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Abstract

Cooperative games on *antimatroids* are cooperative games restricted by a combinatorial structure which generalize the *permission structure.* So, cooperative games on antimatroids group several well-known families of games which have important applications in economics and politics. Therefore, the study of the rectricted games by antimatroids allows to unify criteria of various lines of research. The current paper establishes axioms that determine the restricted *Shapley value* on antimatroids by conditions on the cooperative game v and the structure determined by the antimatroid. This axiomatization generalizes the axiomatizations of both the *conjunctive* and *disjunctive permission value* for games with a permission structure. We also provide an axiomatization of the Shapley value restricted to the smaller class of *poset antimatroids.* Finally, we apply our model to *auction situations.*

Mathematics Subject Classification 2000: 91A12

Key words: Cooperative game, antimatroid, permission structure, Shapley value.

1 Introduction

A cooperative game describes a situation in which a finite set of players N can generate certain payoffs by cooperation. In a cooperative game the players are assumed to be socially identical in the sense that every player can cooperate with every other player. However, in practice there exist social asymmetries among the players. For this reason, the game theoretic analysis of decision processes in which one imposes asymmetric constraints on the behavior of the players has been and continues to be an important subject to study. Important consequences have been obtained of adopting this type of restrictions on economic behavior. Some models which analyze social asymmetries among players in a cooperative game are described in, e.g., Myerson (1977), Owen (1986) and Borm, Owen, and Tijs (1992). In these models the possibilities of coalition formation are determined by the positions of the players in a *communication graph.*

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Another type of asymmetry among the players in a cooperative game is introduced in Gilles, Owen and van den Brink (1992), Gilles and Owen (1999), van den Brink and Gilles (1996) and van den Brink (1997). In these models, the possibilities of coalition formation are determined by the positions of the players in a hierarchical *permission structure*. Two different approaches were introduced for these games: *conjunctive* and *disjunctive*. Algaba, Bilbao, van den Brink and Jiménez-Losada (2000) showed that the feasible coalition systems derived from both approaches were identified to *poset antimatroids* and *antimatroids with the path property*, respectively. Games on antimatroids are introduced in Jiménez-Losada (1998).

On the other hand, Branzei, Fragnelli and Tijs (2000) have introduced *peer group games* as games based on the existence of certain dependences among the players and which are described by a rooted tree. This type of games allows to study particular cases of auction situations, communication situations, sequencing situations or flow games. These games are restricted games on poset antimatroids with the path property. This class of antimatroids are the *permission forest* and *permission tree structures* which are often encountered in the economic literature. So, the study of games on antimatroids allows to unify several research lines in the same one. Another model in which cooperation possibilities in a game are limited by some hierarchical structure on the set of players can be found in Faigle and Kern (1992) who consider feasible rankings of the players.

In Section 2 we discuss some preliminaries on antimatroids and permission structures. An axiomatization of the restricted Shapley value for games on antimatroids is presented in Section 3. Our six axioms generalize the axiomatizations of both the conjunctive and disjunctive permission values for games with a permission structure. In particular, with respect to these we unify the *fairness* axioms used in both conjunctive and disjunctive approaches. In Section 4, we restrict our attention on the special class of poset antimatroids, showing that deleting the fairness axiom characterizes the restricted Shapley value for the class of cooperative games on poset antimatroids. Moreover, it turns out that the class of games on poset antimatroids is characterized as that class of games on which the restricted Shapley value is the unique solution satisfying these axioms. This then also characterizes the Shapley value for games on poset antimatroids satisfying the path property. Finally, an application to auction situations is given in Section 5.

2 Cooperative games on antimatroids

A cooperative game is a pair (N, v), where $N = \{1, \dots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. Since we take the player set N to be fixed we represent a cooperative game by its characteristic function v. A cooperative game v is *monotone* if $v(E) \leq v(F)$ whenever $E \subseteq F \subseteq N$.

We assume that the set of feasible coalitions $\mathcal{A} \subseteq 2^N$ is an *antimatroid*. Antimatroids were introduced by Dilworth (1940) as particular examples of semimodular lattices. A symmetric study of these structures was started by Edelman and Jamison (1985) emphasizing the combinatorial abstraction of convexity. The convex geometries are a dual concept of antimatroids (see Bilbao, 2000).

Definition 1 *An antimatroid $\mathcal{A}$ on N is a family of subsets of 2^N, satisfying*

A1. $\emptyset \in \mathcal{A}$.

A2. *(Accessibility) If $E \in \mathcal{A}$, $E \neq \emptyset$, then there exists $i \in E$ such that $E \setminus \{i\} \in \mathcal{A}$.*

A3. *(Closed under union) If $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$.*

The definition of antimatroid implies the following *augmentation property*: if $E, F \in \mathcal{A}$ with $|E| > |F|$ then there exists $i \in E \setminus F$ such that $F \cup \{i\} \in \mathcal{A}$.

From now on, we only consider antimatroids satisfying

A4. *(Normality) For every $i \in N$ there exists an $E \in \mathcal{A}$ such that $i \in E$.*

In particular, this implies that $N \in \mathcal{A}$. Now we introduce some well-known concepts about antimatroids which can be found in Korte, Lovász and Schrader (1991, Chapter III). Let $\mathcal{A}$ be an antimatroid on N. This set family allows to define the *interior operator* $int_{\mathcal{A}} : 2^N \to \mathcal{A}$, given by $int_{\mathcal{A}}(E) = \bigcup_{F \subseteq E, F \in \mathcal{A}} F \in \mathcal{A}$, for all $E \subseteq N$. This operator satisfies the following properties which characterize it:

I1. $int_{\mathcal{A}}(\emptyset) = \emptyset$,

I2. $int_{\mathcal{A}}(E) \subseteq E$,

I3. if $E \subseteq F$ then $int_{\mathcal{A}}(E) \subseteq int_{\mathcal{A}}(F)$,

I4. $int_{\mathcal{A}}(int_{\mathcal{A}}(E)) = int_{\mathcal{A}}(E)$,

I5. if $i, j \in int_{\mathcal{A}}(E)$ and $j \notin int_{\mathcal{A}}(E \setminus \{i\})$ then $i \in int_{\mathcal{A}}(E \setminus \{j\})$.

Let $\mathcal{A}$ be an antimatroid on N. An *endpoint* or *extreme point* (Edelman and Jamison, 1985) of $E \in \mathcal{A}$ is a player $i \in E$ such that $E \setminus \{i\} \in \mathcal{A}$, i.e., those players that can leave a feasible coalition E keeping feasibility. By condition A2 (Accessibility) every non-empty coalition in $\mathcal{A}$ has at least one endpoint. A set $E \in \mathcal{A}$ is a *path* in $\mathcal{A}$ if it has a single endpoint. The path $E \in \mathcal{A}$ is called a *i-path* in $\mathcal{A}$ if it has $i \in N$ as unique endpoint. A coalition $E \in \mathcal{A}$ if and only if E is a union of paths. Moreover, for every $E \in \mathcal{A}$ with $i \in E$ there exists an i-path F such that $F \subseteq E$. The set of i-paths for a given player $i \in N$ will be denoted by $A(i)$.

The next concept is based on paths in an antimatroid and it is necessary to describe certain permission structures. This notion is closely related to the conditions on paths that are obtained in a tree.

Definition 2 *An antimatroid $\mathcal{A}$ on N is said to have the path property if*

P1. *Every path E has a unique feasible ordering, i.e. $E := (i_1 > \cdots > i_t)$ such that $\{i_1, \ldots, i_k\} \in \mathcal{A}$ for all $1 \leq k \leq t$. Furthermore, the union of these orderings for all paths is a partial ordering of N.*

P2. *If E, F and $E \setminus \{i\}$ are paths such that the endpoint of F equals the endpoint of $E \setminus \{i\}$, then $F \cup \{i\} \in \mathcal{A}$.*

Observe that every path has a unique feasible ordering if and only if for any i-path E with $|E| > 1$ we have that $E \setminus \{i\}$ is a path. A special class of antimatroids are the *poset antimatroids* being antimatroids that are closed under intersection.

Definition 3 *An antimatroid $\mathcal{A}$ is a poset antimatroid if $E \cap F \in \mathcal{A}$ for every $E, F \in \mathcal{A}$.*

For a cooperative game v and an antimatroid $\mathcal{A}$ on N we define the restricted game $v_{\mathcal{A}}$ which assigns to every coalition E the worth generated by the interior of E, i.e., $v_{\mathcal{A}}(E) = v(int_{\mathcal{A}}(E))$, for all $E \subseteq N$. For properties of these restricted games we refer to Algaba *et al.* (2000). A solution for games on antimatroids is a function f that assigns a payoff distribution $f(v, \mathcal{A}) \in \mathbb{R}^n$ to every cooperative game v and antimatroid $\mathcal{A}$ on N. The *restricted Shapley value* $\overline{Sh}(v, \mathcal{A})$ for a cooperative game v and an antimatroid $\mathcal{A}$ on N is obtained by applying the *Shapley value* (Shapley, 1953) to game $v_{\mathcal{A}}$, i.e.,

$$\overline{Sh}_i(v, \mathcal{A}) = Sh_i(v_{\mathcal{A}}) = \sum_{\{E \subseteq N : i \in E\}} \frac{d_{v_{\mathcal{A}}}(E)}{|E|},$$

where

$$d_v(E) = \sum_{T \subseteq E} (-1)^{|E|-|T|} v(T)$$

denotes the *dividend* of the coalition E in game v.

As we have already indicated games on antimatroids generalize cooperative games with an acyclic permission structure. A *permission structure* on N is a mapping $S : N \to 2^N$. The players in $S(i)$ are called the *successors* of i in S. The players in $S^{-1}(i) := \{j \in N : i \in S(j)\}$ are called the *predecessors* of i in S. By $\widehat{S}$ we denote the *transitive closure* of the permission structure S, i.e., $j \in \widehat{S}(i)$ if and only if there exists a sequence of players $(h_1, \ldots, h_t)$ such that $h_1 = i$, $h_{k+1} \in S(h_k)$ for all $1 \leq k \leq t-1$ and $h_t = j$. The players in $\widehat{S}(i)$ are called the *subordinates* of i in S. A permission structure S is *acyclic* if $i \notin \widehat{S}(i)$ for all $i \in N$. In the *conjunctive approach* as developed in Gilles, Owen and van den Brink (1992), it is assumed that each player needs permission from *all* its predecessors before it is allowed to cooperate. This implies that the set of feasible coalitions is given by

$$\Phi_S^c = \left\{E \subseteq N : S^{-1}(i) \subseteq E \text{ for every } i \in E\right\}.$$

Alternatively, in the *disjunctive approach* as discussed in Gilles and Owen (1999) it is assumed that each player that has predecessors only needs permission from *at least one* of its predecessors before it is allowed to cooperate with other players. Consequently, the set of feasible coalitions is given by

$$\Phi_S^d = \{E \subseteq N : S^{-1}(i) = \emptyset \text{ or } S^{-1}(i) \cap E \neq \emptyset \text{ for every } i \in E\}.$$

Algaba *et al.* (2000) show that for every acyclic permission structure S, both Φ_S^c and Φ_S^d are antimatroids. Moreover, the class of all sets of feasible coalitions that can be obtained as conjunctive feasible coalitions is exactly the class of poset antimatroids. The class of all sets of feasible coalitions that can be obtained as disjunctive feasible coalitions is exactly the class of antimatroids satisfying the path property.

A solution for games with a permission structure is a function f that assigns a payoff distribution $f(v, S) \in \mathbb{R}^n$ to every cooperative game v and permission structure S. The *conjunctive permission value* is obtained by applying the Shapley value to the conjunctive restricted games $v_{\Phi_S^c}$, while the *disjunctive permission value* is obtained by applying the Shapley value to the disjunctive restricted games $v_{\Phi_S^d}$, i.e., they are the restricted Shapley values

$$\overline{Sh}(v, \Phi_S^c) = Sh\left(v_{\Phi_S^c}\right) \text{ and } \overline{Sh}\left(v, \Phi_S^d\right) = Sh\left(v_{\Phi_S^d}\right),$$

respectively.

The purpose in the next sections will be to generalize axiomatizations given for the conjunctive and disjunctive permission values to obtain axiomatizations of the restricted Shapley value for cooperative games on antimatroids.

3 An axiomatization of the restricted Shapley value

We provide an axiomatization of the restricted Shapley value for games on antimatroids generalizing the axiomatizations of the conjunctive and disjunctive permission values given in van den Brink (1997, 1999). As we will see, both axiomatizations are special cases of one axiomatization of the restricted Shapley value for games on antimatroids. So, the study of games on antimatroids allows to elaborate common axioms for both approaches, being specially interesting the use of a same fairness axiom.

The first three axioms are straightforward generalizations of efficiency, additivity and the necessary player property for cooperative games (with a permission structure). For two cooperative games v and w the game $(v + w)$ is given by $(v + w)(E) = v(E) + w(E)$ for all $E \subseteq N$.

Axiom 1 (Efficiency) *For every cooperative game v and antimatroid $\mathcal{A}$ on N, $\sum_{i \in N} f_i(v, \mathcal{A}) = v(N)$.*

Axiom 2 (Additivity) *For every pair of cooperative games v, w and antimatroid $\mathcal{A}$ on N, $f(v + w, \mathcal{A}) = f(v, \mathcal{A}) + f(w, \mathcal{A})$.*

Axiom 3 (Necessary player property) *For every monotone cooperative game v and antimatroid $\mathcal{A}$ on N, if $i \in N$ satisfies $v(E) = 0$ for all $E \subseteq N \setminus \{i\}$ then $f_i(v, \mathcal{A}) \geq f_j(v, \mathcal{A})$ for all $j \in N$.*

Note that the necessary player axiom requires that all necessary players get the same payoff. Recall that player i is *inessential* in a game v with permission structure S on N if i and all its subordinates are null players in game v. This concept extends the one of null player in a game v. Player i is a null player in game v if $v(E) = v(E \setminus \{i\})$ for all $E \subseteq N$. Let $\mathcal{A}$ be an antimatroid on N. The *path group* P^i of player i is defined as the set of players that are in some i-path, i.e., $P^i = \bigcup_{E \in A(i)} E$. So, the path group of player i are all players on which i has some dependence. Now, given an antimatroid $\mathcal{A}$ on N, we call $i \in N$ an *inessential player* for $\mathcal{A}$ in v if player i and every player $j \in N$ such that $i \in P^j$ are null players in v. The description of the inessential player axiom is the following.

Axiom 4 (Inessential player property) *For every cooperative game v and antimatroid $\mathcal{A}$ on N, if i is an inessential player for $\mathcal{A}$ in v then $f_i(v, \mathcal{A}) = 0$.*

We generalize *structural monotonicity* for games with a permission structure by introducing a new set. Let $\mathcal{A}$ be an antimatroid on N. The *basic path group* P_i of player i is given by those players that are in every i-path, i.e., $P_i = \bigcap_{E \in A(i)} E$. This set is formed by those players that control totally player i in $\mathcal{A}$, i.e., without them player i can not form any feasible coalition. Obviously, $i \in P_i$ and $P_i \subseteq P^i$.

Axiom 5 (Structural monotonicity) *For every monotone cooperative game v and antimatroid $\mathcal{A}$ on N, if $j \in N$ then for all $i \in P_j$ we have $f_i(v, \mathcal{A}) \geq f_j(v, \mathcal{A})$.*

We can generalize both conjunctive and disjunctive *fairness* for games with a permission structure by requiring that deleting a feasible coalition E from antimatroid $\mathcal{A}$, such that $\mathcal{A} \setminus \{E\}$ is also an antimatroid, changes the payoffs of all players in E by the same amount.

Axiom 6 (Fairness) *For every cooperative game v and antimatroid $\mathcal{A}$ on N, if $E \in \mathcal{A}$ is such that $\mathcal{A} \setminus \{E\}$ is an antimatroid on N, then*

$$f_i(v, \mathcal{A}) - f_i(v, \mathcal{A} \setminus \{E\}) = f_j(v, \mathcal{A}) - f_j(v, \mathcal{A} \setminus \{E\}) \text{ for all } i, j \in E.$$

The next example shows that in general to delete a feasible coalition from an antimatroid does not always give an antimatroid.

Example 1 *Let* $N = \{1,2,3,4\}$ *and the antimatroid given by*

$$\mathcal{A} = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, N\}.$$

If we consider the coalition $E = \{1,2,3\}$ *then* $\mathcal{A} \setminus \{E\}$ *is not an antimatroid because* $\{1,2\} \cup \{1,3\}$ *is not feasible anymore. Considering coalition* $H = \{1,2\}$ *it follows that* $\mathcal{A} \setminus \{H\}$ *is not an antimatroid since there is no* $i \in \{1,2,4\}$ *such that* $\{1,2,4\} \setminus \{i\} \in \mathcal{A} \setminus \{H\}$*. However, taking* $F = \{1,3,4\}$ *it holds that* $\mathcal{A} \setminus \{F\}$ *is an antimatroid.*

Note that given a permission structure S, applying this fairness axiom to the antimatroid Φ_S^d (Φ_S^c) is equivalent to applying disjunctive (conjunctive) fairness to the corresponding game with permission structure (see van den Brink 1997, 1999). We say that coalition $F \in \mathcal{A}$ *covers* coalition $E \in \mathcal{A}$ if $E \subseteq F$ and $|F| = |E| + 1$.

Lemma 1 *Let* $\mathcal{A}$ *be an antimatroid and* $E \in \mathcal{A}$*. Then,* $\mathcal{A} \setminus \{E\}$ *is an antimatroid if and only if* E *is a path,* $E \notin \{\emptyset, N\}$ *and every* $F \in \mathcal{A}$ *that covers* E *is not a path.*

Proof. Suppose that $\mathcal{A} \setminus \{E\}$ is an antimatroid. Then obviously $E \notin \{\emptyset, N\}$. If E would not be a path then there would be $i, j \in E$ such that $E \setminus \{i\}, E \setminus \{j\} \in \mathcal{A} \setminus \{E\}$. This would imply that $E \setminus \{i\} \cup E \setminus \{j\} \in \mathcal{A} \setminus \{E\}$, which is a contradiction with $E \setminus \{i\} \cup E \setminus \{j\} = E$. If there would exist a path F in $\mathcal{A}$ that covers E then $\mathcal{A} \setminus \{E\}$ would fail the accessibility property.

Suppose that E is a path in $\mathcal{A}$, $E \notin \{\emptyset, N\}$ and every $F \in \mathcal{A}$ that covers E is not a path. We have to prove that $\mathcal{A} \setminus \{E\}$ is an antimatroid. Since $E \neq \emptyset$, $\emptyset \in \mathcal{A} \setminus \{E\}$. As $E \neq N$, $\mathcal{A} \setminus \{E\}$ is normal. Let $E_1, E_2 \in \mathcal{A} \setminus \{E\}$. To show that $E_1 \cup E_2 \in \mathcal{A} \setminus \{E\}$ it suffices to show that $E_1 \cup E_2 \neq E$. On the contrary, suppose that $E_1 \cup E_2 = E$. Then it is a path, assume a i-path. We can suppose without loss of generality that $i \in E_1$. But this is a contradiction with the fact that $E_1 \subseteq E, E_1 \neq E$, and E being a i-path. Finally, let $F \in \mathcal{A} \setminus \{E\}$, $F \neq \emptyset$. If there is no $i \in F$ such that $F \setminus \{i\} \in \mathcal{A} \setminus \{E\}$ then there is a unique $i \in F$ such that $F \setminus \{i\} \in \mathcal{A}$ (since $\mathcal{A}$ is an antimatroid). Moreover, $F \setminus \{i\} = E$. But this is a contradiction with the fact that every feasible coalition that covers E is not a path. □

The restricted Shapley value for games on antimatroids satisfies the six axioms introduced above.

Theorem 1 *The restricted Shapley value* $\overline{Sh}$ *satisfies efficiency, additivity, the necessary player property, the inessential player property, structural monotonicity and fairness.*

Proof. Let v be a cooperative game and $\mathcal{A}$ be an antimatroid on N.

1. Since $N \in \mathcal{A}$, efficiency of the Shapley value implies that

$$\sum_{i \in N} \overline{Sh}_i (v, \mathcal{A}) = \sum_{i \in N} Sh_i (v_{\mathcal{A}}) = v_{\mathcal{A}}(N) = v\left(int_{\mathcal{A}}(N)\right) = v(N),$$

showing that $\overline{Sh}$ satisfies efficiency.

2. Additivity of the Shapley value and the fact that

$$\begin{aligned}(v_{\mathcal{A}} + w_{\mathcal{A}})(E) &= v_{\mathcal{A}}(E) + w_{\mathcal{A}}(E) = v\left(int_{\mathcal{A}}(E)\right) + w\left(int_{\mathcal{A}}(E)\right) \\ &= (v+w)\left(int_{\mathcal{A}}(E)\right) = (v+w)_{\mathcal{A}}(E),\end{aligned}$$

for all $E \subseteq N$, imply that

$$\overline{Sh}_i(v,\mathcal{A}) + \overline{Sh}_i(w,\mathcal{A}) = Sh_i(v_{\mathcal{A}}) + Sh_i(w_{\mathcal{A}}) = Sh_i(v_{\mathcal{A}} + w_{\mathcal{A}})$$
$$= Sh_i((v+w)_{\mathcal{A}}) = \overline{Sh}_i(v+w,\mathcal{A}),$$

showing that $\overline{Sh}$ satisfies additivity.

3. Let v be a monotone game and let $i \in N$ be such that $v(E) = 0$ for all $E \subseteq N \setminus \{i\}$. Algaba *et al.* (2000, Proposition 3) show that $v_{\mathcal{A}}$ is monotone. Thus $v_{\mathcal{A}}(E) = v(int_{\mathcal{A}}(E)) \leq v(E) = 0$ for all $E \subseteq N \setminus \{i\}$. Since monotonicity of $v_{\mathcal{A}}$ also implies that $v_{\mathcal{A}}(E) \geq 0$ for all $E \subseteq N$, it must hold that $v_{\mathcal{A}}(E) = 0$ for all $E \subseteq N \setminus \{i\}$. For all $j \in N$ and $e = |E|$ this implies

$$\begin{aligned}\overline{Sh}_i(v,\mathcal{A}) &= \sum_{\{E\subseteq N: i\in E\}} \frac{(e-1)!(n-e)!}{n!}\left(v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E\setminus\{i\})\right) \\ &\geq \sum_{\{E\subseteq N: i,j\in E\}} \frac{(e-1)!(n-e)!}{n!}\left(v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E\setminus\{i\})\right) \\ &\geq \sum_{\{E\subseteq N: i,j\in E\}} \frac{(e-1)!(n-e)!}{n!}\left(v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E\setminus\{j\})\right) \\ &= \sum_{\{E\subseteq N: j\in E\}} \frac{(e-1)!(n-e)!}{n!}\left(v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E\setminus\{j\})\right) \\ &= \overline{Sh}_j(v,\mathcal{A}),\end{aligned}$$

showing that $\overline{Sh}$ satisfies the necessary player property.

4. The Shapley value satisfies the null player axiom (i.e., all null players in a game earn a zero payoff). Therefore it is sufficient to prove that the inessential players are just the null players in the restricted game. Let i be an inessential player for $\mathcal{A}$ in v, and E a coalition such that $i \in E$. Let $F = int_{\mathcal{A}}(E) \setminus int_{\mathcal{A}}(E \setminus \{i\})$. We show that $i \in P^j$ for every $j \in F$. Suppose there exists $j \in F$ with $i \notin P^j$. Then player i is not in any j-path. As $j \in int_{\mathcal{A}}(E)$, then there exists a j-path H contained in $int_{\mathcal{A}}(E)$ in which player i is not, and so H would be contained in $E \setminus \{i\}$. By definition of interior operator and since paths are feasible coalitions in $\mathcal{A}$ we have that H would be contained in $int_{\mathcal{A}}(E\setminus\{i\})$ and in particular $j \in int_{\mathcal{A}}(E\setminus\{i\})$. This gives a contradiction since $j \in F$. If $F = \{j_1, \dots, j_p\}$ then

$$\begin{aligned}v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E\setminus\{i\}) &= v(int_{\mathcal{A}}(E)) - v(int_{\mathcal{A}}(E\setminus\{i\})) \\ &= v(int_{\mathcal{A}}(E)) - v(int_{\mathcal{A}}(E)\setminus F) \\ &= v(int_{\mathcal{A}}(E)) - v(int_{\mathcal{A}}(E)\setminus F) \\ &\quad + \sum_{t=1}^{p-1}\left[v(int_{\mathcal{A}}(E)\setminus\{j_1,\dots,j_t\}) - v(int_{\mathcal{A}}(E)\setminus\{j_1,\dots,j_t\})\right] \\ &= 0,\end{aligned}$$

since as i is inessential then every j_i, $i = 1, \dots, p$ is a null player in v. This show that $\overline{Sh}$ satisfies the inessential player property.

5. Since v being monotone implies that $v_{\mathcal{A}}$ is monotone we can establish the following properties for $j \in N$, $i \in P_j$ and v monotone:

(*i*) $v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \setminus \{i\}) \geq 0$, for all $E \subseteq N$.

(ii) Given $E \subseteq N$ it is satisfied

$$int_{\mathcal{A}}(E \setminus \{i\}) = \bigcup_{\{F \in \mathcal{A}: F \subseteq E \setminus \{i\}\}} F = \bigcup_{\{F \in \mathcal{A}: F \subseteq E \setminus \{i,j\}\}} F$$
$$\subseteq \bigcup_{\{F \in \mathcal{A}: F \subseteq E \setminus \{j\}\}} F = int_{\mathcal{A}}(E \setminus \{j\}),$$

and thus

$$v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \setminus \{i\}) \geq v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \setminus \{j\}), \text{ for all } E \subseteq N.$$

(iii) For every $E \subseteq N \setminus \{i\}$,

$$int_{\mathcal{A}}(E) = \bigcup_{\{F \in \mathcal{A}: F \subseteq E\}} F = \bigcup_{\{F \in \mathcal{A}: F \subseteq E \setminus \{j\}\}} F = int_{\mathcal{A}}(E \setminus \{j\}),$$

and thus

$$v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \setminus \{j\}) = 0, \text{ for all } E \subseteq N \setminus \{i\}.$$

This implies that

$$\begin{aligned}
\overline{Sh}_i(v, \mathcal{A}) = Sh_i(v_{\mathcal{A}}) &= \sum_{\{E \subseteq N: i \in E\}} \frac{(e-1)!(n-e)!}{n!} (v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \setminus \{i\})) \\
&\geq \sum_{\{E \subseteq N: i,j \in E\}} \frac{(e-1)!(n-e)!}{n!} (v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \setminus \{i\})) \\
&\geq \sum_{\{E \subseteq N: i,j \in E\}} \frac{(e-1)!(n-e)!}{n!} (v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \setminus \{j\})) \\
&= \sum_{\{E \subseteq N: j \in E\}} \frac{(e-1)!(n-e)!}{n!} (v_{\mathcal{A}}(E) - v_{\mathcal{A}}(E \setminus \{j\})) \\
&= Sh_j(v_{\mathcal{A}}) = \overline{Sh}_j(v, \mathcal{A}).
\end{aligned}$$

The first inequality follows from (i), the second inequality from (ii), and the first equality after the inequalities follows from (iii). This shows that $\overline{Sh}$ satisfies structural monotonicity.

6. Let $E \in \mathcal{A}$ be such that $\mathcal{A} \setminus \{E\}$ is an antimatroid on N, and let $\{i, j\} \subseteq E$. We establish the following properties:

(i) It follows from Derks and Peters (1992) that $d_{v_{\mathcal{A}}}(F) = 0$ for all $F \notin \mathcal{A}$.

(ii) If $F \in \mathcal{A}$ and $F \not\supseteq \{i, j\}$ then $F \neq E$.

(iii) If $F \not\supseteq \{i, j\}$ then $F \not\supseteq E$, and thus $T \not\supseteq E$ for all $T \subseteq F$. So,

$$int_{\mathcal{A}}(T) = \bigcup_{\{H \in \mathcal{A}: H \subseteq T\}} H = \bigcup_{\{H \in \mathcal{A} \setminus \{E\}: H \subseteq T\}} H = int_{\mathcal{A} \setminus \{E\}}(T) \text{ for all } T \subseteq F.$$

Hence,

$$\begin{aligned}
d_{v_{\mathcal{A}}}(F) &= \sum_{T \subseteq F} (-1)^{|F|-|T|} v_{\mathcal{A}}(T) = \sum_{T \subseteq F} (-1)^{|F|-|T|} v(int_{\mathcal{A}}(T)) \\
&= \sum_{T \subseteq F} (-1)^{|F|-|T|} v(int_{\mathcal{A} \setminus \{E\}}(T)) = \sum_{T \subseteq F} (-1)^{|F|-|T|} v_{\mathcal{A} \setminus \{E\}}(T) \\
&= d_{v_{\mathcal{A} \setminus \{E\}}}(F).
\end{aligned}$$

Defining $\mathcal{A}_i = \{E \in \mathcal{A} : i \in E\}$ it then follows that

$$\begin{aligned}
\overline{Sh}_i(v,\mathcal{A}) - \overline{Sh}_j(v,\mathcal{A}) &= \sum_{F\in\mathcal{A}_i} \frac{d_{v_\mathcal{A}}(F)}{|F|} - \sum_{F\in\mathcal{A}_j} \frac{d_{v_\mathcal{A}}(F)}{|F|} \\
&= \sum_{\{F\in\mathcal{A}_i : j\notin F\}} \frac{d_{v_\mathcal{A}}(F)}{|F|} - \sum_{\{F\in\mathcal{A}_j : i\notin F\}} \frac{d_{v_\mathcal{A}}(F)}{|F|} \\
&= \sum_{\{F\in\mathcal{A}_i\setminus\{E\} : j\notin F\}} \frac{d_{v_\mathcal{A}}(F)}{|F|} - \sum_{\{F\in\mathcal{A}_j\setminus\{E\} : i\notin F\}} \frac{d_{v_\mathcal{A}}(F)}{|F|} \\
&= \sum_{\{F\in(\mathcal{A}\setminus\{E\})_i : j\notin F\}} \frac{d_{v_{\mathcal{A}\setminus\{E\}}}(F)}{|F|} - \sum_{\{F\in(\mathcal{A}\setminus\{E\})_j : i\notin F\}} \frac{d_{v_{\mathcal{A}\setminus\{E\}}}(F)}{|F|} \\
&= \sum_{F\in(\mathcal{A}\setminus\{E\})_i} \frac{d_{v_{\mathcal{A}\setminus\{E\}}}(F)}{|F|} - \sum_{F\in(\mathcal{A}\setminus\{E\})_j} \frac{d_{v_{\mathcal{A}\setminus\{E\}}}(F)}{|F|} \\
&= \overline{Sh}_i(v,\mathcal{A}\setminus\{E\}) - \overline{Sh}_j(v,\mathcal{A}\setminus\{E\}).
\end{aligned}$$

The first equality follows from (i), the third from (ii), and the fourth from (iii). This shows that $\overline{Sh}$ satisfies fairness. □

The six axioms characterize the restricted Shapley value.

Theorem 2 *A solution f for games on antimatroids is equal to the restricted Shapley value $\overline{Sh}$ if and only if it satisfies efficiency, additivity, the necessary player property, the inessential player property, structural monotonicity and fairness.*

Proof. To prove uniqueness, suppose that solution f satisfies the six axioms. Consider antimatroid $\mathcal{A}$ on N and the monotone game $w_T = c_T u_T$, $c_T \geq 0$, where u_T is the unanimity game of $T \subseteq N$, i.e., $w_T(E) = c_T$ if $E \supseteq T$, and $w_T(E) = 0$ otherwise.

We show that $f(w_T, \mathcal{A})$ is uniquely determined by induction on $|\mathcal{A}|$. (Note that $|\mathcal{A}| \geq n+1$ by A3 and A4).

If $|\mathcal{A}| = n+1$ then there is a unique coalition in $\mathcal{A}$ of cardinality i from $i = 1$ until $i = n$. So, there exists a unique i-path for every player i. In this case, $P^i = P_i$ for all $i \in N$. We distinguish the following three cases with respect to $i \in N$:

(i) If $i \in T$ then the necessary player property implies that there exists a $c \in \mathbb{R}$ such that $f_i(w_T, \mathcal{A}) = c$ for all $i \in T$, and $f_i(w_T, \mathcal{A}) \leq c$ for all $i \in N \setminus T$.

(ii) If $i \notin T$ and there is no $j \in T$ such that $i \in P^j$ then the inessential player property implies that $f_i(w_T, \mathcal{A}) = 0$.

(iii) If $i \notin T$ and there is $j \in T$ such that $i \in P^j = P_j$ then structural monotonicity and case (i) imply that $f_i(w_T, \mathcal{A}) = c$.

Now, setting $P^T = \bigcup_{j\in T} P^j$, efficiency implies that $c = c_T / \left|P^T\right|$ and then $f(w_T, \mathcal{A})$ is uniquely determined.

Proceeding by induction assume that $f(w_T, \mathcal{A}')$ is uniquely determined if $|\mathcal{A}'| < |\mathcal{A}|$. Notice that in general $P^i \neq P_i$. Therefore, we can distinguish four cases with respect to $i \in N$, the same three cases as we consider before and moreover, the following case:

(iv) Let $i \notin T$ such that there exists $j \in T$ with $i \in P^j$ and there is no $j \in T$ with $i \in P_j$. Consider then $j \in T$ with $i \in P^j \setminus P_j$. Then there exists a j-path $E \neq N$ such that $i \in E$, and there exists a j-path $F \neq N$ such that $i \notin F$. We define a *chain* from coalition E to N to be a sequence of coalitions $(E_0, E_1, \ldots, E_t)$

satisfying $E_0 = E$, $E_t = N$ and there is a sequence of distinct players $(h_1, \ldots, h_t)$ such that $h_k \in N \setminus E_{k-1}$ and $E_k = E_{k-1} \cup \{h_k\}$ for all $k \in \{1, \ldots, t\}$. If all coalitions in the chain belong to the antimatroid $\mathcal{A}$ it is called a chain in $\mathcal{A}$. The augmentation property implies that there exist chains in $\mathcal{A}$ from E to N and from F to N. We choose a chain from E to N and a chain from F to N in such a way that the first common coalition M of these chains is the largest coalition possible, i.e., there are no other two chains from E and F to N with a first larger common coalition $M' \supset M$. (Note that a first common coalition always exists because coalition N is always a common coalition). Our goal is to find a coalition containing i and j and, under the conditions of Lemma 1 to apply the fairness axiom. If $H \in \mathcal{A}$, $|H| = |E| + 1$, $H \supset E$ imply that H is not a path in $\mathcal{A}$, then define $\mathcal{A}' = \mathcal{A} \setminus \{E\}$. By Lemma 1 $\mathcal{A}'$ is an antimatroid. Otherwise, i.e., if there is a path $E_1 \in \mathcal{A}$, $|E_1| = |E| + 1$, $E_1 \supset E$, it can happen that $H \in \mathcal{A}$, $|H| = |E_1| + 1$, $H \supset E_1$ imply that H is not a path in $\mathcal{A}$. Then define $\mathcal{A}' = \mathcal{A} \setminus \{E_1\}$. In case this does not occur we can proceed in this way, and thus choose a sequence of coalitions labeled by $E_1, E_2, \ldots, E_m$ being paths in $\mathcal{A}$ and such that if $H \in \mathcal{A}$, $|H| = |E_m| + 1$, $H \supset E_m$ then H is not a path in $\mathcal{A}$. In this process as maximum we would get to a path E_m with $|E_m| = |M| - 1, M \supset E_m$. There cannot exist any coalition $Q \in \mathcal{A}$, $Q \neq M$, $|Q| = |E_m| + 1$, $Q \supset E_m$, because if there would be such a coalition Q then the chain chosen from F to N and this alternative chain from E to N through Q would have a larger first common coalition. So, taking $\mathcal{A}' = \mathcal{A} \setminus \{E_m\}$ and applying Lemma 1, $\mathcal{A}'$ is an antimatroid. In any case, by fairness and taking into account that $j \in T$ it follows with case (i) that

$$f_i(w_T, \mathcal{A}) = f_j(w_T, \mathcal{A}) - f_j(w_T, \mathcal{A}') + f_i(w_T, \mathcal{A}') = c - c_i, \tag{1}$$

with $c_i = f_i(w_T, \mathcal{A}') - f_j(w_T, \mathcal{A}')$ already determined by the induction hypothesis (note that as f satisfies fairness we can state that c_i is independent of the coalition E_m deleted to obtain $\mathcal{A}'$). To determine c we apply the efficiency axiom

$$c\left|P^T\right| - \sum_{i \in P^T \setminus P_T} c_i = c_T,$$

where $P^T = \bigcup_{j \in T} P^j$ and $P_T = \bigcup_{j \in T} P_j$. By the induction hypothesis all c_i in the last sum are determined, and so is c. But then $f(w_T, \mathcal{A})$ is uniquely determined by equation (1).

Above we showed that $f(w_T, \mathcal{A})$ is uniquely determined for all (monotone) games $w_T = c_T u_T$ with $c_T \geq 0$. Suppose that $w_T = c_T u_T$ with $c_T < 0$. (Then w_T is not monotone and the necessary player property and structural monotonicity cannot be applied). Let $v_0 \in \mathcal{G}^N$ denote the null game, that is, $v_0(E) = 0$ for all $E \subseteq N$. From the inessential player property it follows that $f_i(v_0, \mathcal{A}) = 0$ for all $i \in N$. Since $-w_T = -c_T u_T$ with $-c_T \geq 0$, and $(v_0)_\mathcal{A} = (w_T)_\mathcal{A} + (-(w_T)_\mathcal{A})$, it follows from additivity of f and the fact that $-w_T$ is monotone that $f(w_T, \mathcal{A}) = f(v_0, \mathcal{A}) - f(-w_T, \mathcal{A}) = -f(-w_T, \mathcal{A})$ is uniquely determined. So, $f(c_T u_T)$ is uniquely determined for all $c_T \in \mathbb{R}$. Since every cooperative game v on N can be expressed as a linear combination of unanimity games it follows with additivity that $f(v, \mathcal{A})$ is uniquely determined. $\square$

We end this section by showing logical independence of the six axioms stated in Theorem 1.

1. The solution g defined in the proof of Theorem 4 (see next section) satisfies efficiency, additivity, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy fairness.

2. The solution given by $f(v, \mathcal{A}) = Sh(v)$ satisfies efficiency, additivity, the inessential player property, the necessary player property and fairness. It does not satisfy structural monotonicity.

3. For antimatroid $\mathcal{A}$ on N, let $B(\mathcal{A}) = \{i \in N : \{i\} \in \mathcal{A}\}$ be the set of atoms in $\mathcal{A}$. Define $\tilde{f}(v, \mathcal{A}) = \sum_{T \subseteq N} d_v(T)\, \tilde{f}(u_T, \mathcal{A})$ with

$$\tilde{f}_i(u_T, \mathcal{A}) = \begin{cases} \dfrac{1}{|T \cup B(\mathcal{A})|} & \text{if } i \in T \cup B(\mathcal{A}), \\ 0 & \text{otherwise.} \end{cases}$$

This solution satisfies efficiency, additivity, the inessential player property, structural monotonicity and fairness. It does not satisfy the necessary player property.

4. The egalitarian solution, $f_i(v, \mathcal{A}) = v(N)/|N|$ for all $i \in N$, satisfies efficiency, additivity, the necessary player property, structural monotonicity and fairness. It does not satisfy the inessential player property.

5. Let u_T^* be the dual of the unanimity game of coalition $T \subseteq N$, i.e.,

$$u_T^*(E) = u_T(N) - u_T(N \setminus E) = \begin{cases} 1 & \text{if } E \cap T \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let the solution f be given by

$$f_i(v, \mathcal{A}) = \begin{cases} \tilde{f}(v, \mathcal{A}) & \text{if } v = u_T^*,\ |T| \geq 2, \\ \overline{Sh}(v, \mathcal{A}) & \text{otherwise,} \end{cases}$$

with $\tilde{f}$ as given above (see 3). This solution satisfies efficiency, the inessential player property, the necessary player property, structural monotonicity and fairness. It does not satisfy additivity.

6. The zero solution given by $f_i(v, \mathcal{A}) = 0$ for all $i \in N$ satisfies additivity, the inessential player property, the necessary player property, structural monotonicity and fairness. It does not satisfy efficiency.

4 Poset antimatroids

In Section 2 we referred to the fact that every set of conjunctive feasible coalitions for some permission structure S is a poset antimatroid. Van den Brink and Gilles (1996) gave an axiomatization of the conjunctive permission value where conjunctive fairness and structural monotonicity for games with a permission structure (see van den Brink, 1999) are replaced by a stronger structural monotonicity axiom. Unlike the characterizations of the permission values, to characterize the restricted Shapley value for games on poset antimatroids without fairness we need not strengthen structural monotonicity. Deleting fairness from the set of axioms stated in Theorem 2 characterizes the restricted Shapley value for games on poset antimatroids. Moreover, poset antimatroids are the unique antimatroids for which it is possible to delete the fairness axiom.

Poset antimatroids are the unique antimatroids such that every player has a unique path. In particular, we can conclude that given an antimatroid $\mathcal{A}$ on N, then $\mathcal{A}$ is a poset antimatroid if and only if $P^i = P_i$ for all $i \in N$.

Theorem 3 *A solution f for games on poset antimatroids is equal to the restricted Shapley value $\overline{Sh}$ if and only if it satisfies efficiency, additivity, the necessary player property, the inessential player property and structural monotonicity.*

Proof. From Theorem 1 it follows that $\overline{Sh}$ satisfies the five axioms. Suppose that solution f satisfies the five axioms on poset antimatroids. Consider a poset antimatroid $\mathcal{A}$ on N and the game $w_T = c_T u_T$, $c_T \geq 0$. Taking into account that for a poset antimatroid $P^j = P_j$ for all $j \in N$, we only have to consider the first three cases from Theorem 2 with respect to $i \in N$. Hence, efficiency implies that $c = c_T / \left|P^T\right|$. Therefore $f\left(w_T, \mathcal{A}\right)$ is uniquely determined. For arbitrary v it follows that $f\left(v, \mathcal{A}\right)$ is uniquely determined in a similar way as in the proof of Theorem 2. □

The last five solutions given at the end of the previous section show logical independence of the axioms stated in Theorem 3. For games on poset antimatroids the restricted Shapley value can be written as follows.

Proposition 1 *If $\mathcal{A}$ is a poset antimatroid on N then*

$$\overline{Sh}_i\left(v, \mathcal{A}\right) = \sum_{\{T \subseteq N : i \in P^T\}} \frac{d_v(T)}{|P^T|}.$$

Proof. Since $d_v(E) = 0$ for every $E \notin \mathcal{A}$ it follows that

$$\overline{Sh}_i\left(v, \mathcal{A}\right) = \sum_{E \in \mathcal{A}_i} \frac{d_{v_\mathcal{A}}(E)}{|E|} = \sum_{E \in \mathcal{A}_i} \frac{\sum_{\{T \subseteq E : E = P^T\}} d_v(T)}{|E|} = \sum_{\{T \subseteq N : i \in P^T\}} \frac{d_v(T)}{|P^T|}.$$

□

We can characterize the class of games on poset antimatroids as the class of games on which the restricted Shapley value satisfies the five axioms of Theorem 3. Above we considered the restricted Shapley value $\overline{Sh}$ which assigns the payoff distribution $\overline{Sh}\left(v, \mathcal{A}\right) \in \mathbb{R}^n$ to every cooperative game v and antimatroid $\mathcal{A}$ on N. Given an antimatroid $\mathcal{A}$ on N, let $\overline{Sh}\left(\cdot, \mathcal{A}\right)$ be the function that assigns to every cooperative game v on N the restricted Shapley value $\overline{Sh}\left(v, \mathcal{A}\right)$. Defining the axioms of Theorem 3 for solutions $f\left(\cdot, \mathcal{A}\right)$ in a straightforward way we give the following result.

Theorem 4 *Let $\mathcal{A}$ be an antimatroid on N. Then $\mathcal{A}$ is a poset antimatroid if and only if $\overline{Sh}\left(\cdot, \mathcal{A}\right)$ is the unique solution satisfying efficiency, additivity, the necessary player property, the inessential player property and structural monotonicity.*

Proof. From Theorem 3 it follows that given a poset antimatroid $\mathcal{A}$, $\overline{Sh}\left(\cdot, \mathcal{A}\right)$ is the unique solution satisfying efficiency, additivity, the necessary player property, the inessential player property and structural monotonicity. Suppose that $\mathcal{A}$ is not a poset antimatroid. Define the solution g for games on antimatroids by

$$g_i\left(u_T, \mathcal{A}\right) = \begin{cases} \dfrac{1}{|P_T|} & \text{if } i \in P_T = \bigcup_{i \in T} P_i, \\ 0 & \text{otherwise,} \end{cases}$$

and for arbitrary game v

$$g_i\left(v, \mathcal{A}\right) = \sum_{T \subseteq N} d_v(T) g_i\left(u_T, \mathcal{A}\right) = \sum_{\{T \subseteq N : i \in P_T\}} \frac{d_v(T)}{|P_T|}.$$

This solution satisfies efficiency, additivity, the necessary player property, the inessential player property and structural monotonicity.

To prove that $g(\cdot, \mathcal{A}) \neq \overline{Sh}(\cdot, \mathcal{A})$ note that, if $\mathcal{A}$ is not a poset antimatroid then there exists a $j \in N$ with $P^j \neq P_j$. By Proposition 1 it then follows that $g(u_T, \mathcal{A}) \neq \overline{Sh}(u_T, \mathcal{A})$ if $j \notin T$ and $(P^j \setminus P_j) \cap T \neq \emptyset$. □

An acyclic permission structure S is a *permission forest structure* if $|S^{-1}(i)| \leq 1$ for all $i \in N$. So, in a permission forest structure every player has at most one predecessor. A permission forest structure is a *permission tree structure* if there is exactly one player i_0 for which $S^{-1}(i_0) = \emptyset$. Algaba *et al.* (2000, Lemma 2) showed that the permission forest structures are exactly those acyclic permission structures for which the sets of conjunctive and disjunctive feasible coalitions coincide. We also showed that the poset antimatroids satisfying the path property are exactly those antimatroids that can be obtained as the set of conjunctive or disjunctive feasible coalitions of some permission forest structure. From Theorem 4 we directly obtain a characterization of the Shapley value restricted to the class of poset antimatroids satisfying the path property, i.e., antimatroids that are obtained as the feasible coalitions for permission forest or tree structures. This result is interesting from an economic point of view since in economic theory we often encounter hierarchical structures that can be represented by *forests* or *trees*.

Corollary 1 *Let $\mathcal{A}$ be a poset antimatroid on N satisfying the path property. Then $\overline{Sh}(\cdot, \mathcal{A})$ is the unique solution satisfying efficiency, additivity, the necessary player property, the inessential player property and structural monotonicity.*

5 An application: auction situations

In the previous sections we mentioned that both the conjunctive and disjunctive permission value for games with a permission structure are characterized by applying the axioms defined in this paper to the specific classes of poset antimatroids and antimatroids satisfying the path property, respectively. We also indicated that Algaba *et al.* (2000) showed that an antimatroid can be the conjunctive feasible coalition set of some permission structure as well as the disjunctive feasible coalition set if and only if it is a poset antimatroid satisfying the path property. A special class of such antimatroids are the feasible sets of *peer group situations* as considered in Branzei, Fragnelli and Tijs (2000). In fact, they consider games with an acyclic permission structure (N, v, S) with $|S^{-1}(i)| \leq 1$ for all $i \in N$. The games v assign zero dividends to all coalitions that are not paths. With acyclicity of the permission structure there is exactly one player, the root i_0, such that $S^{-1}(i_0) = \emptyset$. The restricted *peer group game* then coincides with the (conjunctive or disjunctive) restricted game $v_{\Phi_S^c} = v_{\Phi_S^d}$ arising from this game with permission (tree) structure. Given that all coalitions that are not paths get a zero dividend, the restricted game is equal to the game itself, i.e., $v = v_{\Phi_S^c} = v_{\Phi_S^d}$. (Note that the same restricted game is obtained if we consider the game that assigns to every player i the dividend of its path P_i.) Defining efficiency, additivity, the necessary player property the inessential player property and structural monotonicity restricted to this class, it is shown in van de Brink (1997) that these axioms characterize the restricted Shapley value on this class.

As argued by Branzei *et al.* (2000) peer group situations generalize some other situations such as sealed bid second price auction situations (see Rasmusen, 1994). Consider a seller of an object who has a reservation value $r \geq 0$, and a set $N = \{1, \ldots, n\}$ of n bidders. Each bidder has a valuation $w_i \geq r$ for the object. Assume that the bidders are labelled such that $w_1 > \cdots > w_n$. Using dominant bidding strategies for such auction situations Branzei *et al.* (2000) define the corresponding peer group situation that can be represented as the game with permission (tree) structure (N, v, S) with $S(i) = \{i+1\}$ for $1 \leq i \leq n-1$, $S(n) = \emptyset$, and the game v determined by the dividends $d_v(\{1, \ldots, i\}) = w_i - w_{i+1}$ if $1 \leq i \leq n-1$ and $d_v(N) = w_n - r$. All other coalitions have a zero dividend, $d_v(T) = 0$ for all $T \notin \mathcal{A}$. Clearly these games are determined by the vector of valuations $(w, r) \in \mathbb{R}_+^{n+1}$. Given such a valuation vector let $\overline{Sh}(w, r)$ denote the restricted Shapley value of the corresponding game on the antimatroid $\mathcal{A} = \{\emptyset, \{1\}, \{1,2\}, \ldots, \{1, 2, \ldots n-1\}, N\}$. Allowing the strict inequalities to be weak inequalities $w_1 \geqslant \cdots \geqslant w_n$, applying the axioms stated above to these situations yields that efficiency straightforward says that $\sum_{i \in N} f_i(w, r) = w_1 - r$. The inessential player property states that $f_i(w, r) = 0$ if $w_i = r$. Structural monotonicity states that $f_i(w, r) \geq f_j(w, r)$ if $w_i \geq w_j$. Specifying additivity we must take care that the underlying permission structure does not change. So, we require additivity only for reservation value vectors that 'preserve the order of players', i.e., $f(w+z, r+s) = f(w, r) + f(z, s)$ if $w_i \geq w_j \Leftrightarrow z_i \geq z_j$. We refer to this as additivity over order preserving valuations. Finally, we can give a characterization of the restricted Shapley value for auction situations without using the necessary player property.

Theorem 5 *The restricted Shapley value $\overline{Sh}(w, r)$ is the unique solution for auction situations $(w, r) \in \mathbb{R}_+^{n+1}$ satisfying efficiency, the inessential player property, structural monotonicity and additivity over order preserving valuations. Moreover,*

$$\overline{Sh}_i(w, r) = \frac{w_i}{i} - \sum_{h=i+1}^{n} \frac{w_h}{h(h-1)} - \frac{r}{n}.$$

Proof. It follows from Corollary 1 that $\overline{Sh}(w, r)$ satisfies these axioms. It also follows that the axioms of this corollary characterize $\overline{Sh}(w, r)$ for auction situations. However, we have to prove that we do not have to use the necessary player property and need additivity only over order preserving valuations. Suppose that f is a solution for auction situations that satisfies the axioms, and let $(w, r) \in \mathbb{R}_+^{n+1}$ be an auction situation. For $h = 1, \ldots, n-1$ define the auction situation $(w^h, 0)$ by $w_i^h = w_h - w_{h+1}$ for all $i \in \{1, \ldots, h\}$, $w_i^h = 0$ for all $i \in \{h+1, \ldots, n\}$, and define (w^n, r) by $w_i^n = w_n$ for all $i \in \{1, \ldots, n\}$. Let $h \in \{1, \ldots, n-1\}$. The inessential player property implies that $f_i(w^h, 0) = 0$ for all $i \in \{h+1, \ldots, n\}$. Structural monotonicity implies that all $f_i(w^h, 0)$ are equal for all $i \in \{1, \ldots, h\}$, i.e., $f_i(w^h, 0) = c_h,\ 1 \leq i \leq h$ for some $c_h \in \mathbb{R}$. Similarly, structural monotonicity implies that $f_i(w^n, r) = c_r,\ i \in N$, for some $c_r \in \mathbb{R}$. Efficiency then determines uniquely that $c_h = \dfrac{w_h}{h}$ and $c_r = \dfrac{w_n - r}{n}$. Since all $(w^h, 0),\ h = 1, \ldots, n-1$, and (w^n, r) are order preserving, additivity over order preserving valuations determines $f(w, r)$.

Let $(w, r) \in \mathbb{R}_+^{n+1}$ be an auction situation. Then its corresponding poset antimatroid is $\mathcal{A} = \{\emptyset, \{1\}, \ldots, \{1, 2, \ldots, n-1\}, N\}$. It follows from Proposition 1 that

$$\begin{aligned}
\overline{Sh}_i(w, r) &= \sum_{\{T \subseteq N : i \in P^T\}} \frac{d_v(T)}{|P^T|} \\
&= \frac{w_i - w_{i+1}}{i} + \sum_{h=i+1}^{n-1} \frac{w_h - w_{h+1}}{h} + \frac{w_n - r}{n} \\
&= \frac{w_i}{i} - \sum_{h=i+1}^{n} \frac{w_h}{h(h-1)} - \frac{r}{n}.
\end{aligned}$$

□

From the proof of the above theorem it follows that structural monotonicity could be replaced by *symmetry* stating that $f_i(w, r) = f_j(w, r)$ if $w_i = w_j$. Note that this cannot be done in more general cases as discussed earlier in the paper. Moreover, the inessential player property can be weakened by saying that $f_i(w, r) = 0$ if $w_i = 0$. In a similar way we can characterize solutions for other economic situations such as *airport games* or *hierarchically structured firms*.

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