CentER

Discussion Paper

No. 2004–109

A NEW CHARACTERIZATION OF CONVEX GAMES

By R. Branzei, D. Dimitrov, S.H. Tijs

November 2004

ISSN 0924-7815

Tilburg University

A new characterization of convex games*

Rodica Branzei
Faculty of Computer Science
"Alexandru Ioan Cuza" University, Iasi, Romania

Dinko Dimitrov
CentER and Department of Econometrics and Operations Research
Tilburg University, The Netherlands

Stef Tijs
CentER and Department of Econometrics and Operations Research
Tilburg University, The Netherlands
and
Department of Mathematics
University of Genoa, Italy

November 2004

Abstract

A cooperative game turns out to be convex if and only if all its marginal games are superadditive.

Journal of Economic Literature Classification Numbers: C71.

Keywords: characterization, convex games, marginal games, superadditive games.

*D. Dimitrov gratefully acknowledges financial support from a Marie Curie Research Fellowship of the European Community programme "Improving the Human Research Potential and the Socio-Economic Knowledge Base" under contract number HPMF-CT-2002-02121.

1 Introduction

A cooperative game with transferable utility is a pair $\langle N, v \rangle$, where $N = \{1, \ldots, n\}$ is a *set of players* and $v \colon 2^N \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the *worth* of coalition S, i.e. the members of S can obtain a total payoff of $v(S)$ by agreeing to cooperate.

A game $\langle N, v \rangle$ is called

- *superadditive*, if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$;
- *convex*, if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$.

Clearly, each convex game is also superadditive. The class of convex games was introduced by Shapley (1971) and has attracted a lot of attention because the games in this class have very nice properties: for example, the core of a convex game is the unique stable set and its extreme points can be easily described; moreover, the Shapley value is in the barycenter of the core.

This importance explains also the existence of many characterizations of convex games. For example, the above definition of convexity (that uses the supermodularity property of the characteristic function) is equivalent to the fact that for convex games the gain made when individuals or groups join large coalitions is higher than when they join smaller coalitions. For these and other characterizations of convex games that deal with the relation between the core and the Weber set we refer the reader to Shapley (1971), Ichiishi (1981), Curiel and Tijs (1991), and Curiel (1997).

The purpose of this note is to provide a new characterization (to the best of our knowledge) of convex games that uses the notion of a marginal game.

2 Motivation

Given a game $\langle N, v\rangle$ and a coalition $T \subseteq N$, the *T-marginal game* $v_T : 2^{N\setminus T} \to \mathbb{R}$ is defined by $v_T(S) := v(S \cup T) - v(T)$ for each $S \subseteq N \setminus T$.

Marginal games turned out to be very useful not only for proving the fact that the core of a game is a subset of its Weber set (cf. Weber (1988)) but also when constructing an appropriate algorithm for generating the constrained egalitarian solution for convex games (cf. Dutta and Ray (1989)).

Two of the main questions one can ask with respect to the marginal games $\langle N \setminus T, v_T\rangle$ of an original game $\langle N, v\rangle$ are the following: (1) if the original game is convex, are all its marginal games also convex?, and (2) if the original game is superadditive, are all its marginal games also superadditive?

The answer of the first question is positive (and we show this when providing our characterization result), while the answer of the second question is negative as exemplified below.

Example 1 *Let* $N = \{1,2,3\}$ *and* $v(\{1\}) = 10$, $v(\{1,2\}) = 12$, $v(\{1,3\}) = 11$, $v(\{1,2,3\}) = 12\frac{1}{2}$, *and* $v(S) = 0$ *for all other* $S \subset N$. *Clearly, the game* $\langle N, v\rangle$ *is superadditive. Its* $\{1\}$*-marginal game is given by* $v_{\{1\}}(\{2\}) = v(\{1,2\}) - v(\{1\}) = 2$, $v_{\{1\}}(\{3\}) = 11 - 10 = 1$, *and* $v_{\{1\}}(\{2,3\}) = 2\frac{1}{2}$. *Since* $v_{\{1\}}(\{2,3\}) = 2\frac{1}{2} < 3 = v_{\{1\}}(\{2\}) + v_{\{1\}}(\{3\})$, *the marginal game* $\left\langle \{2,3\}, v_{\{1\}}\right\rangle$ *is not superadditive.*

However, as we will show next, the superadditivity of each marginal game $\langle N \setminus T, v_T\rangle$ of an original game $\langle N, v\rangle$ pushes the game $\langle N, v\rangle$ to satisfy a stronger property - that of convexity.

3 Result

Theorem 1 *A game $\langle N, v\rangle$ is convex if and only if for each $T \in 2^N$ the T-marginal game $\langle N \setminus T, v_T\rangle$ is superadditive.*

Proof. (i) Suppose $\langle N, v\rangle$ is convex and let $T \subseteq N$. Take $S_1, S_2 \subseteq N \setminus T$. Then

$$\begin{aligned} & v_T(S_1 \cup S_2) + v_T(S_1 \cap S_2) \\ = \; & v(T \cup S_1 \cup S_2) + v(T \cup (S_1 \cap S_2)) - 2v(T) \\ = \; & v((T \cup S_1) \cup (T \cup S_2)) + v((T \cup S_1) \cap (T \cup S_2)) - 2v(T) \\ \geq \; & v(T \cup S_1) + v(T \cup S_2) - 2v(T) \\ = \; & (v(T \cup S_1) - v(T)) + (v(T \cup S_2) - v(T)) \\ = \; & v_T(S_1) + v_T(S_2), \end{aligned}$$

where the inequality follows from the convexity of v. Hence, v_T is convex (and superadditive as well).

(ii) Suppose that for each $T \in 2^N$ the game $\langle N \setminus T, v_T\rangle$ is superadditive. Take $S_1, S_2 \subseteq N$. We have to prove that

$$v(S_1 \cup S_2) + v(S_1 \cap S_2) \geq v(S_1) + v(S_2).$$

If $S_1 \cap S_2 = \emptyset$, then the assertion easily follows from the superadditivity of the game $\langle N \setminus \emptyset, v_\emptyset\rangle = \langle N, v\rangle$ and $v(\emptyset) = 0$.

Suppose now $S_1 \cap S_2 \neq \emptyset$ and let $T := S_1 \cap S_2$. Because $\langle N \setminus T, v_T\rangle$ is superadditive, we have that

$$v_T(S_1 \setminus T) + v_T(S_2 \setminus T) \leq v_T((S_1 \cup S_2) \setminus T)$$

iff

$$v(S_1) - v(T) + v(S_2) - v(T) \leq v(S_1 \cup S_2) - v(T)$$

iff

$$v(S_1) + v(S_2) \leq v(S_1 \cup S_2) + v(T)$$

iff

$$v(S_1) + v(S_2) \leq v(S_1 \cup S_2) + v(S_1 \cap S_2).$$

■

Having provided our characterization, we can answer immediately the following question: "Under which conditions are all marginal games of a superadditive original game superadditive?"

Corollary 1 *Let $\langle N, v\rangle$ be a superadditive game. Then $\langle N \setminus T, v_T\rangle$ is superadditive for each $T \in 2^N$ if and only if $\langle N \setminus T, v_T\rangle$ is convex for each $T \in 2^N$.*

References

[1] Curiel, I. (1997): Cooperative Game Theory and Applications: Cooperative Games Arising from Combinatorial Optimization Problems, Kluwer Academic Publishers, Dordrecht.

[2] Curiel, I., and S. Tijs (1991): The minimarg and the maximarg operators, Journal of Optimization Theory and Applications 71, 277-287.

[3] Dutta, B. and D. Ray (1989): A concept of egalitarianism under participation constraints, Econometrica 57, 403-422.

[4] Ichiishi, T. (1981): Super-modularity: Applications to convex games and to the greedy algorithm for LP, Journal of Economic Theory 25, 283-286.

[5] Shapley, L. (1971): Cores of convex games, International Journal of Game Theory 1, 11-26.

[6] Weber, R. (1988): Probabilistic values for games, in: A.E. Roth (Ed.), The Shapley Value: Essays in Honour of LS Shapley, Cambridge University Press, Cambridge.