CentER

Discussion Paper

No. 2004–110

# THE EQUAL SPLIT-OFF SET FOR COOPERATIVE GAMES

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November 2004

ISSN 0924-7815

Tilburg University

# The equal split-off set for cooperative games*


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November 2004


**Abstract**

In this paper the equal split-off set is introduced as a new solution concept for cooperative games. This solution is based on egalitarian considerations and it turns out that for superadditive games the equal split-off set is a subset of the equal division core. Moreover, the proposed solution is single valued on the class of convex games and it coincides with the Dutta-Ray constrained egalitarian solution.

*Journal of Economic Literature* Classification Numbers: C71.



*D. Dimitrov gratefully acknowledges financial support from a Marie Curie Research Fellowship of the European Community programme "Improving the Human Research Potential and the Socio-Economic Knowledge Base" under contract number HPMF-CT-2002-02121.

*Keywords*: convex games, egalitarianism, equal split-off set, superadditive games.

# 1 Introduction

In this paper we propose a new set valued solution concept for cooperative games with transferable utility that we call the equal split-off set. This solution is based on egalitarian considerations and it is inspired by the Dutta-Ray algorithm for finding the constrained egalitarian solution for convex games (cf. Dutta and Ray (1989)).

More precisely, we consider a world $N$ of $n$ players, $N = \{1, \ldots, n\}$, who believe in equal share cooperation. For each coalition $S \subseteq N$, let the real number $v(S)$ represent what the players in $S$ can get if they cooperate (i.e. $v(S)$ is the worth or the value of coalition $S$). We assume that the entire set of players will cooperate and deal with the question how the whole amount of money $v(N)$ generated by $N$ should be divided among the players by considering the following step-wise process.

First, one of the coalitions with maximal average worth, say $T_1$, forms and the players in $T_1$ divide equally the worth $v(T_1)$. In step 2 one of the coalitions in $N \setminus T_1$ with maximal average marginal worth w.r.t. $T_1$, say $T_2$, forms, joins costless $T_1$, and divides equally the increase in value $v(T_2 \cup T_1) - v(T_1)$ among its members. The process stops when a partition of $N$ of the form $\langle T_1, \ldots, T_K \rangle$ for some $1 \leq K \leq n$ is reached. This procedure generates a payoff vector $x \in \mathbb{R}^n$ which we call an equal split-off allocation. The equal split-off set is then defined as consisting of all equal split-off allocations.

Notice that the difference between the above procedure and the Dutta-Ray algorithm for finding the constrained egalitarian solution for convex games is that the corresponding selected coalitions need not to be the largest coalitions with the highest average worth.

The outline of this paper is as follows. After some preliminaries in Section 2, we introduce the equal split-off set for superadditive games in Section 3 and study some of its properties on this class of games. It turns out that any allocation in the equal split-off set is efficient and individually rational, and, in addition, it belongs to the equal division core of the corresponding game. In Section 4 we concentrate on the class of convex games and prove that the equal split-off set of a convex game consists of a unique allocation which is the constrained egalitarian solution of that game. We conclude in Section 5 by pointing out some directions for further research.

# 2 Preliminaries

A TU-game is a pair $(N, v)$, where $N = \{1, \ldots, n\}$ is a *set of players* and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on $N$ satisfying $v(\emptyset) = 0$. Often, we will identify a game $(N, v)$ with its characteristic function $v$. For any coalition $S \subseteq N$, $v(S)$ is the *worth* of coalition $S$, i.e. the members of $S$ can obtain a total payoff of $v(S)$ by agreeing to cooperate.

A game $v$ is called

- *superadditive*, if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$;
- *convex*, if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$.

An *allocation* in a game $v$ is a payoff vector $x \in \mathbb{R}^n$. An allocation of $v(N)$ such that this amount is cleared is called *efficient*, and an allocation $x$ such that $x_i \geq v(i)$ for each $i \in N$ is called *individually rational.* The *imputation set* $I(v)$ of a game $v$ is the set of all efficient and individually rational allocations, i.e.

$$I(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(i) \text{ for each } i \in N \right\}.$$

Further, an allocation is called *stable* if any coalition $S \subseteq N$ receives at least its value $v(S)$. The *core* $C(v)$ of a game $v$ is the set of all efficient and stable allocations (Gillies, 1953), i.e. the set

$$C(v) = \left\{ x \in I(v) \mid \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in 2^N \right\}.$$

For each game $v$ we have that $C(v)$ is a subset of the *equal division core* $EDC(v)$ of $v$. The latter concept was introduced by Selten (1972) as the set

$$\left\{ x \in I(v) \mid \nexists S \in 2^N \setminus \{\emptyset\} \text{ s.t. } \frac{v(S)}{|S|} > x_i \text{ for all } i \in S \right\}$$

consisting of all efficient payoff vectors which cannot be improved upon by the equal division allocation of any subcoalition.

An interesting element of the core $C(v)$ of a convex game $v$ (and, hence, of $EDC(v)$) is the Dutta-Ray *constrained egalitarian solution* $DR(v)$. This solution consists of the unique allocation in $C(v)$ that Lorenz dominates every other core allocation. In their seminal paper, Dutta and Ray (1989) provide an algorithm for generating $DR(v)$ for each convex game $v$. We apply a modified version of the Dutta-Ray algorithm to any superadditive game in order to produce allocations in the equal split-off set of $v$.

# 3 The equal split-off set for superadditive games

Let $v$ be a superadditive game and $\pi = \langle T_1, \ldots, T_K \rangle$ be an ordered partition of the player set $N$. We set $v_1 := v$, and for each $k \in \{2, \ldots, K\}$ we define the *marginal game*

$$v_k : N \setminus \left(\cup_{s=1}^{k-1} T_s\right) \to \mathbb{R}$$

by

$$v_k(S) := v_{k-1}\left(T_{k-1} \cup S\right) - v_{k-1}\left(T_{k-1}\right) = v\left(\left(\cup_{s=1}^{k-1} T_s\right) \cup S\right) - v\left(\cup_{s=1}^{k-1} T_s\right).$$

We call the partition $\pi = \langle T_1, \ldots, T_K \rangle$ of $N$ a *suitable ordered partition with respect to the game* $v$ if $T_k \in \arg\max_{S \in 2^{N \setminus \left(\cup_{s=1}^{k-1} T_s\right)} \setminus \{\emptyset\}} \frac{v_k(S)}{|S|}$ for all $k \in \{1, \ldots, K\}$.

Given such a partition $\pi$, the *equal split-off allocation for* $v$ *generated by* $\pi$ is the payoff vector $x = (x_i)_{i \in N} \in \mathbb{R}^n$, where for all $T_k \in \pi$ and all $i \in T_k$, $x_i = \frac{v_k(T_k)}{|T_k|}$.

Now we define *the equal split-off set* $ESOS(v)$ of the game $v$ as the set

$$\{x \in \mathbb{R}^n \mid \exists \pi \text{ s.t. } x \text{ is an equal split-off allocation for } v \text{ generated by } \pi\}.$$

In order to illustrate this solution concept, let us have a look at the following examples:

**Example 1** *(2-person superadditive games) Let* $v$ *be a game on the player set* $N = \{1, 2\}$ *satisfying* $v(1, 2) \geq v(1) + v(2)$*. Suppose without loss of generality that* $v(1) \geq v(2)$ *and consider the following four cases:*
*(i)* $v(1) > \frac{1}{2} v(1, 2)$*. Then* $\langle \{1\}, \{2\} \rangle$ *is the unique suitable ordered partition and* $ESOS(v) = \{(v(1), v(1, 2) - v(1))\}$*;*
*(ii)* $v(2) < v(1) = \frac{1}{2} v(1, 2)$*. In this case* $ESOS(v) = \left\{\left(\frac{1}{2} v(1, 2), \frac{1}{2} v(1, 2)\right)\right\}$ *corresponding to the suitable ordered partitions* $\langle \{1\}, \{2\} \rangle$ *and* $\langle \{1, 2\} \rangle$*;*
*(iii)* $v(2) = v(1) = \frac{1}{2} v(1, 2)$*. Also here* $ESOS(v) = \left\{\left(\frac{1}{2} v(1, 2), \frac{1}{2} v(1, 2)\right)\right\} = \{(v(1), v(2))\}$ *corresponding to the three suitable ordered partitions* $\langle \{1\}, \{2\} \rangle$*,* $\langle \{2\}, \{1\} \rangle$*, and* $\langle \{1, 2\} \rangle$*;*

*(iv)* $v(1) < \frac{1}{2}v(1,2)$*. Then* $\langle\{1,2\}\rangle$ *is the unique suitable ordered partition and* $ESOS(v) = \left\{\left(\frac{1}{2}v(1,2), \frac{1}{2}v(1,2)\right)\right\}$*.*

**Example 2** *(Simple games) In a simple game* $v$ *on player set* $N$ *we have that for all* $S \subseteq N$*,* $v(S) \in \{0,1\}$ *with* $v(\emptyset) = 0$ *and* $v(N) = 1$*. A coalition* $S \subseteq N$ *is called minimal winning if* $v(S) = 1$ *and* $v(S') = 0$ *for all* $S' \subset S \subseteq N$*. Given a simple game* $v$*, we denote the set of all minimal winning coalitions with a smallest cardinality by* $W^s$*. In the case of simple games* $ESOS(v) = \left\{\frac{1}{|S|}e^S \mid S \in W^s\right\}$ *because for any suitable ordered partition* $\langle T_1, \ldots, T_K\rangle$ *we will have* $T_1 \in W^s$*, and all players in* $T_1$ *will receive* $\frac{1}{|T_1|}$ *whereas the players in* $N \setminus T_1$ *will receive payoff* $0$*.*

**Example 3** *(Glove games) Let* $N = L \cup R$*,* $L \cap R \neq \emptyset$ *and the game* $v$ *be defined by* $v(S) = \min\{|S \cap L|, |R \cap L|\}$ *for each* $S \subseteq N$*. If* $|L| = |R|$*, then* $ESOS(v) = \left\{\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)\right\}$ *that can be generated by many suitable ordered partitions, where each element* $T_k$ *of such a partition has the property that* $|T_k \cap L| = |T_k \cap R|$*. In case* $|L| > |R|$ *each element* $x \in ESOS(v)$ *satisfies* $x_i = \frac{1}{2}$ *for each* $i \in R$ *and for* $|R|$ *elements of* $L$*, and* $x_i = 0$ *for the other elements of* $L$*. Conversely, all elements of this type belong to* $ESOS(v)$*.*

One can easily check that the egalitarian split-off set in Examples 1-3 is a subset of the imputation set of the corresponding game. This fact turns out to be true for all superadditive games as is shown in

**Proposition 1** *Let* $v$ *be a superadditive game. Then* $ESOS(v) \subseteq I(v)$*.*

**Proof.** We have to prove that each allocation in $ESOS(v)$ is efficient and individually rational.

Take $x \in ESOS(v)$ generated by a suitable ordered partition $\langle T_1, \ldots, T_K\rangle$.

Then

$$\begin{aligned}
\sum_{i\in N} x_i &= \sum_{k=1}^{K} |T_k| \frac{v_k(T_k)}{|T_k|} \\
&= \sum_{k=1}^{K} |T_k| \frac{v\left(\cup_{s=1}^{k} T_s\right) - v\left(\cup_{s=1}^{k-1} T_s\right)}{|T_k|} \\
&= \sum_{k=1}^{K} \left(v\left(\cup_{s=1}^{k} T_s\right) - v\left(\cup_{s=1}^{k-1} T_s\right)\right) \\
&= v(N),
\end{aligned}$$

i.e. $x$ is efficient.

Take now an $i \in N$. We have to prove that $x_i \geq v\,(i)$. Suppose $i \in T_r$ for some $r \in \{1, \ldots, K\}$. Then

$$\begin{aligned}
x_i &= \frac{v_r(T_r)}{|T_r|} = \frac{v\left(\cup_{s=1}^{r} T_s\right) - v\left(\cup_{s=1}^{r-1} T_s\right)}{|T_r|} \\
&\geq v\left(\left(\cup_{s=1}^{r-1} T_s\right) \cup \{i\}\right) - v\left(\cup_{s=1}^{r-1} T_s\right) \\
&\geq v(i),
\end{aligned}$$

where the first inequality follows from the definition of $T_r$ and the second inequality from the superadditivity of $v$. Hence, $x$ is individually rational.

We conclude that $x \in I(v)$, $ESOS(v) \subseteq I(v)$. ■

Our next proposition shows a monotonicity property of each suitable ordered partition of $N$ with respect to a superadditive game.

**Proposition 2** *Let $v$ be a superadditive game and let $\langle T_1, \ldots, T_K \rangle$ be a suitable ordered partition of $N$ w.r.t. $v$. Then*

$$\max_{S \in 2^{N \setminus \left(\cup_{s=1}^{k-1} T_s\right)} \setminus \{\emptyset\}} \frac{v_k(S)}{|S|} \geq \max_{S \in 2^{N \setminus \left(\cup_{s=1}^{k} T_s\right)} \setminus \{\emptyset\}} \frac{v_{k+1}(S)}{|S|}$$

*for all $k \in \{1, \ldots, K-1\}$.*

**Proof.** By definition of $T_k$ we have

$$\frac{v\left(\cup_{s=1}^{k}T_s\right)-v\left(\cup_{s=1}^{k-1}T_s\right)}{|T_k|}\geq\frac{v\left(\cup_{s=1}^{k+1}T_s\right)-v\left(\cup_{s=1}^{k-1}T_s\right)}{|T_k|+|T_{k+1}|}.$$

Moreover,

$$\frac{v\left(\cup_{s=1}^{k+1}T_s\right)-v\left(\cup_{s=1}^{k-1}T_s\right)}{|T_k|+|T_{k+1}|}=\frac{v\left(\cup_{s=1}^{k+1}T_s\right)-v(\cup_{s=1}^{k}T_s)+v(\cup_{s=1}^{k}T_s)-v\left(\cup_{s=1}^{k-1}T_s\right)}{|T_k|+|T_{k+1}|},$$

implying that

$$\frac{v\left(\cup_{s=1}^{k}T_s\right)-v\left(\cup_{s=1}^{k-1}T_s\right)}{|T_k|}\geq\frac{v\left(\cup_{s=1}^{k+1}T_s\right)-v(\cup_{s=1}^{k}T_s)+v(\cup_{s=1}^{k}T_s)-v\left(\cup_{s=1}^{k-1}T_s\right)}{|T_k|+|T_{k+1}|}.$$

This inequality is equivalent to

$$\begin{aligned}&\left(v\left(\cup_{s=1}^{k}T_s\right)-v\left(\cup_{s=1}^{k-1}T_s\right)\right)|T_k|+\left(v\left(\cup_{s=1}^{k}T_s\right)-v\left(\cup_{s=1}^{k-1}T_s\right)\right)|T_{k+1}|\\ \geq\;&\left(v\left(\cup_{s=1}^{k+1}T_s\right)-v\left(\cup_{s=1}^{k}T_s\right)\right)|T_k|+\left(v\left(\cup_{s=1}^{k}T_s\right)-v\left(\cup_{s=1}^{k-1}T_s\right)\right)|T_k|,\end{aligned}$$

which is at its turn equivalent to

$$\left(v\left(\cup_{s=1}^{k}T_s\right)-v\left(\cup_{s=1}^{k-1}T_s\right)\right)|T_{k+1}|\geq\left(v\left(\cup_{s=1}^{k+1}T_s\right)-v\left(\cup_{s=1}^{k}T_s\right)\right)|T_k|.$$

■

We show next that the equal split-off set of a superadditive game is a refinement of the equal division core of that game.

**Theorem 1** *Let $v$ be a superadditive game. Then $ESOS(v)\subseteq EDC(v)$.*

**Proof.** Let $x\in ESOS(v)$ be generated by the suitable ordered partition $\langle T_1,\ldots,T_K\rangle$. Take $S\in 2^N\setminus\{\emptyset\}$. We have to prove that there is an $i\in S$ such that $x_i\geq\frac{v(S)}{|S|}$.

Let $m \in \{1, \ldots, K\}$ be the smallest number such that $T_m \cap S \neq \emptyset$. Then

$$\begin{aligned} \frac{v(S)}{|S|} &\leq \frac{v\left(\left(\cup_{s=1}^{m-1} T_s\right) \cup S\right) - v\left(\cup_{s=1}^{m-1} T_s\right)}{|S|} \\ &\leq \frac{v\left(\cup_{s=1}^{m} T_s\right) - v\left(\cup_{s=1}^{m-1} T_s\right)}{|T_m|} \\ &= \frac{v_m(T_m)}{|T_m|} = \max_{T \in 2^{N \setminus \left(\cup_{s=1}^{m-1} T_s\right)} \setminus \{\emptyset\}} \frac{v_m(T)}{|T|}, \end{aligned}$$

where the first inequality follows from the superadditivity of $v$ and the second inequality from the definition of $T_m$. Note that $x_i = \max_{T \in 2^{N \setminus \left(\cup_{s=1}^{m-1} T_s\right)} \setminus \{\emptyset\}} \frac{v_m(T)}{|T|} \geq \frac{v(S)}{|S|}$ for each $i \in T_m \cap S$. So, $x \in EDC(v)$, $ESOS(v) \subseteq EDC(v)$. ■

The next example provides a game for which the equal split-off set is a strict subset of the equal division core.

**Example 4** *Let* $N = \{1, 2, 3\}$ *and* $v$ *be a glove game with* $L = \{1, 2\}$ *and* $R = \{3\}$. *Then* $EDC(v) = \left\{x \in I(v) \mid x_3 \geq \frac{1}{2}\right\}$ *and* $ESOS(v) = \left\{\left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)\right\}$.

# 4 The equal split-off set for convex games

We start this section with a lemma that holds for arbitrary TU-games and will play an important role when proving that the equal split-off set of a convex game consists of a single allocation which is the Dutta-Ray egalitarian solution of that game.

**Lemma 1** *Let* $v$ *be a TU-game and let* $\langle T_1, \ldots, T_K \rangle$ *be a suitable ordered partition of* $N$ *w.r.t.* $v$ *by means of marginal games* $v_1, \ldots, v_K$. *Let* $T_{m+1}, \ldots, T_q$ *be a sequence of coalitions from the ordered partition such that* $v_i(T_i) = \alpha |T_i|$ *for each* $i \in \{m+1, \ldots, q\}$. *Then* $v_{m+1}\left(\cup_{i=m+1}^{q} T_i\right) = \alpha \sum_{i=m+1}^{q} |T_i|$.

**Proof.** Because $v_i(T_i) = \alpha |T_i|$ for each $i \in \{m+1, \ldots, q\}$ we have

$$\begin{aligned} v_{m+1}(T_{m+1}) &= \alpha |T_{m+1}|, \\ v_{m+1}(T_{m+1} \cup T_{m+2}) - v_{m+1}(T_{m+1}) &= \alpha |T_{m+2}|, \\ &\ldots \\ v_{m+1}\left(\cup_{i=m+1}^{q} T_i\right) - v_{m+1}\left(\cup_{i=m+1}^{q-1} T_i\right) &= \alpha |T_q|. \end{aligned}$$

By summing up these equalities we obtain $v_{m+1}\left(\cup_{i=m+1}^{q} T_i\right) = \alpha \sum_{i=m+1}^{q} |T_i|$.
■

Let $\langle D_1, \ldots, D_P \rangle$ be the ordered partition of $N$ according to the Dutta-Ray algorithm for finding the constrained egalitarian solution $DR(v)$ of a convex game $v$. In each step $p \in \{1, \ldots, P\}$ of the Dutta-Ray algorithm $D_p$ is the largest element in the set

$$M^p := \arg \max_{S \in 2^{N \setminus \left\{\cup_{i=1}^{p-1} D_i\right\}} \setminus \{\emptyset\}} \frac{v\left(S \cup \left(\cup_{i=1}^{p-1} D_i\right)\right) - v\left(\left(\cup_{i=1}^{p-1} D_i\right)\right)}{|S|}.$$

We recall that for each $p \in \{1, \ldots, P\}$ the set $M^p$ has a lattice structure w.r.t. the partial ordering of inclusion. So,

$$D_p = \cup \{D \mid D \in M^p\}. \qquad (\#)$$

For further use we introduce also the following notation:

$$\begin{aligned} \overline{D}_{p-1} &: = \cup_{r=1}^{p-1} D_r; \\ v_{\overline{D}_{p-1}}(S) &: = v\left(S \cup \overline{D}_{p-1}\right) - v\left(\overline{D}_{p-1}\right) \text{ for each } S \in 2^{N \setminus \overline{D}_{p-1}}; \\ d_p &: = \frac{v\left(D_p \cup \left(\cup_{r=1}^{p-1} D_i\right)\right) - v\left(\left(\cup_{r=1}^{p-1} D_i\right)\right)}{|D_p|}. \end{aligned}$$

The next proposition will play a key role in this section.

**Proposition 3** *Let* $\langle D_1, \ldots, D_p, \ldots, D_P\rangle$ *be the ordered partition of* $N$ *obtained by applying the Dutta-Ray algorithm to a convex game* $v$. *Let* $T \subsetneq D_p$ *be such that* $v_{\overline{D}_{p-1}}(T) = d_p\,|T|$, *and let* $\overline{T} = \overline{D}_{p-1} \cup T$. *Then*

(1) $\max_{S \in 2^{N\setminus\overline{T}}\setminus\{\emptyset\}} \frac{v_{\overline{T}}(S)}{|S|} = d_p$;

(2) $\arg\max_{S \in 2^{N\setminus\overline{T}}\setminus\{\emptyset\}} \frac{v_{\overline{T}}(S)}{|S|} \subseteq 2^{D_p\setminus T}$.

**Proof.** Take $S \in 2^{N\setminus\overline{T}} \setminus \{\emptyset\}$. Then

$$\begin{aligned} v_{\overline{T}}(S) &= v\left(S \cup \overline{T}\right) - v(\overline{T}) \\ &= \left(v\left(S \cup \overline{T}\right) - v\left(\overline{D}_{p-1}\right)\right) - \left(v\left(\overline{T}\right) - v\left(\overline{D}_{p-1}\right)\right) \\ &= v_{\overline{D}_{p-1}}(S \cup T) - v_{\overline{D}_{p-1}}(T) \\ &= v_{\overline{D}_{p-1}}(S \cup T) - d_p\,|T|\,. \end{aligned}$$

Further, since $D_p$ is the largest set in $N \setminus \overline{D}_{p-1}$ with average worth $d_p$, and $|S \cup T| = |S| + |T|$, we have

$$\begin{aligned} &v_{\overline{D}_{p-1}}(S \cup T) < d_p\left(|S| + |T|\right) \quad \text{if } S \setminus D_p \neq \emptyset, \text{ and} \\ &v_{\overline{D}_{p-1}}(S \cup T) \leq d_p\left(|S| + |T|\right) \quad \text{if } S \subseteq D_p, \end{aligned}$$

implying that

$$\begin{aligned} &\tfrac{v_{\overline{T}}(S)}{|S|} < d_p \quad \text{if } S \setminus D_p \neq \emptyset, \text{ and} \\ &\tfrac{v_{\overline{T}}(S)}{|S|} \leq d_p \quad \text{if } S \subseteq D_p. \end{aligned}$$

To conclude that (1) and (2) hold we have only to show still that there is $S^* \in 2^{D_p\setminus T}$ such that $v_{\overline{T}}(S^*) = d_p\,|S^*|$.

Since (#) holds we can take an $A \in \arg\max_{S \in 2^{N\setminus\overline{D}_{p-1}}\setminus\{\emptyset\}} \frac{v_{\overline{D}_{p-1}}(S)}{|S|}$ and $A \nsubseteq T$. Then $A \subseteq D_p$. If $A \cap T = \emptyset$, we take $S^* = A$; otherwise we take

$S^* = A \setminus T$. So, $S^* \in 2^{D_p \setminus T}$, and

$$\begin{aligned} v_{\overline{T}}(S^*) &= v\left(S^* \cup \overline{T}\right) - v(\overline{T}) \\ &= \left(v\left(S^* \cup \overline{T}\right) - v\left(\overline{D}_{p-1}\right)\right) - \left(v\left(\overline{T}\right) - v\left(\overline{D}_{p-1}\right)\right) \\ &= \left(v\left((S^* \cup T) \cup \overline{D}_{p-1}\right) - v\left(\overline{D}_{p-1}\right)\right) - \left(v\left(T \cup \overline{D}_{p-1}\right) - v\left(\overline{D}_{p-1}\right)\right) \\ &= v_{\overline{D}_{p-1}}(S^* \cup T) - v_{\overline{D}_{p-1}}(T) \\ &= d_p\left(|S^*| + |T|\right) - d_p\,|T| = d_p\,|S^*|, \end{aligned}$$

where the last equality follows from $S^* \cup T = A \cup T \in \arg\max_{S \in 2^{N \setminus \overline{D}_{p-1}} \setminus \{\emptyset\}} \frac{v_{\overline{D}_{p-1}}(S)}{|S|}$, and $|S^* \cup T| = |S^*| + |T|$. ■

We show now that the equal split-off set of a convex game consists of a unique allocation which is precisely the constrained egalitarian solution of that game.

**Theorem 2** *Let $v$ be a convex game. Then $ESOS(v) = \{DR(v)\}$.*

**Proof.** Let $\langle D_1, \ldots, D_P \rangle$ be the ordered partition of $N$ according to the Dutta-Ray algorithm for finding the constrained egalitarian solution $DR(v) = (DR_i(v))_{i \in N}$ of $v$. Let $\langle T_1, \ldots, T_K \rangle$ be a suitable ordered partition of $N$ with respect to $v$ for finding the allocation $x = (x_i)_{i \in N}$ in the equal split-off set $ESOS(v)$ of $v$. We have to show that $x = DR(v)$.

We prove by induction that for each $k \in \{1, \ldots, K\}$ the following property $(P_k)$ holds:

$(P_k)$ There is a unique $s_k \in \{1, \ldots, P\}$ such that $\cup_{r=1}^{k} T_r \subseteq \cup_{p=1}^{s_k} D_p$, $T_k \subseteq D_{s_k}$, and $x_i = DR_i(v)$ for each $i \in \cup_{r=1}^{k} T_r$.

First, note that $(P_1)$ holds with $s_1 = 1$. Indeed $T_1 \subseteq D_1$ since $T_1, D_1 \in M^1$ and $D_1 = \cup \{D \mid D \in M^1\}$, so $x_i = DR_i(v) = d_1$ for each $i \in T_1$.

Further, we prove that $(P_q)$ implies $(P_{q+1})$ for each $q \in \{1, \ldots, K-1\}$. So suppose that $(P_q)$ holds. Then we show $(P_{q+1})$ by distinguishing two cases:

(a) $\cup_{r=1}^{q} T_r = \cup_{p=1}^{s_q} D_p$. In this case $T_{q+1}, D_{s_q+1} \in M^{q+1}$ and $\left|D_{s_q+1}\right| \geq |T_{q+1}|$. Therefore, $(P_{q+1})$ holds with $s_{q+1} = s_q + 1$, and $T_{q+1} \subseteq D_{s_q+1}$ because $D_{s_q+1} = \cup \{D \mid D \in M^{q+1}\}$. Hence, $x_i = DR_i(v)$ for each $i \in \cup_{r=1}^{q+1} T_r$ because of $(P_q)$ and the fact that $x_i = DR_i(v) = d_{q+1}$ for each $i \in T_{q+1}$.

(b) $\cup_{r=1}^{q} T_r \subsetneq \cup_{p=1}^{s_q} D_p$. We prove that in this case $(P_{q+1})$ holds with $s_{q+1} = s_q$ and $T_{q+1} \subseteq D_{s_q}$.

Let $m < q$ be such that $\cup_{r=1}^{m} T_r = \cup_{j=1}^{s_q-1} D_j$ and for each $i \in \{m+1, \ldots, q\}$, $T_i \subsetneq D_{s_q}$ and $v_{\overline{D}_{s_q-1}}(T_i) = d_{s_q} |T_i|$.

Then for $T = \cup_{i=m+1}^{q} T_i \subsetneq D_{s_q}$ we have by Lemma 1 that $v_{\overline{D}_{s_q-1}}(T) = d_{s_q} |T|$. Now we apply Proposition 3 with $s_q$ in the role of $p$, obtaining that $T_{q+1} \subseteq D_{s_q}$ and $v_{\overline{D}_{s_q-1}}(T_{q+1}) = d_{s_q} |T_{q+1}|$. Therefore, $(P_{q+1})$ holds with $s_{q+1} = s_q$ and $T_{q+1} = D_{s_q}$, and $x_i = DR_i(v)$ for each $i \in \cup_{r=1}^{q+1} T_r$.

Since $(P_k)$ holds for each $k \in \{1, \ldots, K\}$, we conclude that each suitable ordered partition $\langle T_1, \ldots, T_K \rangle$ of $N$ w.r.t. a convex game $v$ is a refinement of the Dutta-Ray partition $\langle D_1, \ldots, D_P \rangle$ of $v$. Specifically, there exist $l_1, \ldots, l_P$ such that $D_1 = \cup_{r=1}^{l_1} T_r, D_2 = \cup_{r=l_1+1}^{l_2} T_r, \ldots, D_P = \cup_{r=l_{P-1}+1}^{l_P} T_r$, with $l_P = K$, implying that $ESOS(v) = \{DR(v)\}$. ■

# 5 Concluding remarks

In Section 3 of this paper the equal split-off set has been introduced on the class of superadditive games as a new set valued solution concept based on egalitarian considerations. We have proved that for any superadditive game the new solution concept contains only allocations which are efficient and individually rational, and that it is a refinement of the equal division core. Further, in Section 4, we have proved that for each convex game the equal split-off set consists of a single allocation which is the Dutta-Ray constrained egalitarian solution for that game. When the game $v$ is not superadditive we can extend the equal split-off set to arbitrary TU-games by defining $ESOS(v) := ESOS(\overline{v})$, where $\overline{v}$ denotes the superadditive cover of the game $v$ and study properties of the equal split-off set in this more general setting. Another interesting topic for further research is the relation between the equal split-off set and the constrained egalitarian solution of Dutta and Ray for superadditive games, when the latter solution exists. Further investigations should be done for clarifying possible relations between the equal split-off set and existing egalitarianism-based solution concepts for arbitrary TU-games such as the strong-constrained egalitarian allocations (cf. Dutta and Ray (1991)), the egalitarian set, the preegalitarian set and the stable egalitarian set (cf. Arin and Inarra (2002)) and for balanced games like the Lorenz solution (cf. Hougaard et al. (2001)), the Lorenz stable set and the egalitarian core (cf. Arin and Inarra (2001)).

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