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Abstract. Potential games and supermodular games are attractive games, especially because under certain conditions they possess pure Nash equilibria. Subclasses of games with a potential are considered which are also strategically equivalent to supermodular games. The focus is on two-person zero-sum games and two-person Cournot games.

Keywords: Pure Nash equilibrium, potential game, supermodular game, Cournot game, zero-sum game.

JEL classification: C72, C73.

1. Introduction

The aim of this paper is to investigate two interesting classes of games for which the existence of pure Nash equilibria is obtained under certain conditions, namely:

i) the class of potential games (Monderer and Shapley, 1996);

ii) the class of supermodular games (Topkis, 1998).

The question tackled here is whether there are games belonging to both classes. It turns out that two-person zero-sum supermodular games are potential games and

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conversely that two-person zero-sum potential games can be transformed in a canonical way into supermodular games. Also Cournot games are, under special conditions, members of both classes of games.

A connection between ordinal potential games (Monderer and Shapley, 1996) and supermodular games is also established for certain Cournot games.

In Section 2 the definitions of potential games and of supermodular games are recalled together with some of their properties. In Section 3 the case of two-person zero-sum games is discussed and an example illustrating the connection between the two classes of games is given. Section 4 deals with Cournot duopoly competition and Cournot games. Section 5 contains some concluding remarks.

2. Preliminaries

Let $< A, B, K, L >$ be a *two-person game* with strategy space A for player 1, strategy space B for player 2, and $K: A \times B \mapsto \mathbb{R}$, $L: A \times B \mapsto \mathbb{R}$ the payoff function of player $1, 2$ respectively. If the players 1 and 2 choose $a \in A$ and $b \in B$ respectively, then player 1 obtains a payoff $K(a, b)$ and player 2 obtains $L(a, b)$.

A *Nash equilibrium* for such a game is a point $(\hat{a}, \hat{b}) \in A \times B$ such that $K(a, \hat{b}) \leq K(\hat{a}, \hat{b})$ for each $a \in A$ and $L(\hat{a}, b) \leq L(\hat{a}, \hat{b})$ for each $b \in B$.

Such a game is called a *potential game* (Monderer and Shapley, 1996) if there is a (*potential*) *function* $P: A \times B \mapsto \mathbb{R}$ such that

$$K(a_2, b) - K(a_1, b) = P(a_2, b) - P(a_1, b), \quad \text{for all } a_1, a_2 \in A \text{ and for each } b \in B,$$

$$L(a, b_1) - L(a, b_2) = P(a, b_1) - P(a, b_2), \quad \text{for each } a \in A \text{ and for all } b_1, b_2 \in B.$$

Clearly, elements of argmax(P) are Nash equilibria of the game.

The next lemma will be useful. It states that for a two-person potential game the payoff function of player 1 (player 2) can be written as the sum of a potential and a function on the Cartesian product of the strategy spaces, which only depends on the strategy choice of player 2 (player 1). This is a known result (Slade, 1994; Facchini et al., 1997); an alternative proof is given here.

Lemma 1. Let $< A, B, K, L >$ be a potential game with potential P. Then there exist functions $f: A \mapsto \mathbb{R}$ and $g: B \mapsto \mathbb{R}$ such that

$$K(a, b) = P(a, b) - 2g(b),$$

$$L(a,b) = P(a,b) - 2f(a)$$

for each $a \in A$ and $b \in B$.

Proof. Take $a^* \in A$, $b^* \in B$ and define f and g as follows. For each $a \in A$ and $b \in B$, let

$$f(a) = 1/2\big(P(a,b^*) - L(a,b^*)\big),$$

$$g(b) = 1/2\big(P(a^*,b) - K(a^*,b)\big).$$

Since P is a potential for the game $< A, B, K, L >$, we have

$$K(a,b) - K(a^*,b) = P(a,b) - P(a^*,b) \text{ or } K(a,b) - P(a,b) = -2g(b),$$

and also

$$L(a,b) - L(a,b^*) = P(a,b) - P(a,b^*) \text{ or } L(a,b) - P(a,b) = -2f(a)$$

for all $a \in A$ and $b \in B$. □

The game $< A, B, K, L >$ is called an *ordinal potential game* (Monderer and Shapley, 1996) if there is a (potential) function $P\colon A \times B \mapsto \mathbb{R}$ such that

$$K(a_2,b) - K(a_1,b) > 0 \Longleftrightarrow P(a_2,b) - P(a_1,b) > 0, \text{ for all } a_1, a_2 \in A \text{ and for each } b \in B$$

$$L(a,b_1) - L(a,b_2) > 0 \Longleftrightarrow P(a,b_1) - P(a,b_2) > 0, \text{ for each } a \in A \text{ and for all } b_1, b_2 \in B.$$

We will use the following

Proposition 1 (Monderer and Shapley, 1996). Let $< A, B, K, L >$ be a two-person game. Let A, B be intervals of real numbers and K, L be twice continuously differentiable functions. Then $< A, B, K, L >$ is a potential game if and only if

$$\frac{\partial^2 K}{\partial a \partial b} = \frac{\partial^2 L}{\partial a \partial b}\,.$$

For more information on potential games see Voorneveld (1999) and Mallozzi, Tijs and Voorneveld (2000).

Let us now recall some definitions related to supermodular games. A *partially ordered set* is a set X on which there is a binary relation $\preceq$ that is reflexive, antisymmetric and transitive. Let us consider a partially ordered set X and a subset X' of X. If $x' \in X$ and $x' \preceq x$ for each $x \in X'$, then x' is a *lower bound* for X'; if $x'' \in X$ and $x \preceq x''$ for each $x \in X'$, then x'' is an *upper bound* for X'. When the set of upper bounds of X' has a least element, then this least upper bound of X' is the *supremum* of X' in X; when the set of lower bounds of X' has a greatest element, then this greatest lower bound of X' is the *infimum* of X' in X.

If two elements x_1 and x_2 of a partially ordered set X have a supremum in X, it is called the *meet* of x_1 and x_2 and is denoted by $x_1 \wedge x_2$; if x_1 and x_2 have an infimum in X, it is called the *join* of x_1 and x_2 and is denoted by $x_1 \vee x_2$. A partially ordered set that contains the join and the meet of each pair of its elements is a *lattice*. If a subset X' of a lattice X contains the join and the meet (with respect to X) of each pair of elements of X', then X' is a *sublattice* of X. The real line $\mathbb{R}$ with the natural ordering denoted by $\leq$ is a lattice with $x \vee y = max\{x, y\}$ and $x \wedge y = min\{x, y\}$ for $x, y \in \mathbb{R}$, and $\mathbb{R}^n$ $(n > 1)$ with the natural partial ordering denoted by $\leq$ is a lattice with $x \vee y = (x_1 \vee y_1, ..., x_n \vee y_n)$ and $x \wedge y = (x_1 \wedge y_1, ..., x_n \wedge y_n)$ for $x, y \in \mathbb{R}^n$. Any subset of $\mathbb{R}$ is a sublattice of $\mathbb{R}$, and a subset X of $\mathbb{R}^n$ is a sublattice of $\mathbb{R}^n$ if it has the property that $x, y \in X$ imply that $(max\{x_1, y_1\}, ..., max\{x_n, y_n\})$ and $(min\{x_1, y_1\}, ..., min\{x_n, y_n\})$ are in X.

The game $< A, B, K, L >$ is called a *supermodular game* (Topkis, 1998) if the following three properties are satisfied:

1) A is a sublattice of $\mathbb{R}^{m_1}$ and B is a sublattice of $\mathbb{R}^{m_2}$ for some $m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}$;

2) K, L have increasing differences on $A \times B$, i.e. for all $(a_1, a_2) \in A^2$ and for all $(b_1, b_2) \in B^2$ such that $a_1 \geq a_2$ and $b_1 \geq b_2$,

$$K(a_1, b_1) - K(a_1, b_2) \geq K(a_2, b_1) - K(a_2, b_2),$$

$$L(a_1, b_1) - L(a_2, b_1) \geq L(a_1, b_2) - L(a_2, b_2);$$

3) K is supermodular in the first coordinate and L is supermodular in the second coordinate, i.e. for each $b \in B$, for all $a_1, a_2 \in A$ we have

$$K(a_1, b) + K(a_2, b) \leq K(a_1 \vee a_2, b) + K(a_1 \wedge a_2, b)$$

and for each $a \in A$, for all $b_1, b_2 \in B$ we have

$$L(a, b_1) + L(a, b_2) \leq L(a, b_1 \vee b_2) + L(a, b_1 \wedge b_2).$$

We recall the following propositions:

Proposition 2 (Topkis, 1998). Let f be a differentiable function on $\mathbb{R}^n$, then f has increasing differences on $\mathbb{R}^n$ if and only if $\dfrac{\partial f}{\partial x_i}$ is increasing in x_j for all distinct i and j and all x.

Proposition 3 (Topkis, 1998). Let f be a twice differentiable function on $\mathbb{R}^n$, then f has increasing differences on $\mathbb{R}^n$ if and only if $\dfrac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$, for all distinct i and j.

The following two examples show that the classes of potential games and supermodular games do not coincide. So the study of special subclasses becomes interesting.

Example 1. Let $A = B = [0,1]$ and $K(a,b) = 2ab$, $L(a,b) = a + b$ for all $a, b \in [0,1]$. Then the game $< A, B, K, L >$ is a supermodular game because A and B are sublattices of $\mathbb{R}$, K, L have increasing differences on $[0,1] \times [0,1]$, and K is supermodular in the first coordinate and L in the second coordinate. This game is not an exact potential game because the condition in Proposition 1 is not satisfied since $\dfrac{\partial^2 K}{\partial a \partial b} = 2 \neq \dfrac{\partial^2 L}{\partial a \partial b} = 0$. Let us remark that the game is an ordinal potential game with potential function P given by $P(a,b) = a + b$ for all $a, b \in [0,1]$.

On the other hand there are games that are exact potential games and not supermodular games:

Example 2. Let $A = B = [0,1]$ and $K(a,b) = a^2 - 2a(b - \frac{1}{2})^2 + b$, $L(a,b) = -2a(b - \frac{1}{2})^2$ for all $a, b \in [0,1]$. Then the game $< A, B, K, L >$ is a potential game with potential function P given by $P(a,b) = a^2 - 2a(b - \frac{1}{2})^2$ for all $a, b \in [0,1]$ but it is not a supermodular game in view of Proposition 3 because $\dfrac{\partial^2 K}{\partial a \partial b} = -4(b - \dfrac{1}{2}) < 0$ if $b > \dfrac{1}{2}$.

3. Zero-sum potential games and supermodular games

A two-person game of the form $< A, B, K, -K >$ is called a *zero-sum game*. Such a game will be denoted by $< A, B, K >$. In a zero-sum game one player pays the other. A *saddle point* for such a game is a point $(\hat{a}, \hat{b}) \in A \times B$ such that $K(a, \hat{b}) \leq K(\hat{a}, \hat{b}) \leq K(\hat{a}, b)$ for each $a \in A$ and $b \in B$. We denote by $S(A, B, K)$ the set of all saddle points of $< A, B, K >$. Note that $< A, B, K >$ is a potential game if there is a (potential) function $P \colon A \times B \mapsto \mathbb{R}$ such that

$$K(a_2, b) - K(a_1, b) = P(a_2, b) - P(a_1, b), \quad \text{for all } a_1, a_2 \in A \text{ and for each } b \in B,$$

$$-K(a, b_1) + K(a, b_2) = P(a, b_1) - P(a, b_2), \quad \text{for each } a \in A \text{ and for all } b_1, b_2 \in B.$$

Clearly, elements of argmax(P) are saddle-points of the game. Also the converse turns out to hold as we see in Remark 2.

Useful will be the following

Theorem 1. Let $< A, B, K >$ be a two-person zero-sum game. Then the following assertions are equivalent:

(1_i) $< A, B, K >$ is a potential game;

(1_{ii}) there exists a pair of functions (f, g) with $f\colon A \mapsto \mathbb{R}$ and $g\colon B \mapsto \mathbb{R}$ such that $K(a, b) = f(a) - g(b)$ for all $a \in A$, $b \in B$ (separation property).

Proof. That (1_{ii}) implies (1_i) follows by taking the potential P defined by

$$P(a, b) = f(a) + g(b) \text{ for all } a \in A \text{ and } b \in B.$$

Conversely, suppose (1_i). Then by Lemma 1, there are functions $f\colon A \mapsto \mathbb{R}$ and $g\colon B \mapsto \mathbb{R}$ such that for each $a \in A$ and $b \in B$

$$K(a, b) = P(a, b) - 2g(b), \quad -K(a, b) = P(a, b) - 2f(a).$$

So $K(a, b) = f(a) - g(b)$ for all $(a, b) \in A \times B$. □

Remark 1. This theorem is also proved in Potters, Raghavan and Tijs (1999), in an alternative way. In that paper it was also observed that for 2x2-subgames of a two-person zero-sum potential game the "diagonal property" holds. This is

$$K(a_1, b_1) + K(a_2, b_2) = K(a_1, b_2) + K(a_2, b_1)$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. This property follows easily from (1_{ii}) in Theorem 1. Conversely, it was proved in Potters, Raghavan and Tijs (1999) that the diagonal property for two-person zero-sum games implies also that the game is a potential game.

Remark 2. A pair (f, g) as in (1_{ii}) of Theorem 1 is called a *separating pair* for the potential game $< A, B, K >$. For a potential P of this game we have $P(a, b) = c + f(a) + g(b)$ for each $a \in A$, $b \in B$ and some $c \in \mathbb{R}$. Clearly, $(\hat{a}, \hat{b})$ is a saddle point of $< A, B, K >$ if and only if $\hat{a} \in \operatorname*{argmax}_{a \in A} f(a)$, $\hat{b} \in \operatorname*{argmax}_{b \in B} g(b)$ if and only if $(\hat{a}, \hat{b}) \in \text{argmax}(P)$.

Theorem 1 gives us the possibility to connect a two-person zero-sum potential game with a related game where the strategy spaces are ordered subsets of $\mathbb{R}$ and the payoff function satisfies monotonicity conditions.

Given $< A, B, K >$ with potential function P and separating pair (f, g) such that $P(a, b) = f(a) + g(b)$ for all $a \in A$, $b \in B$, define $< \bar{A}, \bar{B}, \bar{K} >$ as follows. Take $\bar{A} = f(A)$, $\bar{B} = g(B)$ and for $(\bar{a}, \bar{b}) \in \bar{A} \times \bar{B}$ let $\bar{K}(\bar{a}, \bar{b}) = \bar{a} - \bar{b}$.

So we use the real valued functions $f\colon A \mapsto \mathbb{R}$ and $g\colon B \mapsto \mathbb{R}$ to find a game $< \bar{A}, \bar{B}, \bar{K} >$ with strategy spaces in $\mathbb{R}$, which is strategically equivalent to $< A, B, K >$ because

$$K(a,b) = \bar{K}(f(a), g(b)) \quad \text{for all } (a,b) \in A \times B,$$

$$\bar{K}(c,d) = K(a,b) \quad \text{for all } a \in f^{-1}(c),\ b \in g^{-1}(d).$$

From this follows

$$(a,b) \in S(A,B,K) \Longrightarrow (f(a), g(b)) \in S(\bar{A}, \bar{B}, \bar{K}),$$

$$(c,d) \in S(\bar{A}, \bar{B}, \bar{K}) \Longrightarrow (a,b) \in S(A,B,K) \text{ for all } a \in f^{-1}(c),\ b \in g^{-1}(d).$$

The strategy space $\bar{A}$ can be smaller than A because two strategies a_1 and a_2 in A which are equivalent in the sense that

$$K(a_1, b) = K(a_2, b) \ \text{ for all } b \in B$$

are mapped into the same point $f(a_1) = f(a_2) \in \bar{A}$.

Relations between $< A, B, K >$ and $< \bar{A}, \bar{B}, \bar{K} >$ are described in

Theorem 2. Let $< A, B, K >$ a game with potential P and let $< \bar{A}, \bar{B}, \bar{K} >$ be as above. Then

(2_i) $< \bar{A}, \bar{B}, \bar{K} >$ is a potential game with potential $\bar{P}\colon \bar{A} \times \bar{B} \mapsto \mathbb{R}$ such that $\bar{P}(\bar{a}, \bar{b}) = \bar{a} + \bar{b}$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$;

(2_{ii}) $\max(\bar{A}) \times \max(\bar{B}) = \operatorname{argmax}(\bar{P}) = S(\bar{A}, \bar{B}, \bar{K})$;

(2_{iii}) $(a,b) \in S(A,B,K) \Longleftrightarrow f(a) = \max(\bar{A}),\ g(b) = \max(\bar{B})$.

Note that $S(\bar{A}, \bar{B}, \bar{K})$ has cardinality 0 or 1.

Example 3. Consider the matrix game

	L	R	E
T	8	13	13
M	5	10	10
F	10	15	15

corresponding to the two-person zero-sum game $< A, B, K >$ where $A = \{T, M, F\}$, $B = \{L, R, E\}$ and $K(T,L) = 8$, $K(T,R) = K(T,E) = 13$, $K(M,L) = 5$, $K(M,R) =$

$K(M,E) = K(F,L) = 10$, $K(F,R) = K(F,E) = 15$. If we take $f: \{T,M,F\} \mapsto \mathrm{I\!R}$ and $g: \{L,R,E\} \mapsto \mathrm{I\!R}$ as follows: $f(T)=5$, $f(M)=2$, $f(F)=7$, $g(L)=-3$, $g(R)=g(E)=-8$, then $K(a,b)=f(a)-g(b)$ for all $a \in A$, $b \in B$ and $P: A \times B \mapsto \mathrm{I\!R}$ with $P(a,b) = f(a)+g(b)$ for all $a \in A$, $b \in B$ is a potential for this matrix game.

Transforming this game to $< \bar{A}, \bar{B}, \bar{K} >$ with the aid of (f,g) results in $\bar{A} = \{2,5,7\}$, $\bar{B} = \{-8,-3\}$ and $\bar{K}(\bar{a},\bar{b}) = \bar{a} - \bar{b}$ or the "monotonic" matrix game

	-8	-3
2	10	5
5	13	8
7	15	10

with the unique saddle point in $(7,-3)$ corresponding to maximum 4 of the potential $\bar{P}$ which can be written in matrix form as follows

	-8	-3
2	-6	-1
5	-3	-2
7	-1	4

Note that $7 = \max(\bar{A})$, $-3 = \max(\bar{B})$.

Remark 3. If $max(\bar{A})$ (or $max(\bar{B})$) does not exists, then there are no saddle points. If K is bounded, then there are ε-saddle points for each $\varepsilon > 0$ corresponding to points (a', b') with $P(a',b') \geq supP(a,b) - \varepsilon$.

Theorem 3. The game $< \bar{A}, \bar{B}, \bar{K} >$ with $\bar{K}(a,b) = a - b$ for each $a \in \bar{A}$ and $b \in \bar{B}$ is a supermodular game.

Proof. The subsets $\bar{A}$ and $\bar{B}$ are sublattices of $\mathrm{I\!R}$. For each $b \in \bar{B}$ the function $a \mapsto \bar{K}(a,b)$ is supermodular on A and also $b \mapsto -\bar{K}(a,b)$ is supermodular on B for each $a \in A$. We have finished the proof if we show that for each $a_1, a_2 \in A$, $b_1, b_2 \in B$ the functions

$$a \mapsto \bar{K}(a,b_1) - \bar{K}(a,b_2) \quad (a \in \bar{A})$$

$$b \mapsto -\bar{K}(a_1,b) + \bar{K}(a_2,b) \quad (b \in \bar{B})$$

are monotonic. This is true because these functions are in fact constant:

$$\bar{K}(a, b_1) - \bar{K}(a, b_2) = -b_1 + b_2,$$

$$-\bar{K}(a_1, b) + \bar{K}(a_2, b) = -a_1 + a_2.$$

□

We have seen in Theorem 3 that two-person zero-sum potential games can be embedded in the family of supermodular games. The converse is treated in

Theorem 4. Let $< A, B, K >$ be a two-person zero-sum game with $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, which is supermodular. Then $< A, B, K >$ is a potential game.

Proof. The supermodularity implies that for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $a_1 < a_2, b_1 < b_2$ we have

$$K(a_2, b_2) - K(a_2, b_1) \geq K(a_1, b_2) - K(a_1, b_1)$$

$$-K(a_2, b_2) + K(a_1, b_2) \geq -K(a_2, b_1) + K(a_1, b_1)$$

From these two inequalities follows the diagonal property. Then, according to Remark 1, $< A, B, K >$ is a potential game. □

Example 4. Let $< A, B, K, L >$ be the non-zero sum game with $A = \{1, 2\}$, $B = \{1, 2\}$; $K(i, j) = i + j$ for all $i \in A$ and $j \in B$, and $L(1, 1) = 4$, $L(1, 2) = 7$, $L(2, 1) = 5$ and $L(2, 2) = 9$. Then this game is a supermodular game but not a potential game.

Example 5. Let $< A, B, K, L >$ be the non-zero sum game with $A = \{1, 2\}$, $B = \{1, 2\}$; $K(1, 1) = 3$, $K(1, 2) = 1$, $K(2, 1) = 5$, $K(2, 2) = 2$ and $L(1, 1) = 3$, $L(1, 2) = 8$, $L(2, 1) = 6$, $L(2, 2) = 10$. Then the game is a potential game but not a supermodular game.

Remark 4. A subclass of general two-person potential games can be embedded into the class of supermodular games in a similar way as we embedded two-person zero-sum potential games. These are games of the form $< A, B, K, L >$ with separable payoff functions i. e. K and L can be written in the form

$$K(a, b) = f(a) + g(b), \; L(a, b) = h(a) + k(b)$$

for all $a \in A$, $b \in B$, and where f, h are real valued functions on A such that f is injective, and g, k are real valued functions on B such that k is injective. A potential is then P given by $P(a,b) = f(a) + k(b)$ for each $a \in A$ and $b \in B$. A strategically equivalent supermodular game is the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ where $\bar{A} = f(A)$, $\bar{B} = k(B)$ and where for all $c \in \bar{A}$, $d \in \bar{B}$:

$$\bar{K}(c,d) = K(f^{-1}(c), k^{-1}(d)),$$

$$\bar{L}(c,d) = L(f^{-1}(c), k^{-1}(d)).$$

4. Cournot games

Consider Cournot's model of duopoly where the demand arises from a competitive market of a single homogeneous commodity.

Suppose that firm i, $i = 1, 2$, can supply the single homogeneous product in any non negative bounded quantity $q_i \in [0, q_i^0]$ with production cost $c_i(q_i)$. The price of the single homogeneous commodity is given by the inverse demand function $Q(q_1, q_2)$ which is assumed to be twice continuously differentiable function. We suppose that firm i's cost $c_i(q_i)$, $i = 1, 2$, is differentiable.

Given the output level selected by the other firm, the objective of firm i is to maximize its profit

$$\Pi_i(q_1, q_2) = q_i Q(q_1, q_2) - c_i(q_i)$$

by the choice of its output q_i, where $q_i Q(q_1, q_2)$ expresses the revenue (return) of firm i. We assume that the marginal revenue of firm i (i.e. $Q(q_1, q_2) + q_i \dfrac{\partial Q(q_1, q_2)}{\partial q_i}$) is decreasing with respect to q_j ($j \neq i$).

A *Cournot game* is a game of the form $< A, B, K, L >$ where $A = [0, q_1^0]$, $B = [0, q_2^0]$ and

$$K(a,b) = aQ(a,b) - c_1(a),$$

$$L(a,b) = bQ(a,b) - c_2(b)$$

for all $a \in A$ and $b \in B$. If the inverse demand function Q is linear in $a + b$, then the corresponding Cournot duopoly game is also called a *quasi Cournot game.*

Now we put $\bar{a} = a$ and $\bar{b} = -b$ for each $a \in A$ and $b \in B$ and consider the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ where $\bar{A} = A$, $\bar{B} = -B = [-q_2^0, 0]$ and

$$\bar{K}(\bar{a}, \bar{b}) = K(\bar{a}, -\bar{b}), \;\; \bar{L}(\bar{a}, \bar{b}) = L(\bar{a}, -\bar{b})$$

for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$. So

$$\bar{K}(\bar{a}, \bar{b}) = \bar{a}Q(\bar{a}, -\bar{b}) - c_1(\bar{a}),\ \bar{L}(\bar{a}, \bar{b}) = -\bar{b}Q(\bar{a}, -\bar{b}) - c_2(-\bar{b}).$$

The game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is strategically equivalent to $< A, B, K, L >$ because $\bar{K}(\bar{a}, \bar{b}) = K(a, b)$ and $\bar{L}(\bar{a}, \bar{b}) = L(a, b)$ for all $a \in A, b \in B$. We will denote by $NE(A, B, K, L)$ the set of all Nash equilibria of the game $< A, B, K, L >$. Note that $(a, b) \in NE(A, B, K, L)$ if and only if $(a, -b) \in NE(\bar{A}, \bar{B}, \bar{K}, \bar{L})$. Moreover if $< A, B, K, L >$ is a Cournot potential game with potential function P, then the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ as above is also a potential game with potential $\bar{P}$ given by $\bar{P}(\bar{a}, \bar{b}) = P(\bar{a}, -\bar{b})$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$.

Theorem 5. Let $< A, B, K, L >$ be a Cournot game and consider $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ as above. Then

(5_i) if the cost functions c_i are of the form $c_i(q_i) = cq_i$, for $i = 1, 2$, then $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is an ordinal potential game and also a supermodular game;

(5_{ii}) if the inverse demand function Q is linear in the aggregate output level, given by $Q(a, b) = \alpha - \beta(a + b)$, $\alpha, \beta > 0$ (i.e. $< A, B, K, L >$ is a quasi Cournot game), then $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is a potential game and also a supermodular game.

Proof.

(5_i) The Cournot duopoly with cost functions c_i, $i = 1, 2$ is an ordinal potential game with potential function P given by $P(a, b) = ab[Q(a, b) - c]$ for all $a \in [0, q_1^0]$ and $b \in [0, q_2^0]$ (Monderer and Shapley, 1996), so the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is also an ordinal potential game with the potential $\bar{P}$ given by $\bar{P}(\bar{a}, \bar{b}) = P(a, b)$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$. Moreover $\bar{K}(\bar{a}, \bar{b}) = \bar{a}[Q(\bar{a}, -\bar{b}) - c]$ and $\bar{L}(\bar{a}, \bar{b}) = -\bar{b}[Q(\bar{a}, -\bar{b}) - c]$ satisfy the increasing differences property because by Proposition 3 we have $\frac{\partial^2 \bar{K}}{\partial \bar{a} \partial \bar{b}} = -\frac{\partial}{\partial b}\big[Q + a\frac{\partial Q}{\partial a}\big] \geq 0$ and $\frac{\partial^2 \bar{L}}{\partial \bar{a} \partial \bar{b}} = -\frac{\partial}{\partial a}\big[Q + b\frac{\partial Q}{\partial b}\big] \geq 0$, since we assumed that the marginal revenue is decreasing. Moreover the transformed strategy spaces $\bar{A}$ and $\bar{B}$ are sublattices of $\mathbb{R}$, $\bar{K}$ is supermodular in the first coordinate and $\bar{L}$ is supermodular in the second coordinate. Then the Cournot game is a supermodular game.

(5_{ii}) The quasi Cournot competition is a potential game with potential function P given by $P(a, b) = \alpha(a + b) - \beta(a^2 + b^2) - \beta ab - c_1(a) - c_2(b)$ for all $a \in [0, q_1^0]$ and $b \in [0, q_2^0]$ (Monderer and Shapley, 1996), so the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is also a potential game with the potential $\bar{P}$ given by $\bar{P}(\bar{a}, \bar{b}) = P(a, b)$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$. Moreover $\bar{K}(\bar{a}, \bar{b}) = \bar{a}[\alpha - \beta(\bar{a} - \bar{b})] - c_1(\bar{a})$ and $\bar{L}(\bar{a}, \bar{b}) = -\bar{b}[\alpha - \beta(\bar{a} - \bar{b})] - c_2(-\bar{b})$ satisfy the increasing differences property because by Proposition 3 we have $\frac{\partial^2 \bar{K}}{\partial \bar{a} \partial \bar{b}} =$

$\frac{\partial^2 \bar{L}}{\partial \bar{a} \partial \bar{b}} = \beta > 0$. As in the previous case, $\bar{A}$ and $\bar{B}$ are sublattices of $\mathbb{R}$, $\bar{K}$ is supermodular in the first coordinate, $\bar{L}$ is supermodular in the second coordinate and the quasi Cournot game is a supermodular game. □

5. Concluding remarks

Let us first summarize the main results we obtained:

i) a supermodular two-person zero-sum game is a potential game (Theorem 4). Conversely, if a two-person zero-sum game is a potential game then it is strategically equivalent to a supermodular game (Theorems 2 and 3), which is monotonic and has at most one saddle point; the set of pure saddle points of a two-person zero-sum potential game turns out to coincide with the potential maximizers (Remark 2);

ii) two subclasses of Cournot games are described, which are strategically equivalent to supermodular games and which are simultaneously (ordinal or exact) potential games (Theorem 5).

In Remark 4 we discussed a subclass of general two-person potential games which can be embedded in the class of supermodular games. This result holds for a similar subclass of general n-person strategic games with separable payoff functions.

A game of the form $< A_1, ..., A_n, K_1, ..., K_n >$ where $K_i(a_i, a_{-i}) = f_i(a_i) + g_i(a_{-i})$ for all $a_i \in A_i$ and $a_{-i} \in \Pi_{j \in \mathbb{N} - \{i\}} A_j$ is a potential game and it is strategically equivalent to a supermodular game if $f_1, ..., f_n$ are injective functions. A potential is given by

$$P(a) = \sum_{i=1}^{n} f_i(a_i)$$

and the (strategically equivalent) supermodular game is defined as follows:

- for each $i \in N = \{1, ..., n\}$, $\bar{A}_i = f_i(A_i)$;
- for all $b_1 \in \bar{A}_1, ..., b_n \in \bar{A}_n$ and all $i \in N$

$$\bar{K}_i(b_1, ..., b_n) = K_i(f_1^{-1}(b_1), ..., f_n^{-1}(b_n)).$$

Also duopoly results in Section 4 can be extended to multimarket oligopoly (Topkis, 1998). It is interesting to find other economic situations leading to strategic games which are potential games and also supermodular games.

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