

# A Note on “Stability of Tâtonnement Processes of Short Period Equilibria with Rational Expectations”<sup>\*</sup>

P. Jean-Jacques Herings<sup>†</sup>

**Abstract.** In Hens (1997), a new adjustment process is proposed for a setting with reopening spot and asset markets. He argues by means of an intemporal variant of Scarf’s example that this process is more stable than the other processes, although in general it might be more stable or less stable. This note gives further evidence showing that Hens’s process is indeed more stable. The results contradict some of the arguments of Hens (1997), which are corrected.

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It is well known that general equilibrium models with a complete set of contingent contracts available at the outset are allocationally equivalent to models where agents trade sequentially on reopening spot and asset markets, provided there are sufficiently many financial markets, see Arrow (1953) and Magill and Shafer (1991).

A very interesting and important question, investigated in Hens (1997), is whether these two models are equivalent from a stability point of view, where local asymptotic stability is taken as the criterion. Hens (1997) remarks rightly that it is not clear what the appropriate model of tâtonnement should be in a world with reopening spot and asset markets. A good model of tâtonnement should take into account that time plays a serious role in these models, and the adjustment of expectations about future prices is not necessarily the same thing as the adjustment of prices on a spot market. Therefore, four different processes are compared, tâtonnement in contingent contracts prices, Hicks’s notion of perfect stability (see Hicks (1939)), expectational stability (see Balasko (1994)), and a newly proposed

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<sup>†</sup>P.J.J. Herings, Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

process, called Hens's process in the remainder. In Hens (1997), it is argued by means of an intemporal variant of Scarf's example, that the newly proposed process is more stable than the other processes, although in general it might be more stable or less stable. This note gives further evidence showing that Hens's process is indeed more stable.

Suppose there are two time periods,  $t = 1, 2$ ,  $S$  possible states of the world in the second period,  $L_1$  commodities in the first time period,  $L_2$  commodities in each state  $s = 1, \dots, S$  in the second time period, and  $J$  financial assets. An agent  $i = 1, \dots, I$  has an initial endowment  $\omega^i$  and a consumption set  $\mathbb{R}_+^{L_1} \times \mathbb{R}_+^{SL_2}$ . It is assumed that  $J = S$  and that the asset returns matrix  $A \in \mathbb{R}^{S \times J}$  has full rank  $J$ , since otherwise allocational equivalence would not hold and it would be impossible to compare different adjustment processes. The prices  $q \in \mathbb{R}^S$  for the assets are normalized such that  $q_j = 1, \forall j = 1, \dots, J$ , and the prices  $p_1 \in \mathbb{R}^{L_1}$  and  $p_2 \in \mathbb{R}^{SL_2}$  for the spot market are normalized by taking  $p_{2_{SL_2}} = 1$ .

One can look at this economy as being one with markets for all contingent contracts. Then, given prices  $p_1$  and  $p_2$ , the total excess demand for period 1 and period 2 spot market commodities is denoted by  $Z_1(p_1, p_2)$  and  $Z_2(p_1, p_2)$ . The market of commodity  $SL_2$  is taken out of consideration and a hat above a vector denotes truncation of the last component, so  $\widehat{Z}_2$  is the demand function for the first  $SL_2 - 1$  period 2 spot market commodities. It turns out to be very useful to calculate the Jacobian  $J$  of  $\widehat{Z} = (Z_1, \widehat{Z}_2)$  at a competitive equilibrium. We denote  $J_1 = \partial_{p_1} Z_1$ ,  $\widehat{J}_2 = \partial_{\widehat{p}_2} Z_1$ ,  $\widehat{J}_3 = \partial_{p_1} \widehat{Z}_2$ ,  $\widehat{J}_4 = \partial_{\widehat{p}_2} \widehat{Z}_2$ , and

$$\widehat{J} = \begin{bmatrix} J_1 & \widehat{J}_2 \\ \widehat{J}_3 & \widehat{J}_4 \end{bmatrix}.$$

In Hens's tâtonnement process, current prices change in proportion to current period excess demand, and future prices are formed according to the perfect foresight hypothesis, meaning that they are such that the second period spot markets are cleared. One can compare this tâtonnement process with other processes, like the standard tâtonnement in the contingent contracts prices, Hicksian stability, and expectational stability. Under some regularity conditions the following necessary and sufficient conditions for local asymptotic stability are derived in Hens (1997).

Hens's process: all eigenvalues of the matrix  $J_1 - \widehat{J}_2 \widehat{J}_4^{-1} \widehat{J}_3$  have negative real parts, (t)

Contingent contracts: all eigenvalues of the matrix  $\widehat{J}$  have negative real parts, (c)

Hicksian stability:  $-\widehat{J}$  is a  $P$ -matrix, (h)

Expectational stability: norm of all eigenvalues of  $J_1^{-1} \widehat{J}_2 \widehat{J}_4^{-1} \widehat{J}_3$  is less than one. (e)

Recall that a matrix is a  $P$ -matrix if all principal minors of it are positive. Proposition 2 of Hens (1997) claims that if  $\widehat{J}$  is symmetric, then all Conditions (t), (c), (h), and (e) are equivalent. However, we will show by means of examples that one cannot get stronger s-

tatements than those given in Proposition 1. The confusion arises because of the equivocal statement of Theorem 7.7.6 in Horn and Johnson (1985). Therefore, we will avoid using that result in the proof of Proposition 1.

**Proposition 1**

*Let  $\hat{J}$  be symmetric. Then Condition (c) is equivalent to Condition (h). Furthermore, Condition (h) implies both Condition (t) and Condition (e).*

**Proof** If  $\hat{J}$  is symmetric, then both Condition (c) and Condition (h) are equivalent to stating that  $\hat{J}$  is a negative definite matrix, so both conditions are equivalent. That Condition (h) implies Condition (t) follows from the observation that the upper left block of  $\hat{J}^{-1}$  equals  $(J_1 - \hat{J}_2\hat{J}_4^{-1}\hat{J}_3)^{-1}$ , which is then negative definite since the inverse of a negative definite matrix is negative definite, and all principal submatrices of a negative definite matrix are negative definite. So  $J_1 - \hat{J}_2\hat{J}_4^{-1}\hat{J}_3$  is negative definite and all its eigenvalues are negative.

If  $-\hat{J}$  is a symmetric  $P$ -matrix, then it holds that  $-J_1$  and  $-\hat{J}_2\hat{J}_4^{-1}\hat{J}_3$  are positive definite. Moreover, by the previous paragraph  $-J_1 + \hat{J}_2\hat{J}_4^{-1}\hat{J}_3$  is positive definite, so it follows from Theorem 7.7.3 in Horn and Johnson (1985) that the norm of all eigenvalues of  $\hat{J}_2\hat{J}_4^{-1}\hat{J}_3J_1^{-1}$  is less than one. Using symmetry of the matrix  $\hat{J}_2\hat{J}_4^{-1}\hat{J}_3J_1^{-1}$ , this implies that the norm of all eigenvalues of  $J_1^{-1}\hat{J}_2\hat{J}_4^{-1}\hat{J}_3$  is less than one, i.e. Condition (e). Q.E.D.

One of the implications of Proposition 1 is that in the case of a symmetric Jacobian  $\hat{J}$  stability of Hens's process is a weaker requirement than contingent contracts stability and Hicksian stability.

It is even possible to pin down the difference between Conditions (c) and (h) on the one hand and Conditions (t) and (e) on the other hand more precisely. Using the proof of Theorem 7.7.6 of Horn and Johnson (1985) it can be shown, for a symmetric  $\hat{J}$ , that  $-\hat{J}$  is a  $P$ -matrix if and only if  $\hat{J}_4$  is negative definite and  $J_1 - \hat{J}_2\hat{J}_4^{-1}\hat{J}_3$  is negative definite. Also, for a symmetric  $\hat{J}$ ,  $-\hat{J}$  is a  $P$ -matrix if and only if  $J_1$  is negative definite,  $\hat{J}_4$  is negative definite and the norm of all eigenvalues of  $J_1^{-1}\hat{J}_2\hat{J}_4^{-1}\hat{J}_3$  is less than one. So the exact difference is that for Conditions (t) and (e)  $J_1$  and  $\hat{J}_4$  need not be negative definite.

The two examples of Table 1, where  $L_1 = 1$ ,  $L_2 = 2$  and  $S = 1$ , so  $J_1$ ,  $\hat{J}_2$ ,  $\hat{J}_3$ , and  $\hat{J}_4$  are all  $1 \times 1$  matrices, show that no other conclusions than in Proposition 1 can be drawn. A + (−) sign in the table indicates that a specific stability condition is satisfied (violated). Indeed, Conditions (t) and (e) are incomparable and are strictly weaker than Conditions (c) and (h).

	$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
Condition (t)	+	-
Condition (c)	-	-
Condition (h)	-	-
Condition (e)	-	+

In general the stability of one process does not imply stability of any other one, see Proposition 4 in Hens (1997). However, the example with  $S = 1$ ,  $L_1 = 2$  and  $L_2 = 2$  that shows that Condition (h) does not imply Condition (t) is not correct. In fact, even in the general case where  $\hat{J}$  is not symmetric, it is possible to obtain the following result.

**Proposition 2**

*If  $L_1 \leq 2$ , then Condition (h) implies Condition (t).*

**Proof** Suppose Condition (h) is satisfied. If  $L_1 = 1$ , then  $J_1 - \hat{J}_2 \hat{J}_4^{-1} \hat{J}_3$  is a scalar and we have to show it is negative. It is sufficient to show that the inverse of this scalar is negative. The number  $(J_1 - \hat{J}_2 \hat{J}_4^{-1} \hat{J}_3)^{-1}$  is equal to the element in the first row and column of  $\hat{J}^{-1}$ , which is equal to the ratio of the principle minor obtained by deleting the first row and the first column of  $\hat{J}$  and the determinant of  $\hat{J}$ . That ratio is negative if  $-\hat{J}$  is a  $P$ -matrix.

Consider the case where  $L_1 = 2$ . Now  $(J_1 - \hat{J}_2 \hat{J}_4^{-1} \hat{J}_3)^{-1}$  equals the two by two upper left block of  $\hat{J}^{-1}$ . By Formula 0.8.4 in Horn and Johnson (1985), the determinant of this matrix is given by the ratio of the principle minor obtained by deleting the first two rows and the first two columns of  $\hat{J}$  and the determinant of  $\hat{J}$ . That ratio is positive if  $-\hat{J}$  is a  $P$ -matrix, whereas by the arguments of the previous paragraph it holds that the diagonal elements of  $(J_1 - \hat{J}_2 \hat{J}_4^{-1} \hat{J}_3)^{-1}$  are both negative. Now it follows easily that the trace of  $J_1 - \hat{J}_2 \hat{J}_4^{-1} \hat{J}_3$  is negative and its determinant is positive, implying that its eigenvalues have negative real parts. Q.E.D.

Proposition 2 gives further evidence about the good stability properties of Hens's process. For instance, as shown in Hens (1997), it is possible to give an example with  $L_1 = 1$ ,  $L_2 = 2$ , and  $S = 1$  to show that (h) does not imply (e) in general. If the number of first period commodities is less than or equal to two, then (h) does imply (t), irrespective of the number of possible states of the world in the second period and the number of commodities in each state. Unlike any of the other processes, time plays a serious role in Hens's process. This also explains why the condition  $L_1 \leq 2$  in Proposition 2 is asymmetric with respect to the number of first and second period commodities.

Proposition 2 cannot be strengthened further. When  $S = L_2 = 1$ , then due to the normalization of the second period price,  $p_{2SL_2} = 1$ , expectations about the future play no

role, and so Condition (t) and Condition (c) coincide. But then the example given in Hens (1997) to show that Condition (h) does not imply Condition (c) can be used to show that Condition (h) does not imply Condition (t). Indeed, take  $S = 1$ ,  $L_1 = 3$ , and  $L_2 = 1$ , and

$$\hat{J} = \begin{bmatrix} -1 & 0 & -3 \\ -3 & -1 & 0 \\ 0 & -3 & -1 \end{bmatrix}.$$

It is easily verified that  $-\hat{J}$  is a  $P$ -matrix, so Hicksian stability is satisfied. The eigenvalues of  $\hat{J}$  are given by  $-4$ ,  $\frac{1}{2} - \frac{1}{2}i\sqrt{27}$ , and  $\frac{1}{2} + \frac{1}{2}i\sqrt{27}$ , so Condition (t) is not satisfied.

Clearly, this example seems to be contrived since it relies on the absence of period 2 effects. However, this example can easily be extended to one with  $SL_2 \geq 2$ . Take  $S = 1$ ,  $L_1 = 3$ ,  $L_2 = 2$ , and

$$\hat{J} = \left[ \begin{array}{ccc|c} -1 & 0 & -3 & 0 \\ -3 & -1 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ \hline 0 & 0 & 0 & \alpha \end{array} \right].$$

It is straightforward to verify that the conditions for Hicksian stability are satisfied if and only if  $\alpha < 0$ . The matrix  $J_1 - \hat{J}_2 \hat{J}_4^{-1} \hat{J}_3$  is identical to the corresponding one for the example given before, so its eigenvalues are given by  $-4$ ,  $\frac{1}{2} - \frac{1}{2}i\sqrt{27}$ , and  $\frac{1}{2} + \frac{1}{2}i\sqrt{27}$ , and Condition (t) is not satisfied. Small perturbations of the zeroes in the last row and the last column will leave this conclusion unchanged.

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