also partial geometric, with parameters $(nr, mk, mnt, mn(c + r + k - 1) - nr - mk + 1)$.

3.2 SPBIBDs and transversal designs

A (two-class) partially balanced incomplete block design (PBIBD) is a design with constant replication and constant block size, and whose incidence matrix N satisfies the equation

$$NN^T = rI + \lambda_1 A + \lambda_2 (J - I - A),$$

where A is the adjacency matrix of a strongly regular graph (i.e., a regular graph with three eigenvalues) and $\lambda_1 > \lambda_2$ are suitable integers. Consequently, a (two-class) PBIBD has at most three distinct singular values. We refer to [13] for tables and other information on PBIBDs.

Bridges and Shrikhande [7] called a (two-class) PBIBD special if, for suitable integers μ_1, μ_2 the equation $AN = \mu_1 N + \mu_2 (J - N)$ is satisfied. It is clear that these SPBIBDs are precisely the partial geometric PBIBDs. Bridges and Shrikhande [7] already realized that a PBIBD is special if and only if the rank of the incidence matrix is less than v (since then it has only two singular values). If θ_2 is the negative eigenvalue of the strongly regular graph (A), then this is equivalent to the equation $\theta_2 = (r - \lambda_2)/(\lambda_2 - \lambda_1)$, as one can check by computing the eigenvalues of NN^T from the equation $NN^T = rI + \lambda_1 A + \lambda_2 (J - I - A)$.

A well investigated family of PBIBDs is the family of group-divisible designs, that is, the PBIBDs where the underlying graph is a complete multipartite graph. Among the group-divisible designs, the so-called singular and semiregular ones are precisely the partial geometric designs.

Bose, Bridges, and Shrikhande [4] showed that under some conditions a partial geometric design must be an SPBIBD. We use their proof, but weaken these conditions somewhat.

Proposition 3 *Let D be a partial geometric design on v points, with incidence matrix N and parameters (r, k, t, c). If there are integers λ_1, λ_2 such that*

$$r(k - 1 + c) + \lambda_2((v - 1)\lambda_2 - 2r(k - 1)) = (\lambda_1 - \lambda_2)(r(k - 1) - \lambda_2(v - 1))$$

and such that each entry of $Y = NN^T - rI - \lambda_2(J - I)$ is a multiple of $\lambda_1 - \lambda_2$, then D is an SPBIBD or a (v, k, λ_i) design for $i = 1$ or 2.

Proof. By using the assumption and the fact that $(NN^T)^2 = (NN^T N)N^T = (r + k - 1 + c - t)NN^T + rtJ$, we find after some tedious calculations that Y^2 has diagonal entries $(Y^2)_{ii} = (\lambda_1 - \lambda_2)(r(k - 1) - \lambda_2(v - 1))$. The row sums of Y equal $r(k - 1) - \lambda_2(v - 1)$. Since each entry of Y is a multiple of $\lambda_1 - \lambda_2$, it follows that $(Y_{ij})^2 \geq (\lambda_1 - \lambda_2)Y_{ij}$ with equality if and only if Y_{ij} equals 0 or $\lambda_1 - \lambda_2$.

Now $v(\lambda_1 - \lambda_2)(r(k - 1) - \lambda_2(v - 1)) = \mathrm{Trace}(Y^2) = \sum_{ij}(Y_{ij})^2 \geq \sum_{ij}(\lambda_1 - \lambda_2)Y_{ij} = v(\lambda_1 - \lambda_2)(r(k - 1) - \lambda_2(v - 1))$, hence Y_{ij} equals 0 or $\lambda_1 - \lambda_2$ for all i, j, from which the result follows. □

As a consequence, we find the following by taking $\lambda_1 = \lambda_2 + 1$.

Corollary 1 *Let D be a partial geometric design on v points and parameters (r, k, t, c). If there is an integer λ_2 such that*

$$r(k - 1 + c) + \lambda_2((v - 1)\lambda_2 - 2r(k - 1)) = r(k - 1) - \lambda_2(v - 1)$$

then D is an SPBIBD with $\lambda_1 = \lambda_2 + 1$ or a (v, k, λ) design, where λ equals λ_2 or $\lambda_2 + 1$.

For the parameter set $(r, k, t, c) = (6, 4, 10, 4)$ on 8 points, we find from this corollary that all examples must be SPBIBDs with $\lambda_1 = 3, \lambda_2 = 2$ (on the graph $K_{4,4}$). There are 3 such examples; see Section 3.4 ($N = 7$).

Typical examples of SPBIBDs are transversal designs. A transversal design $\mathrm{TD}_\lambda(k, m)$ is a (group-divisible) design on $v = km$ points with replication $r = \lambda m$, and which can be partitioned into k groups of size m, with $b = \lambda m^2$ blocks of size k, such that each block is incident to one point from each group ($\lambda_2 = 0$), and a pair of points from different groups are contained in $\lambda_1 = \lambda$ blocks. The strongly regular graph involved is a complete k-partite graph. It follows from Corollary 1 that any partial geometric design with the same parameters as a transversal design with $\lambda = 1$ must be such a transversal design. In general this is not true for larger λ. Even stronger, the parameter set $(r, k, t, c) = (6, 5, 12, 8)$ (on 10 points) is realized by two transversal designs $\mathrm{TD}_3(5, 2)$, but also by two SPBIBDs with $\lambda_1 = 4, \lambda_2 = 2$ on the Petersen graph (so-called triangular designs); see Section 3.4 ($N = 14$).

Other examples of SPBIBDs arise in some strongly regular graphs having a partition into two regular subgraphs, of which (at least) one is strongly regular. The adjacencies between these two subgraphs form the incidences of a PBIBD (on the strongly regular subgraph), as is easily checked. Depending on the parameters of the strongly regular graphs, this PBIBD is special. It suffices for example that $\min\{f, g\} < v - 1$, where f and g are the multiplicities of the restricted eigenvalues of the (large) strongly regular graph, and v is the number of points of the design. An interesting example is obtained from the Higman-Sims graph, which has a partition into two Hoffman-Singleton subgraphs (cf. [8, p. 391]). For more on strongly regular graphs with strongly regular subgraphs we refer to [18].

Also some strongly regular graphs with a strongly regular subconstituent (the induced subgraph on the set of neighbours of a vertex) give examples. The adjacencies between the set of neighbours and the set of non-neighbours form the incidences of a PBIBD, and depending on the parameters this PBIBD is special. Also here the condition $\min\{f, g\} < v-1$ is sufficient. For more on strongly regular graphs with strongly regular subconstituents we refer to [11].

3.3 Some characterizations

In Proposition 2 we characterized the partial geometric designs with $m_1 = v - 1$. The ones with $m_1 = v - 2$ can be characterized as follows.

Proposition 4 *A partial geometric design with $m_1 = v - 2$ is an SPBIBD based on the strongly regular complete bipartite graph $K_{v/2,v/2}$, where two points in the same part of the bipartition are contained in $\lambda_2 = \frac{t}{k} - \frac{\sigma_1^2}{v}$ blocks, and two points in different parts are contained in $\lambda_1 = \frac{t}{k} + \frac{\sigma_1^2}{v}$ blocks.*

Proof. Consider the matrix $B = NN^T - \sigma_1^2 I - \frac{t}{k}J$, where N is the incidence matrix of a partial geometric design with $m_1 = v - 2$. Then B is a rank 1 matrix with diagonal elements $r - \sigma_1^2 - \frac{t}{k} = -\frac{\sigma_1^2}{v}$. It follows that the off-diagonal entries of NN^T can only take the values $\frac{t}{k} \pm \frac{\sigma_1^2}{v}$. It follows that the matrix $A = \frac{v}{2\sigma_1^2}(NN^T - \sigma_1^2 I + (\frac{\sigma_1^2}{v} - \frac{t}{k})J)$ is the adjacency matrix of a graph with eigenvalues $\pm\frac{v}{2}$ (both with multiplicity one) and 0. This graph must hence be the complete bipartite graph $K_{v/2,v/2}$. □

Besides the partial geometric designs with large rank, we can also describe the ones with small rank. Rank two cannot occur among the partial geometric designs, as we can see from the rank two designs in Section 2.1.

Proposition 5 *Let D be a partial geometric design with rank three. Then D is the complement of three disjoint copies of a complete design, or it is of the form* $\mathrm{TD}_1(2,2) \otimes J_{m,n}$.

Proof. Let N be the incidence matrix of D, and let $M = bN - rJ$, then $MJ = O$, hence M has rank 2. The matrix M has two distinct entries; each row has r entries $l = b - r$, and $b - r$ entries $o = -r$; each column has k entries l and $v - k$ entries o.

Consider two points p_1 and p_2 that are not incident to the same blocks, but have at least one common incident block (such points exist since D is non-complete and connected). Let $V_i, i = 1, 2$ be the set of points that are incident to the same blocks as p_i. Let B_{11} be the set of blocks that are incident to both p_1 and p_2, B_{10} be the set of blocks incident to p_1 but not to p_2, B_{01} be the set of blocks incident to p_2 but not to p_1, and B_{00} be the set of blocks incident to neither p_1 nor p_2. By assumption B_{11}, B_{10}, and B_{01} are nonempty. Since M has rank two, it follows now that each row is a linear combination of the rows corresponding to p_1 and p_2. Hence each row is constant on the columns corresponding to each of the $B_{ij}, i, j = 0, 1$. Since all blocks have the same size k, there must be a point p_3 that is not incident to the blocks in B_{11}. Let V_3 now be the set of points that are incident to the same blocks as p_3.

Let's first consider now the case that p_3 is not incident to the blocks in B_{10}. From the fact that M has rank two it follows that p_3 is incident to the blocks in B_{01} and B_{00}, and that $o = \pm l$. Since $o \neq l$, this is equivalent to $b = 2r$. By comparing block sizes in B_{11} and B_{10}, there must be a point p_4 that is incident to the blocks in B_{10}, but not to the blocks in B_{11}. It follows (from the rank of M) that p_4 is incident to the blocks in B_{00}, but not to the blocks in B_{01}. Let V_4 be the set of points that are incident to the same blocks as p_4. Then it follows (again from the rank of M) that no other rows can occur in M, i.e., each point is in one of the sets $V_i, i = 1, 2, 3, 4$. Thus N can be rearranged into the form

$$N = \begin{bmatrix} J & J & O & O \\ J & O & J & O \\ O & O & J & J \\ O & J & O & J \end{bmatrix}.$$

From checking the conditions for a partial geometric design, it follows that each block in this matrix has the same size, hence $D = TD_1(2,2) \otimes J_{v/4,b/4}$. The case that p_3 is not incident to the blocks in B_{01} is similar.

Finally, we consider the case that p_3 is incident to the blocks in both B_{10} and B_{01}. By considering the rank of M we find that $o = -2l$, which is equivalent to $2b = 3r$. In this case, B_{00} must be empty. Moreover, each point must occur in one of the sets $V_i, i = 1, 2, 3$. Thus N can be rearranged into the form

$$N = \begin{bmatrix} J & J & O \\ J & O & J \\ O & J & J \end{bmatrix},$$

and also here it follows that all blocks have the same size. So D is the complement of the union of three disjoint copies of a complete design. □

Partial geometric designs with block size two and three are characterized in the following propositions.

Proposition 6 *A partial geometric design with parameters (r, k, t, c) with $k = 2$ must be of the form $D \otimes J_{1,c+1}$, where D is either the design of all pairs on v points, or the transversal design* $\mathrm{TD}_1(2, r/(c+1))$.

Proof. By considering an incident point-block pair, it follows that each block (seen as pair of points) occurs exactly $c+1$ times. Thus the design is of the form $D \otimes J_{1,c+1}$. It follows that D is also partial geometric with parameters $(r' = r/(c+1), k' = 2, t' = t/(c+1), c' = 0)$. It follows easily that $t' = 1$ or $t' = 2$ (since in D each pair occurs at most once as block). If $t' = 2$, then it follows that D is the design of all pairs. For $t' = 1$, consider the incidence matrix M of D. The matrix $A = MM^T - r'I$ is the adjacency matrix of a graph with possible eigenvalues $r', 0$, and $-r'$. This graph must be the complete bipartite graph $K_{r',r'}$. Hence the design consists of all pairs of points, with one point in one (fixed) half of the point set, and the other point in the other half, i.e., D is the transversal design $\mathrm{TD}_1(2, r')$).
□

Proposition 7 *A partial geometric design with parameters (r, k, t, c) with $k = 3$ is an SPBIBD with $\lambda_1 = \frac{c}{2} + 1$ and $\lambda_2 = 0$ or a $(v, 3, \lambda = \frac{c}{2} + 1)$ design. Moreover, in the case of an SPBIBD, $\frac{c}{2} + 1$ divides both r and $\sigma_1^2 - r$.*

Proof. Consider a partial geometric design with $k = 3$. Consider two points p_1, p_2 that are contained in at least one block, say B. Consider also the third point p_3 of B. Let b_0 be the number of blocks containing p_1, p_2, and p_3. Let $b_i, i = 1, 2, 3$ be the number of blocks containing all of p_1, p_2, p_3 except p_i. Then it follows from considering the point-block pair (p_i, B) that $c = 2(b_0 - 1) + b_j + b_h$, where $i \neq j \neq h \neq i$. Thus $b_1 = b_2 = b_3$, and $c = 2(b_0 - 1) + 2b_3$. The number of blocks containing p_1 and p_2 thus equals $b_0 + b_3 = \frac{c}{2} + 1$. We have shown that two points are contained in either $\lambda_2 = 0$ or $\lambda_1 = \frac{c}{2} + 1$ blocks, from which the first part of the proposition follows. In the case of an SPBIBD, the matrix $\frac{1}{\lambda_1}(NN^T - rI)$ is the adjacency matrix of a strongly regular graph with eigenvalues $\frac{1}{\lambda_1} r(k-1)$, $\frac{1}{\lambda_1}(\sigma_1^2 - r)$, and $-\frac{1}{\lambda_1} r$, and these should be integer. □

3.4 Enumeration of small designs

By computer we enumerated all possible parameter sets (r, k, t, c) for partial geometric designs with v points and b blocks, with $v + b \leq 35$, $r \geq k$, $\frac{v}{2} \geq k > 2$, $2 < m_1 < v - 1$ (thus excluding (v, k, λ) designs, rank 3 designs, and the case $k = 2$; see Propositions 2, 5, and 6). We also checked that kc and rc are both even (see Section 3.1). The obtained parameter sets are displayed in Table 1. For most parameters sets (all with $v + b \leq 31$) we determined, partly by computer, the number of corresponding designs. The column "#" gives the number of designs for each parameter set; in between brackets we give, for the square designs, the number of designs up to duality. We shall comment on some parameter sets in the following.

- $N = 1$. By Proposition 7 these designs must be transversal designs $\mathrm{TD}_2(3, 2)$. It is easily found that there are two such designs, one of which is $(4, 2, 1)^* \otimes J_{1,2}$, where D^* denotes the dual design of D.

- $N = 2$. The incidence graphs of such designs are cospectral to the Hamming graph $\mathrm{H}(4, 2)$. It is known that besides the Hamming graph, there is one such graph (cf. [8, p. 263]). Both graphs can be described by a transversal design $\mathrm{TD}_2(4, 2)$. The design corresponding to $\mathrm{H}(4, 2)$ is self-dual, the other is not. The dual of the latter provides the third design (this one is not transversal) with these parameters.

- $N = 7$. By Corollary 1 or Proposition 4 this is an SPBIBD with $\lambda_1 = 3$ and $\lambda_2 = 2$ on the complete bipartite graph. By computer we enumerated all 3 such designs. One is the dual of $\mathrm{TD}_2(6, 2)$.

- $N = 12$. By Proposition 7 such a design is an SPBIBD, with $\lambda_1 = 2, \lambda_2 = 0$, on the lattice graph $\mathrm{L}_2(3)$. It is easily found that there is a unique such design, $\mathrm{TD}_1(2, 3)^* \otimes J_{1,2}$.

- $N = 16$. Five of the six designs are $\mathrm{TD}_4(4, 2)$. The remaining design is one of the three designs obtained as $D_2 \otimes J_{1,2}$, where D_2 stands for a design with parameter set under $N = 2$.

- $N = 19$. Both designs have a polarity with no absolute points, i.e. they have a symmetric incidence matrix with zero diagonal. The corresponding graphs have 4 distinct eigenvalues. One of these is the line graph of the Cube.

- $N = 20$. By Corollary 1 all designs with these parameters are SPBIBDs with $\lambda_1 = 3, \lambda_2 = 2$, on $K_{4,4,4}$. There is one example with a polarity with no absolute points. The corresponding graph is the line graph of the cocktail-party graph $\mathrm{CP}(3)$.

- $N = 26$. There are 15 examples, 11 of which are $\mathrm{TD}_4(5, 2)$. Among the dual designs there is one SPBIBD, with $\lambda_1 = 3, \lambda_2 = 1$ on the Clebsch graph.

- $N = 36$. Both designs are SPBIBDs on the Petersen graph with $\lambda_1 = 6, \lambda_2 = 3$. One of the designs is $(6, 3, 2)^* \otimes J_{1,3}$. In [13] it is incorrectly stated that the latter design is the unique triangular design with these parameters.

- $N = 37$. By Corollary 1, such a design must be a $\mathrm{TD}_1(3, 4)$, which is the same as a Latin square of side 4. There are 2 such Latin squares.

- $N = 59$. By Proposition 4, a design with these parameters must be an SPBIBD on $K_{6,6}$ with $\lambda_1 = 5$ and $\lambda_2 = 3$. This implies that on each of the two parts of the graph $K_{6,6}$, we must have a $(6, 3, 3)$ design. Such a design does not exist, however.

N	v,b	r,k	t,c	σ_1^2	m_1	# (u.t.d.)	remarks
1	6, 8	4, 3	4, 2	4	3	2	$\mathrm{TD}_2(3,2)$, Prop. 7
2	8, 8	4, 4	6, 3	4	4	3 (2)	$\mathrm{TD}_2(4,2)^{(*)}$, $\mathrm{H}(4,2)$
3	6,12	6, 3	6, 4	6	3	2	$\mathrm{TD}_3(3,2)$, Prop. 7
4	8,10	5, 4	8, 4	4	5	2	$\mathrm{TD}_2(5,2)^*$, computer
5	9, 9	3, 3	2, 0	3	6	1 (1)	$\mathrm{TD}_1(3,3)$, Cor. 1, Prop. 7
6	9, 9	3, 3	1, 2	6	3	0	Prop. 7
7	8,12	6, 4	10, 5	4	6	3	$\mathrm{TD}_2(6,2)^*$, Cor. 1, Prop. 4, computer
8	8,12	6, 4	9, 6	6	4	1	$\mathrm{TD}_3(4,2)$, computer
9	8,12	6, 4	8, 7	8	3	2	$\mathrm{TD}_2(3,2)^* \otimes J_{1,2}$, computer
10	10,10	4, 4	4, 3	6	4	1 (1)	$(5,2,1) \otimes J_{2,1}$, computer
11	10,10	5, 5	10, 6	5	5	0 (0)	computer
12	9,12	4, 3	2, 2	6	4	1	$\mathrm{TD}_1(2,3)^* \otimes J_{1,2}$, Prop. 7
13	6,16	8, 3	8, 6	8	3	3	$\mathrm{TD}_4(3,2)$, Prop. 7
14	10,12	6, 5	12, 8	6	5	4	$(6,3,2)^* \otimes J_{1,2}$, $\mathrm{TD}_3(5,2)$, computer
15	10,12	6, 5	10,10	10	3	0	computer
16	8,16	8, 4	12, 9	8	4	6	$D_2 \otimes J_{1,2}$, $\mathrm{TD}_4(4,2)$, computer
17	9,15	5, 3	3, 2	6	5	0	Prop. 7
18	10,14	7, 5	14,10	7	5	0	computer
19	12,12	4, 4	4, 1	4	8	2 (2)	4ev graphs, computer
20	12,12	6, 6	16, 9	4	9	8 (6)	4ev graph, Cor. 1, computer
21	12,12	6, 6	15,10	6	6	8 (4)	$\mathrm{TD}_3(6,2)$, computer
22	12,12	6, 6	12,13	12	3	4 (2)	$\mathrm{TD}_3(3,2) \otimes {J_{2,1}}^{(*)}$, computer
23	10,15	6, 4	6, 6	9	4	1	$(5,2,1)^* \otimes J_{1,3}$, computer
24	6,20	10, 3	10, 8	10	3	3	$\mathrm{TD}_5(3,2)$, Prop. 7
25	8,18	9, 4	12,12	12	3	2	$\mathrm{TD}_2(3,2)^* \otimes J_{1,3}$, computer
26	10,16	8, 5	16,12	8	5	15	$\mathrm{TD}_4(5,2)$, computer
27	12,14	7, 6	18,12	6	7	7	$\mathrm{TD}_3(7,2)^*$, computer
28	12,14	7, 6	14,16	14	3	0	computer
29	9,18	6, 3	4, 2	6	6	4	$\mathrm{TD}_2(3,3)$, Prop. 7, computer
30	9,18	6, 3	3, 4	9	4	1	$\mathrm{TD}_1(2,3)^* \otimes J_{1,3}$, Prop. 7
31	9,18	6, 3	2, 6	12	3	0	Prop. 7
32	12,15	5, 4	5, 2	5	8	0	computer
33	12,15	5, 4	4, 4	8	5	2	$(6,2,1) \otimes J_{2,1}$, computer
34	8,20	10, 4	16,11	8	5	6	$D_4 \otimes J_{1,2}$, computer
35	8,20	10, 4	15,12	10	4	3	$\mathrm{TD}_5(4,2)$, computer
36	10,18	9, 5	18,14	9	5	2	$(6,3,2)^* \otimes J_{1,3}$, computer
37	12,16	4, 3	2, 0	4	9	2	$\mathrm{TD}_1(3,4)$, Cor. 1, Prop. 7
38	12,16	8, 6	21,14	6	8	19	$\mathrm{TD}_3(8,2)^*$, computer
39	12,16	8, 6	20,15	8	6	55	$\mathrm{TD}_4(6,2)$, $D_7^* \otimes J_{1,2}$, computer
40	12,16	8, 6	18,17	12	4	5	$\mathrm{TD}_3(4,2)^* \otimes J_{1,2}$, computer
41	12,16	8, 6	16,19	16	3	3	$\mathrm{TD}_4(3,2) \otimes J_{2,1}$, computer
42	14,14	6, 6	12, 9	8	6	11 (7)	$(7,3,2) \otimes J_{2,1}$, computer

N	v, b	r, k	t, c	σ_1^2	m_1	# (u.t.d.)	remarks
43	6,24	12, 3	12,10	12	3	4	$\mathrm{TD}_6(3,2)$, Prop. 7
44	9,21	7, 3	5, 2	6	7	0	Prop. 4, 7
45	10,20	8, 4	8, 9	12	4	2	$(5,2,1) \otimes J_{2,2}, (5,2,1)^* \otimes J_{1,4}$, computer
46	10,20	10, 5	20,16	10	5	11	$\mathrm{TD}_5(5,2)$, computer
47	12,18	6, 4	6, 3	6	8	15	$\mathrm{TD}_2(4,3)$, computer
48	12,18	6, 4	4, 7	12	4	1	$\mathrm{TD}_1(2,3) \otimes J_{2,2}$, computer
49	12,18	9, 6	24,16	6	9	127	$\mathrm{TD}_3(9,2)^*$, computer
50	12,18	9, 6	18,22	18	3	2	$\mathrm{TD}_3(3,2)^* \otimes J_{1,3}$, computer
51	14,16	8, 7	24,18	8	7	169	$\mathrm{TD}_4(7,2)$, computer
52	15,15	3, 3	1, 0	4	9	1 (1)	GQ(2,2), Prop. 7
53	15,15	5, 5	5, 6	10	5	0 (0)	computer
54	15,15	6, 6	12, 7	6	9	0 (0)	computer
55	8,24	12, 4	20,13	8	6	56	Prop. 4, computer
56	8,24	12, 4	18,15	12	4	11	$D_2 \otimes J_{1,3}$, computer
57	8,24	12, 4	16,17	16	3	2	$\mathrm{TD}_2(3,2)^* \otimes J_{1,4}$, computer
58	10,22	11, 5	22,18	11	5	0	computer
59	12,20	10, 6	27,18	6	10	0	Prop. 4
60	12,20	10, 6	25,20	10	6	83	$\mathrm{TD}_5(6,2)$, computer
61	12,20	10, 6	24,21	12	5	≥ 14	$D_{14}^* \otimes J_{1,2}$, computer
62	12,20	10, 6	20,25	20	3	3	$D_{24} \otimes J_{2,1}$, computer
63	14,18	9, 7	28,20	7	9	0	computer
64	16,16	4, 4	3, 0	4	12	1 (1)	$\mathrm{TD}_1(4,4)$, Cor. 1
65	16,16	4, 4	2, 3	8	6	2 (1)	$\mathrm{TD}_1(2,4) \otimes J_{2,1}{}^{(*)}$, computer
66	16,16	4, 4	1, 6	12	4	0 (0)	
67	16,16	6, 6	9,10	12	5	0 (0)	computer
68	16,16	6, 6	6,15	20	3	0 (0)	computer
69	16,16	8, 8	28,21	8	8	642 (327)	$\mathrm{TD}_4(8,2)^{(*)}$, computer
70	16,16	8, 8	24,25	16	4	9 (5)	$D_{16} \otimes J_{2,1}{}^{(*)}$, computer
71	9,24	8, 3	4, 6	12	4	1	$\mathrm{TD}_1(2,3)^* \otimes J_{1,4}$, Prop. 7
72	12,21	7, 4	7, 4	7	8	0	computer
73	15,18	6, 5	8, 4	6	10	25	$\mathrm{TD}_2(5,3)$, computer
74	15,18	6, 5	6, 8	12	5	1	$(6,2,1)^* \otimes J_{1,3}$, computer
75	15,18	6, 5	5,10	15	4	0	computer
76	6,28	14, 3	14,12	14	3	4	$\mathrm{TD}_7(3,2)$, Prop. 7
77	10,24	12, 5	24,20	12	5	≥ 4	$\mathrm{TD}_6(5,2)$, $D_{14} \otimes J_{1,2}$
78	10,24	12, 5	20,24	20	3		
79	12,22	11, 6	22,28	22	3		
80	14,20	10, 7	30,24	10	7	≥ 1	$\mathrm{TD}_5(7,2)$
81	14,20	10, 7	28,26	14	5		
82	16,18	9, 8	32,24	8	9	≥ 1	$\mathrm{TD}_4(9,2)^*$
83	16,18	9, 8	30,26	12	6		
84	16,18	9, 8	24,32	24	3	≥ 3	$D_{13}^* \otimes J_{1,3}, D_{25} \otimes J_{2,1}$
85	10,25	10, 4	12, 9	10	6		
86	10,25	10, 4	10,12	15	4	≥ 1	$(5,2,1)^* \otimes J_{1,5}$
87	10,25	10, 4	8,15	20	3		
88	14,21	6, 4	4, 5	10	6	≥ 1	$(7,2,1) \otimes J_{2,1}$
89	14,21	9, 6	18,16	12	6	≥ 10	$(7,3,3) \otimes J_{2,1}$
90	15,20	8, 6	16,11	8	9		
91	15,20	8, 6	12,17	18	4	≥ 1	$(5,2,1) \otimes J_{3,2}$

Table 1: Small partial geometric designs

- $N = 66$. The parameters $r = k = 4, t = 1, c = 6$ easily give a contradiction, so there can be no such design.

Acknowledgments We thank Willem Haemers for the stimulating conversations we had on the topic of this paper.

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