CentER

Discussion Paper

No. 2010–101

A TECHNICAL NOTE ON LORENZ DOMINANCE IN COOPERATIVE GAMES

By J. Sanchez-Soriano, R. Branzei, N. Llorca and S.H. Tijs

October 2010

ISSN 0924-7815

Tilburg University

A technical note on Lorenz dominance in cooperative games*

J. Sanchez-Soriano[†], R Branzei[‡], N. Llorca[§], S.H. Tijs[¶]

September 24, 2010

Abstract: In this paper we provide some technical results related to the Lorenz dominance, which allow to prove that the allocation obtained by the algorithm in Dutta and Ray (1989), when exists, and the elements of the equal split-off set always Lorenz dominate every allocation in the core of the game.

AMS classification: 91A12
JEL Classification: C71
Keywords: Cooperative games, Lorenz dominance, egalitarianism, constrained egalitarian solution, equal split-off set.

1 Introduction

In many societies the egalitarianism is a desirable end and is considered as a social value. However, what „egalitarianism" means for each individual or society can be different. In this sense, in the context of cooperative game theory the concept of egalitarianism has generated several so-called egalitarian solutions. For example, for arbitrary cooperative games, the equal division core (Selten, 1972), the constrained egalitarian solution (Dutta and Ray,

*The authors acknowledge the financial support from the Ministerio de Educacion y Ciencia of Spain through the project MTM2008-06778-C02-01 and from the Conselleria d'Educacio of Generalitat Valenciana through the project ACOMP/2010/102.

[†]Corresponding author. CIO and Department of Statistics, Mathematics and Computer Science. University Miguel Hernandez of Elche, Spain. e-mail: joaquin@umh.es

[‡]Faculty of Computer Science, Iaşi, Romania. e-mail: branzeir@info.uaic.ro

[§]CIO and Department of Statistics, Mathematics and Computer Science. University Miguel Hernandez of Elche, Spain. e-mail: nllorca@umh.es

[¶]CentER and Department of Econometrics and Operations Research. Tilburg University, The Netherlands. e-mail: s.h.tijs@uvt.nl

1989), the strong constrained egalitarian solution (Dutta and Ray, 1991), the egalitarian set, the preegalitarian set and the stable egalitarian set (Arin and Inarra, 2002) and the equal split-off set (Branzei et al., 2006).

Dutta and Ray (1989) combined the Lorenzian concept of inequality[1] and the values for each coalition in a cooperative game to introduce the constrained egalitarian solution. They introduced the Lorenz map to select the non Lorenz dominated allocations of a given set. Thus, using this Lorenz map and a principle similar to that behind of the definition of the equal division core (Selten, 1972), they defined the Lorenz core of a game through a recursive procedure and the result of applying the Lorenz map on the Lorenz core is called the set of constrained egalitarian solutions which contains at most one allocation. Likewise, Dutta and Ray (1989) provided an algorithm to find the constrained egalitarian solution for convex games. Indeed, for general games, when this algorithm gives an allocation in the Lorenz core then it coincides with the constrained egalitarian solution. Based on this algorithm, Branzei et al (2006) introduced the equal split-off set for cooperative games which is contained in the equal division core for superadditive games and coincides with the constrained egalitarian solution for convex games. Furthermore, for convex games, the constrained egalitarian solution belongs to the core of the game and Lorenz dominates every other allocation in the core (Dutta and Ray, 1989).

The allocation obtained by the algorithm, when exists, does not always coincide with the constrained egalitarian solution. In this paper we provide some technical results related to the Lorenz dominance in cooperative games which allow to prove that the allocation obtained by the algorithm, when exists, always Lorenz dominates every core allocation. Likewise, these results allow to prove that every allocation in the equal split-off set also Lorenz dominates every allocation in the core of the game. On the other hand, if we consider that minimal requirements of an allocation are the efficiency and the individual rationality, i.e., to belong to the imputation set, then these technical results provide insights about the structure of the less unequal allocations in the imputation set of the game in the Lorenzian sense.

The rest of the paper is organized as follows. In Section 2 we provide the basic definitions, concepts and solutions of cooperative games used along the paper. Section 3 is devoted to an example from Dutta and Ray (1989) which is useful to illustrate some aspects mentioned in the Introduction. Section 4 is the main part of the paper, where we prove some technical results related to the Lorenz dominance. In Section 5 we conclude and discuss some implications of the technical results with regards to cooperative games.

[1] See Sen (1973) for a review on economic inequality.

2 Preliminaries

A *transferable utility cooperative game* (TU-game in short) is defined by a pair (N, v), where $N = \{1, 2, ..., n\}$ is a *finite set of players or agents* and v, called the *characteristic function*, is a map from the set of all possible subsets of N to $\mathbb{R}$, such that $v(\emptyset) = 0$. For each coalition S, $v(S)$ represents what the players in S can obtain if they cooperate.

Depending on the properties of the characteristic function we have different classes of games. Two relevant classes of games are the following:

- *Superadditive games*: if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subset N$ such that $S \cap T = \emptyset$.
- *Convex games*: if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subset N$.

Given a game (N, v) an *allocation or distribution* for it is a vector $x \in \mathbb{R}^n$. We will denote by $x(S) = \sum_{i \in S} x_i$. An allocation is called *efficient* if $x(N) = v(N)$ and an allocation is called *individually rational* if $x_i \geq v(i)$ for all $i \in N$. Three well-known sets of allocations for a game (N, v) are the *imputation set* $(I(N, v))$, the *core* $(C(N, v))$ and the *equal division core* $(EDC(N, v))$, which are defined as follows:

$$\begin{aligned} I(N,v) &= \{x \in \mathbb{R}^n : x(N) = v(N) \text{ and } x_i \geq v(i) \text{ for all } i \in N\}, \\ C(N,v) &= \{x \in I(N,v) : x(S) \geq v(S) \text{ for all } S \subset N\}, \\ EDC(N,v) &= \{x \in I(N,v) : \nexists S \subset N, S \neq \emptyset \text{ such that } \frac{v(S)}{|S|} > x_i \text{ for all } i \in S\}. \end{aligned}$$

On the other hand, let $(x_{(1)}, x_{(2)}, ..., x_{(n)})$ be the vector obtained from $x \in \mathbb{R}^n$ by ordering its components in decreasing order: $x_{(1)} \geq x_{(2)} \geq ... \geq x_{(n)}$. Given two vectors $x, y \in \mathbb{R}^n$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ we say that x *Lorenz dominates* y if $\sum_{i=1}^k x_{(i)} \leq \sum_{i=1}^k y_{(i)}$ for each $k = 1, 2, ..., n$, with at least one strict inequality. Based in this concept we can consider two additional sets introduced by Dutta and Ray (1989), the *Lorenz core* $(L(N, v))$ and the *set of egalitarian allocations* $(EL(N, v))$.

The Lorenz core and the set of egalitarian allocations are defined iteratively in the cardinality of the coalitions in the following way:

- Initial step: $L(i, v) = \{v(i)\}$ for all $i \in N$ and $EL(i, v) = \{v(i)\}$ for all $i \in N$.
- General step: $L(S, v) = \{x \in \mathbb{R}^{|S|} : x(S) = v(S)$, and $\nexists T \subsetneq S$ and $y \in EL(T, v)$ such that $y_i \geq x_i$ for all $i \in T$ with at least one strict

inequality}, where $EL(T,v) = \{x \in L(T,v) : \nexists y \in L(T,v)$ such that $\sum_{i=1}^{k} y_{(i)} \leq \sum_{i=1}^{k} x_{(i)}$ for each $k = 1, 2, ..., |T|$, with at least one strict inequality}.

The final result of the described recursive procedure are precisely $L(N,v)$ and $EL(N,v)$. In Dutta and Ray (1989) it is proved that $EL(N,v)$ contains at most one allocation and this is called the *constrained egalitarian solution* ($CES(N,v)$). Furthermore, Dutta and Ray (1989) provided the following algorithm to compute the constrained egalitarian solution for convex games:

Step 1: Let $S_1 \in \arg\max_{S \subset N} \frac{v(S)}{|S|}$ such that $|S_1| > |T|$ for all $T \in \arg\max_{S \subset N} \frac{v(S)}{|S|}$. (For convex games S_1 exists). Define $x_i^* = \frac{v(S_1)}{|S_1|}$ for all $i \in S_1$.

Step *k*: Let us assume that $S_1, S_2, ..., S_{k-1}$ have been defined recursively and $\bigcup_{j=1}^{k-1} S_j \neq N$. Define the game with set of player $N \setminus \bigcup_{j=1}^{k-1} S_j$ and characteristic function given by $v_k(S) = v(\bigcup_{j=1}^{k-1} S_j \cup S) - v(\bigcup_{j=1}^{k-1} S_j)$. (If v is convex, then v_k is convex). Now let $S_k \in \arg\max_{S \subset N \setminus \bigcup_{j=1}^{k-1} S_j} \frac{v_k(S)}{|S|}$ such that $|S_1| > |T|$ for all $T \in \arg\max_{S \subset N \setminus \bigcup_{j=1}^{k-1} S_j} \frac{v_k(S)}{|S|}$. Define $x_i^* = \frac{v_k(S_k)}{|S_k|}$ for all $i \in S_k$.

This algorithm always ends in a finite number of iterations, and we will denote x^* by $DR(N,v)$. It is clear that for convex games $DR(N,v) = CES(N,v)$, but this is not true in general. Furthermore, in general, S_k could not be unique, in that case we could obtain more than one allocation and there would be more than one $DR(N,v)$. Thus, in a general sense, $DR(N,v)$ could be considered a set of allocations.

Following the scheme of the Dutta and Ray algorithm, Branzei et al (2006) defined the *equal split-off set* ($ESOS(N,v)$). For each partition $\pi = \{S_1, S_2, ..., S_K\}$ of N such that $S_k \in \arg\max_{S \subset N \setminus \bigcup_{j=1}^{k-1} S_j} \frac{v_k(S)}{|S|}$, $k = 1, 2, ..., K$, they consider the allocation x^π given by $x_i^\pi = \frac{v_k(S_k)}{|S_k|}$ for all $S_k \in \pi$ and all $i \in S_k$. Given a game (N,v), the equal spli-off set is given by

$$ESOS(N,v) = \{x \in \mathbb{R}^n : \exists \pi \text{ such that } x = x^\pi\}.$$

Clearly, in general, there are more than one partition π in the described conditions, therefore $ESOS(N,v)$ could contain more than one allocation. In Branzei et al (2006) it is proved that $ESOS(N,v)$ coincides with $CES(N,v)$ for convex games. Finally, it is straightforward that $DR(N,v) \subset ESOS(N,v)$.

3 An example

In this section we are going to analyze Example 4 in Dutta and Ray (1989). This example was used to illustrate that the egalitarian allocation does not Lorenz dominate all core allocations when the game is not convex.
Example 1 (Dutta and Ray, 1989): $N = \{1, 2, 3, 4\}$ and the characteristic function v is defined as follows:

S	1	2	3	4	12	13	14	23
$v(S)$	1	1.5	2.5	4	3	4	5	4
S	24	34	123	124	134	234	1234	
$v(S)$	6	7	5.5	7	8	8.5	10	

The constrained egalitarian allocation is $CES(N, v) = (1, 2, 3, 4)$ and belongs to the core of the game, but it does not Lorenz dominate the allocation $(1\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 4\frac{1}{2})$ which is also in the core. However, the allocation obtained by the Dutta and Ray algorithm is $DR(N, v) = (1\frac{1}{2}, 1\frac{1}{2}, 3, 4)$ which Lorenz dominates every core allocation. Of course, it belongs to neither the core nor the Lorenz core. Furthermore, it is easy to check that $ESOS(N, v) = DR(N, v)$.

Therefore this example suggests the following questions: Does the allocation obtained by the algorithm, when exists, Lorenz dominate every core allocation? The answer is positive and one proof of that is obtained using the technical results in the next section.

4 Technical results

In this section we provide some technical results related to Lorenz dominance which could be useful to analyze the Lorenz dominance in cooperative games.

Let $(x_{(1)}, x_{(2)}, ..., x_{(n)})$ be the vector obtained from $x \in \mathbb{R}^n$ by ordering its components in decreasing order: $x_{(1)} \geq x_{(2)} \geq ... \geq x_{(n)}$.

In the sequel, we use the following equivalent definition of Lorenz dominance: x Lorenz dominates y if

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \tag{1}$$

for each $k = 1, 2, ..., n$, with at least one strict inequality.

Lemma 1 *The vector $a = 1_n a$ Lorenz dominates each other element of the set $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = na > 0\}$, where 1_n represents the vector whose n coordinates are equal to 1.*

Proof. Let us consider $x \neq a$. It is obvious that $a = a_{(1)} < x_{(1)}$. Let us assume that $ka \leq \sum_{i=1}^{k} x_{(i)}$. Now, we are going to prove that $(k+1)a \leq \sum_{i=1}^{k+1} x_{(i)}$. Let us suppose on the contrary that $(k+1)a > \sum_{i=1}^{k+1} x_{(i)}$. This implies that $\sum_{i=k+2}^{n} x_{(i)} > (n-k-1)a$. Therefore, there exists an $i^* > k+1$ such that $x_{(i^*)} > a$. On the other hand, we have

$$(n-1)a = ka + (n-k-1)a < \sum_{i=1}^{k} x_{(i)} + \sum_{i=k+2}^{n} x_{(i)} = na - x_{(k+1)}.$$

Hence, we obtain $x_{(k+1)} < a$. Further, since $i^* > k+1$ implies $x_{(k+1)} \geq x_{(i^*)}$, we have $a > x_{(k+1)} \geq x_{(i^*)} > a$. Therefore, the result holds. ■

Lemma 2 *The vector $a = 1_n a$ satisfies (1) for each other element of the set $\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \geq na\}$.*

Proof. Let us consider $x_{(1)} \geq x_{(2)} \geq ... \geq x_{(n)}$ and define $x'_{(n)} = x_{(n)} - (\sum_{i=1}^{n} x_i - na)$. Then, $x_{(1)} \geq x_{(2)} \geq ... \geq x'_{(n)}$ and $\sum_{i=1}^{n-1} x_i + x'_{(n)} = na$. Thus, applying Lemma 1 we obtain the desired result. ■

Proposition 1 *The vector $a = (1_{n_1}a_1, 1_{n_2}a_2, ..., 1_{n_t}a_t)$, such that $a_1 \geq a_2 \geq ... \geq a_t > 0$ and $\sum_{i=1}^{t} n_i = n$, Lorenz dominates each other element of the set*

$$\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n_1} x_i = n_1 a_1, \sum_{i=n_1+1}^{n_2} x_i = n_2 a_2, ..., \sum_{i=\sum_{j=1}^{t-1} n_j + 1}^{n} x_i = n_t a_t\}. \quad (2)$$

Proof. Let us consider $x \neq a$ and the vector obtained from vector $\mathbf{x}$ by rearranging in decreasing order the elements inside each block displayed in (2): $(x_{(1,1)}, ..., x_{(1,n_1)}; x_{(2,1)}, ..., x_{(2,n_2)}; ...; x_{(t,1)}, ..., x_{(t,n_t)})$ with

$$\begin{aligned} x_{(1,1)} &\geq ... \geq x_{(1,n_1)} \\ x_{(2,1)} &\geq ... \geq x_{(2,n_2)} \\ &... \\ x_{(t,1)} &\geq ... \geq x_{(t,n_t)}. \end{aligned}$$

By Lemma 1 we know that for each block $i = 1, 2, ..., t$ we have

$$\sum_{j=1}^{k} a_i \leq \sum_{j=1}^{k} x_{(i,j)} \text{ for each } k = 1, 2, ..., n_i, \quad (3)$$

and at least one inequality is strict, where $x_{(i,j)}$ is the j-th element in the decreasing order within the i-th block of vector x.

Now, we are going to prove that x is Lorenz dominated by a. For each k, $1 \leq k \leq n$, we have

$$k = \sum_{h=0}^{r} n_h + r(k)$$

for some $r \leq t$, $n_0 = 0$ and $0 \leq r(k) \leq n_{r+1}$. Then,

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{h=1}^{r} \sum_{j=1}^{n_h} x_{(h,j)} + \sum_{j=1}^{r(k)} x_{(r+1,j)} \geq \sum_{h=1}^{r} \sum_{j=1}^{n_h} a_{(h,j)} + \sum_{j=1}^{r(k)} a_{(r+1,j)} = \sum_{i=1}^{k} a_{(i)},$$

where the first inequality follows from the decreasing order inside each block, the second inequality follows from (3) and the equality from the definition of a. Note that we use the alternative notations for ordered vectors $a_{(\cdot)}$ and $a_{(\cdot,\cdot)}$, where the former refers to blocks and the latter to the elements inside each block. Now, since $x \neq a$ there exists a block q and an l-th coordinate inside of it such that $\sum_{j=1}^{l} a_q < \sum_{j=1}^{l} x_{(q,j)}$. Take $\hat{k} = \sum_{h=0}^{q-1} n_h + l$. Then,

$$\sum_{i=1}^{\hat{k}} x_{(i)} \geq \sum_{h=1}^{q-1} \sum_{j=1}^{n_h} x_{(h,j)} + \sum_{j=1}^{l} x_{(q,j)} > \sum_{h=1}^{q-1} \sum_{j=1}^{n_h} a_{(h,j)} + \sum_{j=1}^{l} a_{(q,j)} = \sum_{i=1}^{\hat{k}} a_{(i)},$$

where the strict inequality follows from (3) and the above comment.

Therefore, the result holds. ■

Lemma 3 *The vector $a = (1_{n_1}a_1, 1_{n_2}a_2, ..., 1_{n_t}a_t)$, such that $a_1 \geq a_2 \geq ... \geq a_t > 0$ and $\sum_{i=1}^{t} n_i = n$, satisfies (1) for each other element of the set*

$$\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n_1} x_i \geq n_1 a_1, \sum_{i=n_1+1}^{n_2} x_i \geq n_2 a_2, ..., \sum_{i=\sum_{j=1}^{t-1} n_j+1}^{n} x_i \geq n_t a_t\}.$$

Proof. The proof can be derived straightforwardly taking into account Lemma 2 and Proposition 1. ■

Proposition 2 *The vector $a = (1_{n_1}a_1, 1_{n_2}a_2, ..., 1_{n_t}a_t)$, where $a_1 \geq a_2 \geq ... \geq a_t > 0$ and $\sum_{i=1}^{t} n_i = n$, Lorenz dominates each other element $x \in \mathbb{R}^n$ such that*

$$\sum_{i=1}^{n_1} x_i \geq n_1 a_1, \sum_{i=1}^{n_1+n_2} x_i \geq \sum_{i=1}^{2} n_i a_i, ..., \sum_{i=1}^{n-n_t} x_i \geq \sum_{i=1}^{t-1} n_i a_i, \sum_{i=1}^{n} x_i = \sum_{i=1}^{t} n_i a_i. \tag{4}$$

Proof. Let us consider $x \neq a$. If for this vector x every inequality in (4) is an equality, then applying Proposition 1 we obtain the result. Therefore, let us suppose that at least one inequality in (4) is strict. In this case, there exist $r < s \leq t$ such that

$$\sum_{j=1}^{n_r} x_{(r,j)} > n_r a_r \text{ and } \sum_{j=1}^{n_s} x_{(s,j)} < n_s a_s.$$

For the r-th block, by applying an analogous argument as in Lemma 1, we obtain that $\sum_{j=1}^{k} a_r \leq \sum_{j=1}^{k} x_{(r,j)}$ for each $k = 1, ..., n_r$ and at least one inequality is strict (for example the last one). Let s be the first block for which $\sum_{j=1}^{n_s} x_{(s,j)} < n_s a_s$. Now, we consider the vector $(x'_{(1)}, x'_{(2)}, ..., x'_{(n_1+n_2+...+n_{s-1})})$ obtained by arranging in decreasing order the first $n_1 + n_2 + ... + n_{s-1}$ coordinates of vector x, that is

$$x'_{(1)} \geq x'_{(2)} \geq ... \geq x'_{(n_1+n_2+...+n_{s-1})}.$$

Applying Proposition 1 we obtain

$$\sum_{i=1}^{k} x'_{(i)} \geq \sum_{i=1}^{k} a_{(i)},$$

for each $k \leq n_1 + n_2 + ... + n_{s-1}$, with at least one strict inequality. In particular, we have

$$\sum_{i=1}^{n_1+n_2+...+n_{s-1}} x_{(i)} = \sum_{i=1}^{n_1+n_2+...+n_{s-1}} x'_{(i)} > \sum_{i=1}^{n_1+n_2+...+n_{s-1}} a_{(i)}.$$

Let $A = \sum_{i=1}^{n_1+n_2+...+n_{s-1}} x'_{(i)} - \sum_{i=1}^{n_1+n_2+...+n_{s-1}} a_{(i)}$; then, from the definition of the set (4), we obtain

$$A \geq n_s a_s - \sum_{j=1}^{n_s} x_{(s,j)} = B > 0.$$

Now, we take $x_{(s,1)}$, which is a largest element of block s, and distinguish between two cases:

- $B \geq a_s - x_{(s,1)} > 0$. We construct a new decreasing ordered vector x'' using the first $n_1 + n_2 + ... + n_{s-1}$ coordinates of vector $\mathbf{x}$ and $x_{(s,1)}$, that is

$$x''_{(1)} \geq x''_{(2)} \geq ... \geq x''_{(n_1+n_2+...+n_{s-1}+1)}.$$

We have

$$\sum_{i=1}^{n_1+n_2+...+n_{s-1}+1} x_{(i)} \geq \sum_{i=1}^{n_1+n_2+...+n_{s-1}+1} x''_{(i)} = \sum_{i=1}^{n_1+n_2+...+n_{s-1}} x'_{(i)} + x_{(s,1)}.$$

On the other hand, we have

$$\begin{aligned} & \sum_{i=1}^{n_1+n_2+...+n_{s-1}+1} x''_{(i)} - \sum_{i=1}^{n_1+n_2+...+n_{s-1}+1} a_{(i)} \\ = & \sum_{i=1}^{n_1+n_2+...+n_{s-1}} x'_{(i)} + x_{(s,1)} - \sum_{i=1}^{n_1+n_2+...+n_{s-1}} a_{(i)} - a_s \\ = & \; A + x_{(s,1)} - a_s \geq A - B \geq 0. \end{aligned}$$

Therefore, we obtain

$$\sum_{i=1}^{n_1+n_2+...+n_{s-1}+1} x_{(i)} \geq \sum_{i=1}^{n_1+n_2+...+n_{s-1}+1} x''_{(i)} \geq \sum_{i=1}^{n_1+n_2+...+n_{s-1}+1} a_{(i)}.$$

- $x_{(s,1)} \geq a_s$. The proof of this case is straightforward.

At this point, we can do the same for each other element in the s-th block and we obtain a new decreasing ordered vector with the first $n_1 + n_2 + ... + n_{s-1} + n_s$ coordinates of vector x. Repeating the reasoning for each block p such that $\sum_{j=1}^{n_p} x_{(p,j)} < n_p a_p$, we can conclude that a Lorenz dominates each other element x of the set (4). ■

5 Some remarks on the Lorenz dominance

We shall start this section with considerations upon the constrained egalitarian solution. Theorem 3 in Dutta and Ray (1989) states that $CES(N, v)$ Lorenz dominates every allocation in $C(N, v)$ for convex games, however, in its proof nothing about convexity is mentioned. Therefore, one could understand that this result is also true in other situations. Furthermore, a „rich-to-poor" transfer reasoning is used for proving that $CES(N, v)$ Lorenz dominated every allocation in $C(N, v)$. In our opinion, this „rich-to-poor" reasoning, although intuitively clear, is little precise from a mathematical point of view.

Next using the technical results in Section 4, we provide an alternative and more general proof for Theorem 3 in Dutta and Ray (1989).

Theorem 1 *In a balanced game, if $DR(N,v)$ is univocally defined, then $DR(N,v)$ Lorenz dominates every allocation in $C(N,v)$.*

Proof. Let $x \in C(N,v)$ such that $x \neq DR(N,v)$. Since for all $k = 1,2,...,K$, $x(S_k) \geq v(S_k)$, two cases are possible:

- $x(S_k) = v(S_k)$ for all $k = 1,2,...,K$. Then, it is easy to check that we are in the conditions of Proposition 1; hence, $DR(N,v)$ Lorenz dominates x.

- $x(S_k) > v(S_k)$ for some $k = 1,2,...,K$. Then it is easy to check that we are in the conditions of Proposition 2; hence, $DR(N,v)$ Lorenz dominates x.

Therefore, the proof is finished. ■

Corollary 2 *(Theorem 3, Dutta and Ray, 1989) In a convex game, $CES(N,v)$ Lorenz dominates every other allocation in $C(N,v)$.*

Proof. It is straightforward taking into account that in convex games $CES(N,v) = DR(N,v)$ and Theorem 1. ■

These results can be easily extended to $ESOS(N,v)$ as the following theorem shows.

Theorem 3 *In a balanced game, every allocation in $ESOS(N,v)$ Lorenz dominates every allocation in $C(N,v)$.*

Proof. The proof is analogous to the proof of Theorem 1. ■

On the other hand, in superadditive games we have that $DR(N,v) \subset ESOS(N,v) \subset EDC(N,v) \subset I(N,v)$. Thus, if a society believes that a „fair" allocation should be in the imputation set and the Lorenz criterion is a good concept of inequality, then the core would be in trouble many times. If the coalitional rationality is included as other desirable criterion, then the ideas and results in Arin and Inarra (2001) would be very useful.

A simple algorithm to obtain an allocation in the imputation set which Lorenz dominates each other allocation in the imputation set is the following:

Step 1: Let $S_1 = \{i \in N : v(i) \geq \frac{v(N)}{|N|}\}$. If $S_1 \neq \emptyset$ then define $x_i^* = v(i)$ for all $i \in S_1$, otherwise $x_i^* = \frac{v(N)}{|N|}$ for all $i \in N$.

Step *k*: Let us assume that $S_1, S_2, ..., S_{k-1}$ have been defined recursively and $\bigcup_{j=1}^{k-1} S_j \neq N$. Let $S_k = \{i \in N \setminus \bigcup_{j=1}^{k-1} S_j : v(i) \geq \frac{v(N) - \sum_{j \in \bigcup_{j=1}^{k-1} S_j} v(i)}{\left| N \setminus \bigcup_{j=1}^{k-1} S_j \right|}\}$. If $S_k \neq \emptyset$ then define $x_i^* = v(i)$ for all $i \in S_k$, otherwise $x_i^* = \frac{v(N) - \sum_{j \in \bigcup_{j=1}^{k-1} S_j} v(i)}{\left| N \setminus \bigcup_{j=1}^{k-1} S_j \right|}$ for all $i \in N \setminus \bigcup_{j=1}^{k-1} S_j$.

References

[1] Arin, J. and E. Inarra (2001) Egalitarian solutions in the core. International Journal of Game Theory 30:187-193.

[2] Arin, J. and E. Inarra (2002) Egalitarian sets for TU-games. International Game Theory Review 4:183-199.

[3] Branzei, R., D. Dimitrov and S. Tijs (2006) The equal split-off set for cooperative games. Game Theory and Mathematical Economics, Banach Center Publications 71:39-46.

[4] Dutta, B. and D. Ray (1989) A concept of egalitarianism under participation constraints. Econometrica 57:615-635.

[5] Dutta, B. and D. Ray (1991) Constrained egalitarian allocations. Games and Economic Behavior 3:403-422.

[6] Selten, R. (1972) Equal share analysis of characteristic function experiments, in: Sauermann, H. (Ed.), Contributions to Experimentation in Economics, vol. 3, Mohr Verlag, pp.130-165.

[7] Sen, A.K. (1973) On Economic Inequality. Oxford: Clarenden Press.