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By S.Z. Alparslan Gök, R. Brânzei, S.Tijs

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Tilburg University

Cooperative interval games arising from airport situations with interval data

S.Z. Alparslan Gök [*,†] R. Branzei [‡] S. Tijs [§]

Abstract

This paper deals with the research area of cooperative interval games arising from airport situations with interval data. We also extend to airport interval games some results from classical theory.

JEL Classification: C71

Keywords: cooperative interval games, concave games, airport games, cost games, interval data

1 Introduction

In literature much attention is paid to airport situations and related games. We refer here to Littlechild and Owen (1977), Littlechild and Thompson (1977) and Tijs and Driessen (1986). In airport situations costs of the coalitions are considered. A cost game $< N, c >$ is a cooperative game, where N

[*]Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey and Süleyman Demirel University, Faculty of Arts and Sciences, Department of Mathematics, 32 260 Isparta, Turkey, e-mail: alzeynep@metu.edu.tr

[†]This author acknowledges the support of TUBITAK (Turkish Scientific and Technical Research Council).

[‡]Faculty of Computer Science, "Alexandru Ioan Cuza" University, Iaşi, Romania, e-mail: branzeir@infoiasi.ro

[§]CentER and Department of Econometrics and OR, Tilburg University, Tilburg, The Netherlands and Department of Mathematics, University of Genoa, Italy, e-mail: S.H.Tijs@uvt.nl

is the set of players, and $c : 2^N \to \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a real number in which $c(S)$ is the cost of the coalition S with $c(\emptyset) = 0$. A game $< N, c >$ is called concave (or submodular) if and only if $c(S \cup T) + c(S \cap T) \leq c(S) + c(T)$ for all $S, T \in 2^N$. It is well known that airport games are concave. The economists Baker (1965) and Thompson (1971) proposed an appealing rule now called the Baker-Thompson rule. The idea is that only users of a piece of the runway pay for that piece and they share the cost of it equally. The core C (Gillies (1959)) and the Shapley value ϕ (Shapley (1953)) are central solution concepts defined on the class of classical cooperative games. The Shapley value is a core element on the class of concave games. Littlechild and Owen (1973) showed that the Baker-Thompson rule corresponds to the Shapley value. For a game $< N, c >$ and a coalition $K \in 2^N \setminus \{\emptyset\}$ the dual K-unanimity game u_K^* is defined by

$$u_K^*(S) = \begin{cases} 1, & K \cap S \neq \emptyset \\ 0, & \text{otherwise}, \end{cases}$$

and the Shapley value $\phi(u_K^*)$ of the dual K-unanimity game u_K^* is defined by

$$\phi_i(u_K^*) = \begin{cases} 1/|K|, & i \in K \\ 0, & i \in N \setminus K. \end{cases}$$

We recall that the decomposition of an airport game $< N, c >$ using dual unanimity games is given by $c = \sum_{k=1}^{m} t_k u^*_{\cup_{r=k}^m N_r}$, where t_k is the extra cost to extend a runway which is already suitable for landings of planes of type $k-1$ to the one which is suitable for landings of planes of type k, and $N = \cup_{r=1}^m N_r$ is the set of all users of the runway.
In this paper we consider airport situations, where cost of pieces of the runway are intervals. Then, we associate as in the classical case to such a situation an interval cost game and extend to airport interval games the results presented above.
The rest of the paper is organized as follows. We recall in Section 2 basic notions and facts from interval calculus and the theory of cooperative interval games. Section 3 is devoted to airport situations with interval data and related airport interval games. We conclude in Section 4 with some final remarks on other economic and OR situations with interval data.

2 Preliminaries

We start with some preliminaries from interval calculus (Alparslan Gök, Branzei and Tijs (2008a)). We denote by $I(\mathbb{R})$ the set of all closed intervals in $\mathbb{R}$, and by $I(\mathbb{R})^N$ the set of all n-dimensional vectors with elements in $I(\mathbb{R})$.

Let $I, J \in I(\mathbb{R})$ with $I = \left[\underline{I}, \overline{I}\right]$, $J = \left[\underline{J}, \overline{J}\right]$, $|I| = \overline{I} - \underline{I}$ and $\alpha \in \mathbb{R}_+$. Then, $I + J = \left[\underline{I} + \underline{J}, \overline{I} + \overline{J}\right]$; $\alpha I = \left[\alpha \underline{I}, \alpha \overline{I}\right]$. The partial substraction operator $I - J$ is defined, only if $|I| \geq |J|$, by $I - J = \left[\underline{I} - \underline{J}, \overline{I} - \overline{J}\right]$. We say that I is weakly better than J, which we denote by $I \succcurlyeq J$, if and only if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. We also use the reverse notation $J \preccurlyeq I$, if and only if $\underline{J} \leq \underline{I}$ and $\overline{J} \leq \overline{I}$.

Now, we give basic definitions and some useful results of cooperative interval cost games inspired by the theory of cooperative interval reward games (Alparslan Gök, Miquel and Tijs (2008), Alparslan Gök, Branzei and Tijs (2008b)).

An interval cost game is an ordered pair $< N, d >$ where $N = \{1, 2, \ldots, n\}$ is the set of players, and $d : 2^N \to I(\mathbb{R})$ is the characteristic function such that $d(\emptyset) = [0, 0]$. For each $S \in 2^N$, the worth set (or worth interval) $d(S)$ of the coalition S in the interval game $< N, d >$ is of the form $[\underline{d}(S), \overline{d}(S)]$, where $\underline{d}(S)$ is the lower bound and $\overline{d}(S)$ is the upper bound of $d(S)$. Some classical cooperative cost games associated with an interval game $< N, d >$ will play a key role, namely the border games $< N, \underline{d} >$, $< N, \overline{d} >$ and the length game $< N, |d| >$, where $|d|(S) = \overline{d}(S) - \underline{d}(S)$ for each $S \in 2^N$.

Let $< N, d_1 >$ and $< N, d_2 >$ be interval cost games. We say that $d_1 \preccurlyeq d_2$ if $d_1(S) \preccurlyeq d_2(S)$ for each $S \in 2^N$. We define $< N, d_1 + d_2 >$ by $(d_1 + d_2)(S) = d_1(S) + d_2(S)$ for each $S \in 2^N$. For $< N, d_1 >$ and $< N, d_2 >$ with $|d_1(S)| \geq |d_2(S)|$ for each $S \in 2^N$, $< N, d_1 - d_2 >$ is defined by $(d_1 - d_2)(S) = d_1(S) - d_2(S)$. Given $< N, d >$ and $\lambda \in \mathbb{R}_+$ we define $< N, \lambda d >$ by $(\lambda d)(S) = \lambda \cdot d(S)$ for each $S \in 2^N$.

Let $< N, d >$ be an interval cost game. Then, the interval core $\mathcal{C}(d)$ is defined by

$$\mathcal{C}(d) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i \in N} I_i = d(N), \sum_{i \in S} I_i \preccurlyeq d(S), \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

A game $< N, d >$ is called size monotonic if $< N, |d| >$ is monotonic, i.e. $|d|(S) \leq |d|(T)$ for all $S, T \in 2^N$ with $S \subset T$. We denote by $SMIG^N$ the

class of size monotonic interval games with player set N.
An interval game $< N, d >$ is called concave if $< N, d >$ is submodular and $< N, |d| >$ is concave (or submodular), i.e.

$$d(S) + d(T) \succcurlyeq d(S \cup T) + d(S \cap T), \quad (1)$$

and $|d|\,(S) + |d|\,(T) \geq |d|\,(S \cup T) + |d|\,(S \cap T)$ for all $S, T \in 2^N$.
In the following we give a characterization for concave interval games.

Proposition 2.1. *Let $< N, d >$ be a concave interval game. Then the following assertions hold:*

(i) *A game $< N, d >$ is submodular if and only if $< N, \underline{d} >$ and $< N, \overline{d} >$ are concave (or submodular);*

(ii) *A game $< N, d >$ is concave if and only if $< N, |d| >$ and $< N, \underline{d} >$, $< N, \overline{d} >$ are concave (or submodular);*

(iii) *A game $< N, d >$ is concave if and only if $< N, \underline{d} >$ and $< N, |d| >$ are concave (or submodular).*

Proof. (i) This assertion follows from formula (1).

(ii) By definition $< N, d >$ is concave if and only if $< N, d >$ and $< N, |d| >$ are both submodular. By (i), $< N, d >$ is submodular if and only if its border games are concave (or submodular). Now, since submodularity of $< N, |d| >$ is the same with its concavity, we conclude that $< N, d >$ is concave if and only if $< N, \underline{d} >$, $< N, \overline{d} >$ and $< N, |d| >$ are concave (or submodular).

(iii) This assertion follows easily from (ii) by noting that $< N, |d| >$, $< N, \underline{d} >$ and $< N, \overline{d} >$ are concave (or submodular) if and only if $< N, |d| >$ and $< N, \underline{d} >$ are concave (or submodular) because $\overline{d} = \underline{d} + |d|$.

□

Note that the fact that $< N, |d| >$ is concave (or submodular) implies that $< N, |d| >$ is a monotonic game because for each $S, T \in 2^N$ with $S \subset T$ we have $|d|\,(S) + |d|\,(T \setminus S) \geq |d|\,(T) + |d|\,(\emptyset)$, and this implies that $|d|\,(S) \geq |d|\,(T)$. As a by-product we have each concave interval game $< N, d >$ is size monotonic.

Denote by $\Pi(N)$ the set of permutations $\sigma : N \to N$. Let $d \in SMIG^N$ and $\sigma \in \Pi(N)$. The interval marginal vector of d with respect to σ, $m^\sigma(d)$, corresponds to a situation, where the players enter a room one by one in the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$ and each player is given the marginal contribution he/she creates by entering. We denote the set of predecessors of i in σ by $P_\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$, where $\sigma^{-1}(i)$ denotes the entrance number of player i, and define

$$m_i^\sigma(d) = d(P_\sigma(i) \cup \{i\}) - d(P_\sigma(i)) \text{ for each } i \in N.$$

The interval Shapley value $\Phi : SMIG^N \to I(\mathbb{R})^N$ is defined by $\Phi(d) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(d)$, for each $w \in SMIG^N$. We notice that the interval Shapley value $\Phi(d)$ is an interval core element on the class of concave interval games.

3 Airport situations with interval data and related games

Consider the aircraft fee problem of an airport with one runway. Suppose that the planes which are to land are classified into m types. One can think that the runway is divided into m consecutive pieces, namely $P_1, \ldots, P_m$, and P_1 is sufficient for planes of type 1, P_1 and P_2 together are sufficient for planes of type 2, P_1, P_2 and P_3 are together sufficient for planes of type 3 etc.. Let $T_j \succcurlyeq [0, 0]$ be the interval cost of piece P_j. Let N_j be the set of players who own a plane of type j. Then, $N = \cup_{j=1}^m N_j$ is the set of all users of the runway. Let n_j denote the number of planes of type j. Let $S \subset N$ be a coalition with $S \cap N_j \neq \emptyset$ and $S \cap N_{j+1} = \emptyset$. Then, this coalition needs the pieces $P_1, \ldots, P_j$ of the runway. The interval cost of the used pieces of the runway is equal to $\sum_{i=1}^j T_i$. The characteristic function d of the corresponding cost game $< N, d >$ is given by $d(\emptyset) = [0, 0]$ and $d(S) = \sum_{i=1}^j T_i$ if $S \cap N_j \neq \emptyset$ and $S \cap N_{j+1} = \emptyset$ for each $S \subset N$.
To give a formal description of d we introduce interval games of the form $< N, Tc >$, where $< N, c >$ is the classical cooperative cost game and $T \in I(\mathbb{R}_+)$. We denote by $I(\mathbb{R}_+)$ the set of all closed intervals in $\mathbb{R}_+$.
Let $T \in I(\mathbb{R}_+)$ and $c : 2^N \to \mathbb{R}$. Then, the interval game $< N, Tc >$ is defined by $(Tc)(S) = c(S)T$ for each $S \in 2^N$.
We notice that the $\Phi(Tc)$ for the interval game $< N, Tc >$ is related with

the Shapley value $\phi(c)$ of the classical game $< N, c >$ as follows:

$$\Phi_i(Tc) = \phi_i(c)T \text{ for each } i \in N. \quad (2)$$

Now we give the description of the airport interval game as follows:

$$d = \sum_{k=1}^{m} T_k u^*_{\cup_{r=k}^m N_r}. \quad (3)$$

In the following proposition we show that airport interval games are concave.

PROPOSITION 3.1. *Let $< N, d >$ be an airport interval game. Then, $< N, d >$ is concave.*

Proof. It is well known that non-negative multiples of classical dual unanimity games are concave (or submodular). By (3) we have, $\underline{d} = \sum_{k=1}^{m} \underline{T}_k u^*_{\cup_{r=k}^m N_r}$ and $|d| = \sum_{k=1}^{m} |T_k| \, u^*_{\cup_{r=k}^m N_r}$ are concave because $\underline{T}_k \geq 0$ and $|T_k| \geq 0$ for each k. By Proposition 2.1 $< N, d >$ is concave. □

Next we propose an interval cost allocation rule β, which we call the interval Baker-Thompson rule. For a given airport interval situation the Baker-Thompson allocation for a player i of type j is as follows:

$$\beta_i(d) = \sum_{k=1}^{j} (\sum_{r=k}^{m} n_r)^{-1} T_k. \quad (4)$$

Note that for the piece P_k of the runway the users are $\cup_{r=k}^m N_r$, i.e. there are $\sum_{r=k}^m n_r$ users. So, $(\sum_{r=k}^m n_r)^{-1} T_k$ is the equal cost share of each user. This means that a player i of type j contributes to the cost of the pieces $P_1, \ldots, P_j$.
We notice that it is helpful to implement the interval Baker-Thompson allocation when the uncertainty regarding the costs of pieces of the runway is removed. For details regarding the use of interval solutions for determining the distribution of achieved common costs we refer the reader to Branzei, Tijs and Alparslan Gök (2008b).
In the following proposition we show that the interval Baker-Thompson allocation rule coincides with the interval Shapley value of the corresponding airport interval game.

PROPOSITION 3.2. *Let $< N, d >$ be an airport interval game with d as in (3). Then, the interval allocation $\beta(d)$ whose components are given by (4) corresponds to $\Phi(d)$.*

Proof. Notice that $\Phi(d)$ is additive. Then, for $i \in N_j$ we have

$$\begin{aligned} \Phi_i(d) &= \Phi_i(\sum_{k=1}^{m} T_k u^*_{\cup_{r=k}^m N_r}) = \sum_{k=1}^{m} \Phi_i(T_k u^*_{\cup_{r=k}^m N_r}) \\ &= \sum_{k=1}^{m} \phi_i(u^*_{\cup_{r=k}^m N_r}) T_k \\ &= \sum_{k=1}^{j} (\sum_{r=k}^{m} n_r)^{-1} T_k = \beta_i(d), \end{aligned}$$

where the second equality follows from the additivity of $\Phi(d)$ and the third equality follows from (2). □

Note that if we consider the special case $N_1 = \{1\}, N_2 = \{2\}, \ldots, N_n = \{n\}$. Then, $\beta(d) = (\frac{T_1}{n}, \frac{T_1}{n} + \frac{T_2}{n-1}, \ldots, \frac{T_1}{n} + \frac{T_2}{n-1} + \ldots + \frac{T_n}{1})$. Here, each piece of the runway is completely paid by the users and all users of the same piece contribute equally. It is proved in Alparslan Gök, Branzei and Tijs (2008b) that the interval Shapley value is an interval core element for convex interval games. Since the airport interval games are concave by Proposition 3.1 the proof of the following proposition is straightforward.

PROPOSITION 3.3. *Let $< N, d >$ be an airport interval game with d as in (3). Then, $\Phi(d) \in \mathcal{C}(d)$.*

EXAMPLE 3.1. *Let $< N, d >$ be a three-person airport interval game corresponding to the airport interval situations depicted in Figure 1. The interval costs of the pieces are given by $T_1 = [30, 45]$, $T_2 = [20, 40]$ and $T_3 = [100, 120]$. Then, $d(\emptyset) = [0, 0]$, $d(1) = [30, 45]$, $d(2) = d(1, 2) = [50, 85]$ and $d(3) = d(1, 3) = d(2, 3) = d(N) = [150, 205]$. Clearly, $< N, \underline{d} >$ and $< N, |d| >$ are submodular (or concave). So, $< N, d >$ is concave by Proposition 2.1. The following table shows the interval marginal vectors of the game, where rows correspond to orderings of players and columns correspond*

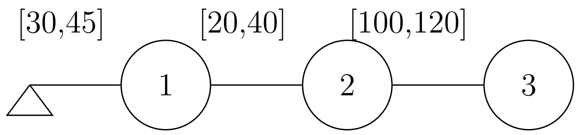

Figure 1: An airport situation with interval data

to players

$$\begin{array}{c} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{bmatrix} [30,45] & [20,40] & [100,120] \\ [30,45] & [0,0] & [120,160] \\ [0,0] & [50,85] & [100,120] \\ [0,0] & [50,85] & [100,120] \\ [0,0] & [0,0] & [150,205] \\ [0,0] & [0,0] & [150,205] \end{bmatrix}.$$

Note that $d = [30,45]u^*_{\{1,2,3\}} + [20,40]u^*_{\{2,3\}} + [100,120]u^*_{\{3\}}$ *and*

$$\Phi(d) = ([10,15]\,, [20,35]\,, [120,155]) \in \mathcal{C}(d).$$

Notice also that

$$\begin{aligned} \Phi(d) &= \Phi(\sum_{k=1}^{3} T_k u^*_{\cup_{r=k}^{3} N_r}) = \phi(u^*_{\{1,2,3\}})T_1 + \phi(u^*_{\{2,3\}})T_2 + \phi(u^*_{\{3\}})T_3 \\ &= (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})[30,45] + (0, \frac{1}{2}, \frac{1}{2})[20,40] + (0,0,1)[100,120] = \beta(d). \end{aligned}$$

4 Final remarks

In this paper we studied airport situations with interval data and related games. Other economic and Operations research situations with interval

data and related interval games have been also studied. We refer here to Branzei, Tijs and Alparslan Gök (2008a), Branzei and Dall'Aglio (2008a,b) for bankruptcy situations, Alparslan Gök et al. (2008) for sequencing situations and Moretti et al. (2008) for minimum cost spanning tree situations. Weber, Alparslan Gök and Söyler (2007) and Weber et al. (2007) considered environmental and gene-network problems with interval uncertainty. It is a topic for further research to associate cooperative interval games with such situations.
Other OR situations and combinatorial optimization problems with interval data among which flow situations, linear production situations and holding situations can give rise to interesting interval games. The existing literature on related classical games can be an inspiration source for further research (Borm, Hamers and Hendrickx (2001), Curiel (1997), Kalai and Zemel (1982), Owen (1975), Tijs, Meca and López (2005).

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