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On the core of multiple longest traveling salesman games

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Abstract:

In this paper we introduce multiple longest traveling salesman (MLTS) games. An MLTS game arises from a network in which a salesman has to visit each node (player) precisely once, except its home location, in an order that maximizes the total reward. First it is shown that the value of a coalition of an MLTS game is determined by taking the maximum of suitable combinations of one and two person coalitions. Secondly it is shown that MLTS games with five or less players have a nonempty core. However, a six player MLTS game may have an empty core. For the special instance where the reward between a pair of nodes is equal to 0 or 1, we provide relations between the structure of the core and the underlying network.

Keywords: Game theory, longest traveling salesman problem/game, multiple longest traveling salesman problem/game, Core.

1 Introduction

A traveling salesman (TS) problem can be described by a complete undirected graph in which the vertices represent the cities and the cost between two cities can be represented by a cost function on the edges. The objective is to find a Hamiltonian cycle that minimizes total cost. A Hamiltonian cycle is a tour that starts in a specific city, referred to as home city, visits all other cities exactly once and then returns to the home city. For a review on TS problems we refer to Lawler et al. (1997). An important variant of the TS problem is the longest traveling salesman (LTS) problem. Here the cost function on the edges is replaced by a reward function. The objective in an LTS problem is to find a Hamiltonian cycle

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that maximizes total rewards. For a review on LTS problems we refer to Lewenstein and Sviridenko (2003), Blokh and Gutin (1995) and Barvinok et al. (2002). Multiple traveling salesman (MTS) and multiple longest traveling salesman (MLTS) problems are relaxations of TS and LTS problems respectively. The objective is to find a weak Hamiltonian cycle that minimizes costs in case of MTS problems and maximizes rewards in case of MLTS problems. A weak Hamiltonian cycle is a tour that starts and ends in the home city and visits each city once except the home city that can be revisited several times.

Cooperative transferable utility MLTS (LTS) games arising from MLTS (LTS) problems are the object of study of this paper. Given an MLTS (LTS) problem we identify each vertex in the graph, except the one corresponding to the home location, with a player. Then the value of a coalition in an MLTS (LTS) game is defined as the maximum reward this coalition can obtain by solving the MLTS (LTS) problem on the complete subgraph on the vertices corresponding to this coalition and the home location. It is shown that the the value of a coalition of an MLTS game is determined as the maximum of suitable combinations of 1- and 2-person subcoalitions.

We show that MLTS games have a nonempty core if the number of players is at most five. MLTS games need not have core elements if the number of players is at least six. For the special case when the reward function on the edges only takes values 0 or 1 we provide relations between the structure of the core and the underlying 1-graph. Here the 1-graph is the subgraph of the complete graph that contains exactly those edges with reward equal to 1. We show that if the 1-graph is Hamiltonian then the core is the convex hull of specific $(0, 1)$-vectors. If the 1-graph is a line graph we show that the game is convex and therefore the core is nonempty. If the 1-graph is a traceable 1-sum of Hamiltonian and line graphs, we can characterize a nonempty subset of the core. If the 1-graph is a tree the core is nonempty (cf. LeBreton et al. (1992)).

Finally we discuss the relation between TS games, introduced by Potters, Curiel and Tijs (1992), and MTS games, LTS games and MLTS games.

The paper is organized as follows. In section 2 we formally introduce MLTS games. Section 3 provides the various results on the core and section 4 considers the relation between MLTS games and MTS games.

2 Multiple longest traveling salesman games

In this section we introduce longest traveling salesman games and multiple longest traveling salesman games.

Let $N_0 = \{0, 1, 2, \ldots, n\}$ denotes the set of cities that a salesman has to visit, where city 0 is the home city of the salesman. Let $T = (t_{ij})$ be an $N_0 \times N_0$-matrix where t_{ij} denotes the rewards of going from city i to city j, with $i, j \in N_0$. We assume $t_{ij} \geq 0$, $t_{ij} = t_{ji}$ and $t_{ii} = 0$ for every $i, j \in N_0$. The network (N_0, T) is usually represented by the complete graph on N_0 with rewards t_{ij} on the edges.

In a longest traveling salesman (LTS) problem a salesman, starting in city 0, has to visit each of the other cities exactly once and has to return to city 0 at the end of the journey. The order of the cities is selected in such a way that the total rewards are maximized.

In a multiple longest traveling salesman (MLTS) problem again the salesman has to visit each city exactly once but now the home city can be revisited several times instead of only at the start and the end of the journey.

LTS (MLTS) games arise from LTS (MLTS) problems if one associates players to the set of cities $N = \{1, \ldots, n\}$ and each coalition of players faces an LTS (MLTS) problem.

Before formally introducing LTS and MLTS games we will fix some notation. For any $S \subset N$ we denote by $\Pi(S)$ the set of bijective functions $\pi : \{1, \ldots, |S|\} \to S$, where $\pi(k)$ denotes the player that is visited in k^{th} position in the tour on cities in S induced by π. Associated to each $\pi \in \Pi(S)$ we define the function $\overline{\pi} : \{0, 1, \ldots, |S|, |S|+1\} \to S \cup \{0\}$ such that $\overline{\pi}(k) = \pi(k)$ if $k \in \{1, \ldots, |S|\}$ and $\overline{\pi}(0) = \overline{\pi}(|S|+1) = 0$.

The *LTS game*, (N, r), corresponding to (N_0, T), is defined by

$$r(S) = \max_{\pi \in \Pi(S)} \sum_{k=0}^{|S|} t_{\overline{\pi}(k)\overline{\pi}(k+1)}$$

for every $S \subset N$.

The *MLTS game*, (N, v), corresponding to (N_0, T), is defined by

$$v(S) = \max_{\langle S_1, \ldots, S_l \rangle \in \mathcal{P}(S)} \sum_{m=1}^{l} r(S_m)$$

for every $S \subset N$, where $\mathcal{P}(S)$ denotes the set of partitions of S.

It is readily seen that MLTS games are superadditive, i.e. $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$, and monotonic, i.e. $v(T) \geq v(S)$ for all $S, T \in 2^N$ with $S \subset T$.

Recall that the *core* of a cooperative game (N, v), is given by

$$Core(v) = \{x \in \mathbb{R}^N : x(N) = v(N), x(S) \geq v(S) \text{ for all } S \in 2^N\},$$

where $x(T) = \sum_{i \in T} x_i$ for any $T \subset N$. An element of the core gives an allocation of v(N) in such a way that there is no coalition with an incentive to split off.

The following example illustrates that LTS games and MLTS games can be different.

Example 2.1. Let the network (N_0, T) be represented by the graph of Figure 1.

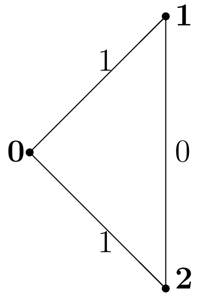

Figure 1: A complete graph representing the network (N_0, T).

The associated LTS game is given by $r(\{1\}) = 2$, $r(\{2\}) = 2$, $r(\{1, 2\}) = t_{01} + t_{12} + t_{20} = 2$. Note that $r(\{1\}) + r(\{2\}) = 4 > 2 = r(\{1, 2\})$. So $Core(r) = \emptyset$.
The associated MLTS game is given by $v(\{1\}) = 2$, $v(\{2\}) = 2$, $v(\{1, 2\}) = \max\{\text{r}(\{1\})+\text{r}(\{2\}), r(\{1, 2\})\} = r(\{1\}) + r(\{2\}) = 4$. In this case $Core(v) = \{(2, 2)\}$. □

It is readily seen that $v = r$ if and only if $t_{0i} + t_{0j} \leq t_{ij}$ for every $i, j \in N$.

The following theorem states that every MLTS game with at most five players has a nonempty core. The proof of this theorem can be found in section 4.

Theorem 2.2. Let (N, v) be an MLTS game with $|N| \leq 5$, then the core of (N, v) is nonempty.

The following example provides a 6 player MLTS game with an empty core.

Example 2.3. Let the network (N_0, T) be represented by the graph in Figure 2. All edges $\{i, j\}$ with $t_{ij} = 0$ are omitted for convenience.

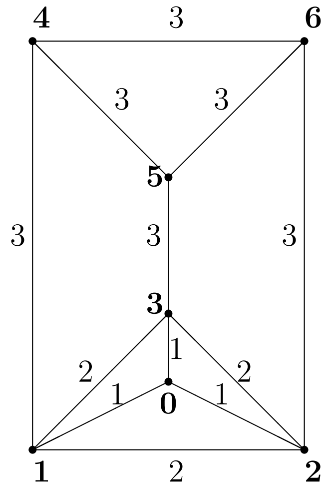

Figure 2: A graph representing the network (N_0, T).

It is easy to check that the optimal tour of coalition $\{1,2,4,6\}$ is $(0,1,4,6,2,0)$ and consequently $v(\{1,2,4,6\}) = 11$. Because of symmetry, $v(\{1,3,4,5\}) = v(\{2,3,5,6\}) = 11$. An optimal tour of N is $(0,1,4,5,6,2,3,0)$. Hence $v(N) = 16$.

Assume that x is a core element of this game. Then $x(\{1,2,4,6\}) \geq 11$, $x(\{1,3,4,5\}) \geq 11$ and $x(\{2,3,4,5\}) \geq 11$. So if we sum these inequalities we obtain $2x(N) \geq 33$ which contradicts $x(N) = 16$. Hence, the core is empty. □

The following Theorem shows that the value of every coalition only depends on the values of one and two person subcoalitions.

Theorem 2.4. Let (N, v) be an MLTS game corresponding to a network (N_0, T). Then

$$v(S) = \max_{\pi \in \Pi(S)} \Big[\sum_{k=1}^{|S|-1} v(\{\pi(k), \pi(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\pi(k)\}) \Big]$$

for all $S \subset N$.

Proof: Let $S \subset N$. First we show that

$$v(S) \geq \max_{\pi \in \Pi(S)} \Big[\sum_{k=1}^{|S|-1} v(\{\pi(k), \pi(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\pi(k)\}) \Big].$$

Let $\rho \in \Pi(S)$ be the order in which the maximum on the right hand side is attained. Then ρ induces a tour on S in the following way: from 0 go to $\rho(1)$ and from $\rho(1)$ go directly to $\rho(2)$ if $t_{\rho(1)\rho(2)} > t_{0\rho(1)} + t_{0\rho(2)}$, otherwise go via 0 to $\rho(2)$. The decision to go from $\rho(k)$ directly to $\rho(k+1)$ or indirectly via 0 is made analogously depending on $t_{\rho(k)\rho(k+1)} > t_{0\rho(k)} + t_{0\rho(k+1)}$ or not, with $k = 2, \ldots, |S|-1$. Finally from $\rho(|S|)$ go back to 0. Clearly, this tour generates a set B of positions in $\{1, \ldots, |S|-1\}$ for which the tour goes back to zero; $F = \{1, \ldots, |S|-1\} \setminus B$ is the set of positions where the tour moves forward. The total rewards of the induced tour are

$$R = \sum_{k \in B} (t_{0\rho(k)} + t_{0\rho(k+1)}) + t_{0\rho(1)} + t_{0\rho(|S|)} + \sum_{k \in F} t_{\rho(k)\rho(k+1)}.$$

It is sufficient to show that $R = \sum_{k=1}^{|S|-1} v(\{\rho(k), \rho(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\rho(k)\})$.

$$\begin{aligned}
&\sum_{k=1}^{|S|-1} v(\{\rho(k), \rho(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\rho(k)\}) = \\
&= \sum_{k \in B} v(\{\rho(k), \rho(k+1)\}) + \sum_{k \in F} v(\{\rho(k), \rho(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\rho(k)\}) \\
&= \sum_{k \in B} (2t_{0\rho(k)} + 2t_{0\rho(k+1)}) + \sum_{k \in F} (t_{0\rho(k)} + t_{\rho(k)\rho(k+1)} + t_{0\rho(k+1)}) - \sum_{k=2}^{|S|-1} 2t_{0\rho(k)} \\
&= R + \sum_{k \in B} (t_{0\rho(k)} + t_{0\rho(k+1)}) + \sum_{k \in F} (t_{0\rho(k)} + t_{0\rho(k+1)}) - 2\sum_{k=2}^{|S|-1} t_{0\rho(k)} - t_{0\rho(1)} - t_{0\rho(|S|)} \\
&= R + \sum_{k=1}^{|S|-1} (t_{0\rho(k)} + t_{0\rho(k+1)}) - \sum_{k=1}^{|S|-1} t_{0\rho(k)} - \sum_{k=2}^{|S|} t_{0\rho(k)} \\
&= R.
\end{aligned}$$

To show the inverse we first suppose that there is a tour ρ such that

$$v(S) = r(S) = \sum_{k=0}^{|S|} t_{\overline{\rho}(k)\overline{\rho}(k+1)}. \tag{1}$$

Since $\rho \in \Pi(S)$ is optimal for S, it holds that $t_{\rho(k)\rho(k+1)} \geq t_{0\rho(k)} + t_{0\rho(k+1)}$ for $k \in \{1, \ldots, |S|-1\}$. Hence $v(\{\rho(k), \rho(k+1)\}) = r(\{\rho(k), \rho(k+1)\}) = t_{\rho(k)\rho(k+1)} + t_{0\rho(k)} + t_{0\rho(k+1)}$.

Then,

$$
\begin{aligned}
v(S) &= \sum_{k=0}^{|S|} t_{\overline{\rho}(k)\overline{\rho}(k+1)} = t_{0\rho(1)} + \sum_{k=1}^{|S|-1} t_{\rho(k)\rho(k+1)} + t_{0\rho(|S|)} \\
&= t_{0\rho(1)} + \sum_{k=1}^{|S|-1} \left(t_{\rho(k)\rho(k+1)} + t_{0\rho(k)} + t_{0\rho(k+1)}\right) + t_{0\rho(|S|)} - \sum_{k=1}^{|S|-1} t_{0\rho(k)} - \sum_{k=2}^{|S|} t_{0\rho(k)} \\
&= \sum_{k=1}^{|S|-1} v(\{\rho(k), \rho(k+1)\}) - 2\sum_{k=2}^{|S|-1} t_{0\rho(k)} \\
&= \sum_{k=1}^{|S|-1} v(\{\rho(k), \rho(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\rho(k)\}) \\
&\leq \max_{\pi\in\Pi(S)} \Big[\sum_{k=1}^{|S|-1} v(\{\pi(k), \pi(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\pi(k)\})\Big].
\end{aligned}
$$

So we know that $v(S) \geq \max_{\pi\in\Pi(S)} \Big[\sum_{k=1}^{|S|-1} v(\{\pi(k), \pi(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\pi(k)\})\Big]$ for every $S \subset N$ and that the equality holds when $v(S) = r(S)$.

Finally, if there is no ρ such that (1) is satisfied, then there exists a partition $\langle S_1, \ldots, S_t\rangle$ of S such that $v(S) = \sum_{j=1}^{t} r(S_j)$, and we can use the same argument for all partition elements $S_1, \ldots, S_t$ separately. Aggregating the t formulas the result follows. □

Observe that Theorem 2.4 is also true for LTS games.

In the following we will denote by (N_0, T^{01}) a network in which the rewards of edges of T equal either 0 or 1. The corresponding MLTS (LTS) games will be referred to as 0-1 MLTS (LTS) games. It turns out that an optimal tour for a coalition S will visit the cities $i \in S$ with $t_{0i} = 1$ separately.

Theorem 2.5. Let (N, v) be an MLTS game corresponding to a network (N_0, T^{01}). Then,

$$
v(S) = \sum_{i\in S\setminus S^*} v(\{i\}) + v(S^*),
$$

for every $S \subset N$, where $S^* = \{i \in S : t_{0i} = 0\}$.

Proof: Given $S \subset N$ it holds that $v(S) \geq \sum_{i\in S\setminus S^*} v(\{i\}) + v(S^*)$ because $\langle S^*, (\{i\})_{i\in S\setminus S^*}\rangle$ is a partition of S and (N, v) is superadditive.

Next, we have to show that $v(S) \leq \sum_{i\in S\setminus S^*} v(\{i\})+v(S^*)$. Using Theorem 2.4 it is sufficient to prove that

$$\max_{\pi\in\Pi(S)}\Big[\sum_{k=1}^{|S|-1} v(\{\pi(k),\pi(k+1)\}) - \sum_{k=2}^{|S|-1} v(\{\pi(k)\})\Big] \leq \sum_{i\in S\setminus S^*} v(\{i\}) + v(S^*).$$

Let $\rho \in \Pi(S)$ be the order in which the maximum is reached, then ρ induces a tour, that is optimal in the corresponding MLTS problem, and sets B and F just as in the proof of Theorem 2.4. Moreover, as $t_{0i}+t_{0j} \geq t_{ij}$ for every $i \in S\setminus S^*$, $j \in S$ it holds that $\rho(F) \subset S^*$. So, denoting by R the total rewards of the induced tour,

$$\begin{aligned}
R &= \sum_{k\in B}(t_{0\rho(k)} + t_{0\rho(k+1)}) + t_{0\rho(1)} + t_{0\rho(|S|)} + \sum_{k\in F} t_{\rho(k)\rho(k+1)} \\
&= \sum_{k\in B} t_{0\rho(k)} + t_{0\rho(|S|)} + \sum_{k\in B} t_{0\rho(k+1)} + t_{0\rho(1)} + \sum_{k\in F} t_{\rho(k)\rho(k+1)} \\
&= \sum_{\substack{k\in B\\ \rho(k)\in S^*}} t_{0\rho(k)} + \sum_{\substack{k\in B\\ \rho(k)\in S\setminus S^*}} t_{0\rho(k)} + t_{0\rho(|S|)} + \\
&\quad \sum_{\substack{k\in B\\ \rho(k+1)\in S^*}} t_{0\rho(k+1)} + \sum_{\substack{k\in B\\ \rho(k+1)\in S\setminus S^*}} t_{0\rho(k+1)} + t_{0\rho(1)} + \sum_{k\in F} t_{\rho(k)\rho(k+1)} \\
&= \sum_{\substack{k\in B\\ \rho(k)\in S\setminus S^*}} t_{0\rho(k)} + t_{0\rho(|S|)} + \sum_{\substack{k\in B\\ \rho(k+1)\in S\setminus S^*}} t_{0\rho(k+1)} + t_{0\rho(1)} + \sum_{k\in F} t_{\rho(k)\rho(k+1)} \\
&\leq \sum_{i\in S\setminus S^*} t_{0i} + \sum_{i\in S\setminus S^*} t_{0i} + \sum_{k\in F} t_{\rho(k)\rho(k+1)} \\
&= 2\sum_{i\in S\setminus S^*} t_{0i} + \sum_{k\in F} t_{\rho(k)\rho(k+1)} = \sum_{i\in S\setminus S^*} v(\{i\}) + \sum_{k\in F} t_{\rho(k)\rho(k+1)} \\
&\leq \sum_{i\in S\setminus S^*} v(\{i\}) + r(F) \leq \sum_{i\in S\setminus S^*} v(\{i\}) + r(S) \leq \sum_{i\in S\setminus S^*} v(\{i\}) + v(S).
\end{aligned}$$

where the fourth equality holds since $i \in S^*$ implies that $t_{0i} = 0$. □

3 0-1 MLTS games

In this section we investigate the structure of the core of 0-1 MLTS games.

For a network (N_0, T^{01}) we denote by N^* the set of players that are connected to city 0 with reward 0. So, $N\setminus N^*$ is the set of players that are connected to city 0 with reward 1. From

Theorem 2.5 it follows that every element of the core assigns to each player in $N \backslash N^*$ exactly 2. For this reason we will restrict attention to *standard networks* with $N = N^*$ in the remaining part of this section. Observe that in this case 0-1 MLTS and 0-1 LTS games coincide.

A network (N_0, T^{01}) induces a graph G_1, where G_1 is the undirected graph with set of vertices N and set of edges $E = \{\{i, j\} \subset N : t_{ij} = 1\}$. The graph G_1 is called *traceable* if there exists a complete Hamiltonian path. A *Hamiltonian path* between vertices i and j is a succession of connected vertices starting in i and ending in j such that all the vertices in the succession appear exactly once. We say that a *Hamiltonian path* between vertices i and j is *complete* if it visits all the vertices in the graph. A *Hamiltonian path* is *closed* if the starting vertex and the end vertex coincide. A *Hamiltonian cycle* is a complete closed Hamiltonian path. A graph is *Hamiltonian* if it contains a Hamiltonian cycle.

The following proposition states that traceability of G_1 in a standard 0-1 MLTS problem is equivalent to $v(N) = |N| - 1$ for the corresponding MLTS game. The proof is straightforward and therefore omitted.

Proposition 3.1. Let (N, v) be an MLTS game corresponding to a standard network (N_0, T^{01}). Then $v(N) = |N| - 1$ if and only if the graph G_1 is traceable.

The following result implies that a 0-1 MLTS game has a nonempty core whenever the associated graph G_1 is traceable. Here $e^S \in \mathbb{R}^N$ denotes the vector where $e^S_j = 1$ if $j \in S$ and $e^S_j = 0$ otherwise.

Theorem 3.2. Let (N, v) be an MLTS game corresponding to a standard network (N_0, T^{01}). If G_1 is traceable, then $conv\{e^N - e^{\{i\}} : i \in N\} \subset Core(v)$.

PROOF: Let $i \in N$ and let $x^i = e^N - e^{\{i\}}$. It is sufficient to show that $x^i \in Core(v)$. By definition $x^i(N) = |N| - 1 = v(N)$ and $x^i(S) \geq |S| - 1 \geq v(S)$ for every $S \subset N$. □

The following theorem shows that the inclusion of Theorem 3.2 is an equality if G_1 is Hamiltonian.

Theorem 3.3. Let (N, v) be an MLTS game corresponding to a standard network (N_0, T^{01}). If G_1 is Hamiltonian, then $Core(v) = conv\{e^N - e^{\{i\}} : i \in N\}$.

Proof: By Theorem 3.2 it is sufficient to show that $Core(v) \subset conv\{e^N - e^{\{i\}} : i \in N\}$. Since G_1 is Hamiltonian, it holds for $i \in N$ that $v(N \setminus \{i\}) = |N| - 2$. Since $v(N) = |N| - 1$ it follows that $0 = v(\{i\}) \leq x_i \leq v(N) - v(N \setminus \{i\}) = 1$ for every $x \in Core(v)$. Hence $C(v) \subset \{x \in \mathbb{R}^N : x_i \in [0,1], \sum_{i \in N} x_i = |N| - 1\}$. Because $conv\{e^N - e^{\{i\}} : i \in N\} = \{x \in \mathbb{R}^N : x_i \in [0,1], \sum_{i \in N} x_i = |N| - 1\}$ we can conclude that $Core(v) \subset conv\{e^N - e^{\{i\}} : i \in N\}$.□

Next we show that an 0-1 MLTS game is convex whenever G_1 is a line. Recall that a cooperative game (N, v) is *convex* if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for every $S \subset T \subset N \setminus \{i\}$. For the proof of this result we need the concept of *connected components* of a graph G, i.e. maximally connected subgraphs of G.

Theorem 3.4. Let (N, v) be an MLTS game corresponding to a standard network (N_0, T^{01}). If G_1 is a line, then (N, v) is convex.

Proof: Let $S \subset T \subset N \setminus \{i\}$. We prove that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. First observe that for any $i \in N$ and $U \subset N \setminus \{i\}$ it holds $v(U \cup \{i\}) - v(U) \in \{0, 1, 2\}$. Hence if $v(T \cup \{i\}) - v(T) = 2$, the inequality is satisfied by the observation. If $v(T \cup \{i\}) - v(T) = 1$, then i is connected to exactly one component of T in G_1. Because G_1 is a line it holds that i is connected, at most, to one component of S in G_1. So $v(S \cup \{i\}) - v(S) \leq 1$. If $v(T \cup \{i\}) - v(T) = 0$, then i is not connected to any component of T in G_1. Hence i is not connected to any component of S in G_1. So $v(S \cup \{i\}) - v(S) = 0$. □

Before we can state our next result we need the notion of 1-sum of two graphs that arises from standard situations. Let $G = (V, E)$ and $H = (V', E')$ be two graphs such that $|V \cap V'| = 1$. Then we define the *1-sum* of G and H by $G \oplus H = (V \cup V', E \cup E')$. The 1-sum of more than two graphs is defined in a recursive way.

Example 3.5. The graph in Figure 3 is 1-sum of the line L_1 with set of players $N_{L_1} = \{1, 2, 3\}$ and the Hamiltonian graphs H_1, H_2 with set of players $N_{H_1} = \{3, 4, 5, 6, 7\}$ and $N_{H_2} = \{7, 8, 9, 10\}$ respectively.

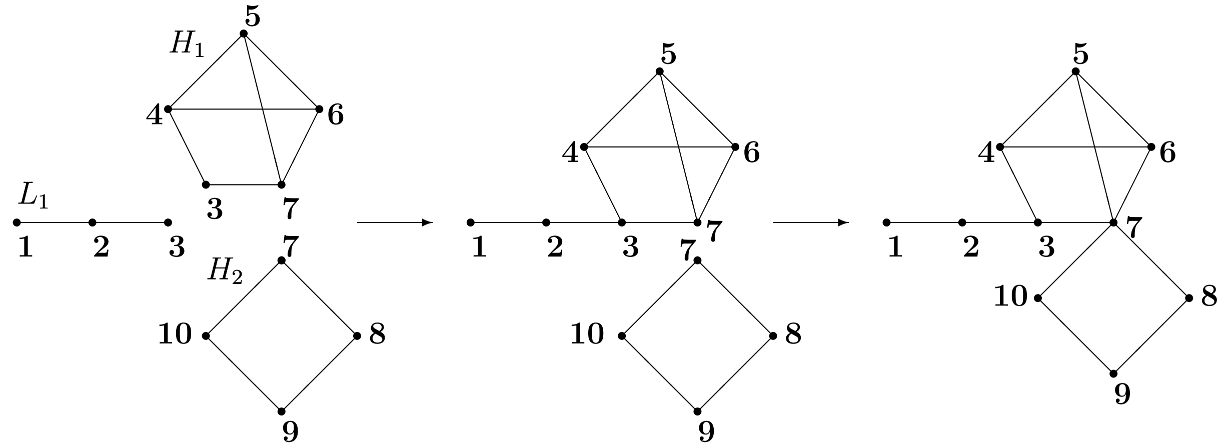

Figure 3: 1-sum of Hamiltonian graphs and lines that is traceable.

Observe that in the associated MLTS game, (N, v), an optimal tour for N is given by $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0)$. So, $v(N) = 9$ which implies that the 1-sum of L_1, H_1, H_2 is traceable. $\square$

We will only consider traceable graphs that are 1-sum of line graphs $L_1, \ldots, L_q$ and Hamiltonian graphs $H_1, \ldots, H_p$. In Figure 4 we illustrate some 1-sums that are not traceable.

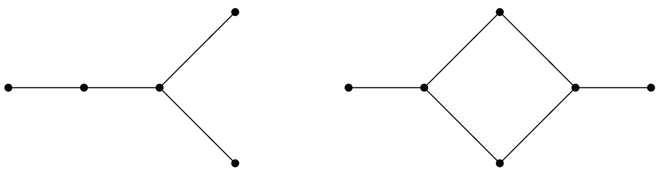

Figure 4: 1-sums that are not traceable.

It is readily verified that to achieve traceability each line is coalesced to, at most, two Hamiltonian graphs using its end vertices. Moreover, each Hamiltonian graph is coalesced to, at most, two different graphs (either two lines, or one line and one Hamiltonian graph, or two Hamiltonian graphs) using two consecutive vertices in a Hamiltonian cycle.

Now consider a standard network (N_0, T^{01}) such that G_1 is traceable and such that G_1 is 1-sum of Hamiltonian graphs $\{H_r\}_{r=1}^{p}$ and lines $\{L_s\}_{s=1}^{q}$. We denote by N_{L_s} (N_{H_r}) the set of players of L_s (H_r) and by v_{L_s} (v_{H_r}) the characteristic function of the MLTS game corresponding to the standard network $(N_{L_s}, T^{01}_{|L_s})$ ($(N_{H_r}, T^{01}_{|H_r})$). Let $\{x^{H_r}\}_{r=1}^{p}$ and $\{x^{L_s}\}_{s=1}^{q}$ be such that $x^{H_r} \in Core(v_{H_r})$ for every $r \in \{1, \ldots, p\}$, $x^{L_s} \in Core(v_{L_s})$ for every

$s \in \{1, \ldots, q\}$. We define $x \in \mathbb{R}^N$ by

$$x_i = \begin{cases} x_i^{L_s} + x_i^{H_r} & \text{if } i \in N_{L_s} \cap N_{H_r}, \\ x_i^{H_r} + x_i^{H_t} & \text{if } i \in N_{H_R} \cap N_{H_t}, \\ x_i^{L_s} & \text{if } i \in N_{L_s} \text{ and } i \notin N_{H_r} \text{ for every } r \in \{1, \ldots, p\}, \\ x_i^{H_r} & \text{if } i \in N_{H_r},\ i \notin N_{L_s} \text{ for every } s \in \{1, \ldots, q\} \text{ and } i \notin N_{H_t},\ t \neq r \end{cases} \tag{2}$$

Since G_1 is traceable, (2) covers all cases.

Theorem 3.6. Let (N, v) be an MLTS game corresponding to a standard network (N_0, T^{01}). Moreover, let G_1 be traceable and the 1-sum of Hamiltonian graphs $\{H_r\}_{r=1}^p$ and lines $\{L_s\}_{s=1}^q$. Then each vector $x \in \mathbb{R}^N$ as defined in (2) is an element of $Core(v)$.

Proof: Observe that there are exactly $p + q - 1$ players that are in exactly two of the underlying Hamiltonian and line graphs. Then

$$\begin{aligned} \sum_{i \in N} x_i &= \sum_{r=1}^{p} \sum_{i \in H_r} x_i^{H_r} + \sum_{s=1}^{q} \sum_{i \in L_S} x_i^{L_s} = \sum_{r=1}^{p} (|N_{H_r}| - 1) + \sum_{s=1}^{q} (|N_{L_s}| - 1) \\ &= \sum_{r=1}^{p} |N_{H_r}| + \sum_{s=1}^{q} |N_{L_s}| - p - q = (|N| + p + q - 1) - (p + q) = |N| - 1. \end{aligned}$$

Next we show stability. Let ρ be an optimal tour of $S \subset N$ in G_1 and let $E(\rho)$ denote the set of edges in G_1 covered by ρ. Hence $v(S) = |E(\rho)|$. Obviously, $E(\rho) \cap E(L_1), \ldots, E(\rho) \cap E(L_q), E(\rho) \cap E(H_1), \ldots, E(\rho) \cap E(H_p)$ is a partition of $E(\rho)$, where $E(L_s)$ and $E(H_r)$ denote the set of edges of L_s and H_r, respectively. Let $A \in \{L_1, \ldots, L_q, H_1, \ldots, H_p\}$, we can define an induced tour in $S \cap A$ in the following way. Let $k_1, \ldots, k_{|S \cap N_A|}$ be such that $k_1 < k_2 < \ldots < k_{|S \cap N_A|}$ and $\rho(k_l) \in S \cap N_A$ for every l. Then the induced tour is given by: from 0 go to $\rho(k_1)$ and from $\rho(k_1)$ go directly to $\rho(k_2)$ if $\{\rho(k_1), \rho(k_2)\} \in E(\rho) \cap E(A)$, otherwise go via 0 to $\rho(k_2)$. The decision to go from $\rho(k_l)$ directly to $\rho(k_{l+1})$ or indirectly via 0 is made analogously depending on $\{\rho(k_l), \rho(k_{l+1})\} \in E(\rho) \cap E(A)$ or not, with $l = 2, \ldots, |S \cap N_A| - 1$. Finally from $\rho(k_{|S \cap N_A|})$ go back to 0. Then, the total reward corresponding to this tour is $|E(\rho) \cap E(A)|$. Hence $v(S \cap N_A) \geq |E(\rho) \cap E(A)|$. Therefore

$$\begin{aligned} v(S) = |E(\rho)| &= |E(\rho) \cap E(L_1)| + \ldots + |E(\rho) \cap E(L_q)| + \\ &+ |E(\rho) \cap E(H_1)| + \ldots + |E(\rho) \cap E(H_p)| \\ &\leq v(S \cap N_{L_1}) + \ldots + v(S \cap N_{L_q}) + v(S \cap N_{H_1}) + \ldots + v(S \cap N_{H_p}) \\ &\leq x^{L_1}(S \cap N_{L_1}) + \ldots + x^{L_q}(S \cap N_{L_q}) + x^{H_1}(S \cap N_{H_1}) + \ldots + x^{H_p}(S \cap N_{H_p}) \\ &= x(S). \end{aligned}$$

□

The following example shows that if G_1 is not traceable the core of the corresponding MLTS game can be empty.

Example 3.7. Let (N_0, T^{01}) be the standard network of which G_1 is represented in Figure 5.

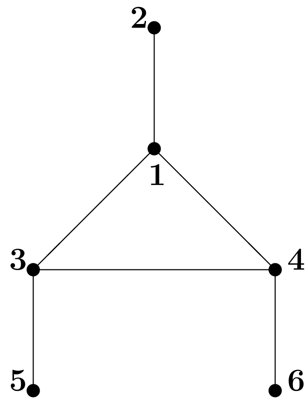

Figure 5: A non-traceable graph representing the standard network (N_0, T^{01}).

In this case is easy to check that an optimal tour for N is (0,5,3,1,4,6,0,2,0) so $v(N) = 4$. An optimal tour for $\{1, 2, 3, 5\}$ is $(0, 2, 1, 3, 5, 0)$ and $v(\{1, 2, 3, 5\}) = 3$. By symmetry $v(\{1, 2, 4, 6\}) = 3$ and $v(\{3, 4, 5, 6\}) = 3$. Hence if $x \in Core(v)$ then, $x(\{1, 2, 3, 5\}) \geq 3$, $x(\{1, 2, 4, 6\}) \geq 3$ and $x(\{3, 4, 5, 6\}) \geq 3$. Therefore, if we sum all the inequalities, we find $2x(N) \geq 9$ which contradicts $x(N) = 4$. Hence, the core is empty. □

Finally we have

Theorem 3.8. Let (N, v) be an MLTS game corresponding to a standard network (N_0, T^{01}). If G_1 is a tree, then (N, v) has a nonempty core.

The proof is straightforward because if G_1 is a tree, then the corresponding 0-1 MLTS game is Γ-component additive (see LeBreton et al. (1992)).

4 Relation between MLTS games and MTS games

In this section we prove Theorem 2.2. For this purpose we define multiple traveling salesman (MTS) games and we show that they are related to MLTS games.

Let (N_0, T) be a network as before, but with a different interpretation: t_{ij} now represents the *costs* of going from city i to city j, with $i, j \in N_0$.

In a traveling salesman (TS) problem corresponding to the cost network (N_0, T) a salesman, starting in city 0, has to visit each of the other cities exactly once and has to return to city 0 at the end of the journey. The order of the cities is selected in such a way that the total costs are minimized.

In a multiple traveling salesman (MTS) problem the salesman has to visit each city exactly once but now the home city can be revisited several times instead of only at the start and the end of the journey.

In Potters, Curiel and Tijs (1992) TS games arising from TS problems are introduced. Analogously one can define MTS games arising from MTS problems where the players are identified with the set of cities $N = \{1, \ldots, n\}$ and each coalition of cities faces an MTS problem.

The *TS game* (N, κ), corresponding to the cost network (N_0, T) is given by

$$\kappa(S) = \min_{\pi \in \Pi(S)} \sum_{k=0}^{|S|} t_{\overline{\pi}(k)\overline{\pi}(k+1)}$$

for every $S \subset N$ and the *MTS game*, (N, c), corresponding to (N_0, T) is defined by

$$c(S) = \min_{\langle S_1, \ldots, S_l \rangle \in \mathcal{P}(S)} \sum_{m=1}^{l} \kappa(S_m)$$

for every $S \subset N$.

It is readily seen that given a cost network (N_0, T) the corresponding TS and MTS coincide, if and only if $t_{0i} + t_{0j} \geq t_{ij}$ for every $i, j \in N$.

Analogously to Theorem 2.4 it can be proved that for an MTS game (N, c) it holds that

$$c(S) = \min_{\pi \in \Pi(S)} \Big[\sum_{k=1}^{|S|-1} c(\{\pi(k), \pi(k+1)\}) - \sum_{k=2}^{|S|-1} c(\{\pi(k)\}) \Big] \tag{3}$$

for all $S \subset N$.

Observe that this result also holds for TS games.

Using the results in Potters, Curiel and Tijs (1992), Tamir (1989) and Kuipers (1991) we can state the following result on the nonempty core of the traveling salesman game.

Theorem 4.1. Let (N, κ) be a TS game with $|N| \leq 5$ corresponding to a cost network (N_0, T) such that $t_{0i} + t_{0j} \geq t_{ij}$ for every $i, j \in N$. Then, $Core(\kappa) \neq \emptyset$.

Now we show that every MTS game with at most five players has a nonempty core.

Theorem 4.2. Let (N, c) be an MTS game with $|N| \leq 5$ corresponding to a cost network (N_0, T). Then, $Core(c) \neq \emptyset$.

PROOF: We are going to construct a cost network (N_0, U) such that the associated TS game, (N, κ), and (N, c) coincide.

Let U be defined by $u_{0i} = t_{0i}$, $u_{ij} = t_{0i} + t_{0j}$ if $t_{ij} > t_{0i} + t_{0j}$ and $u_{ij} = t_{ij}$ otherwise, for every $i, j \in N$.

The matrix of costs U satisfies the triangular inequalities with respect to the home location. Hence, using Theorem 4.1, (N, κ) has nonempty core and it suffices to show that $\kappa(S) = c(S)$ for all $S \subset N$. Using (3) it is sufficient to prove that $\kappa(S) = c(S)$ for all $S \subset N$ with $|S| \leq 2$.

Case 1: $S = \{i\}$, with $i \in N$.

$$\kappa(\{i\}) = 2u_{0i} = 2t_{0i} = c(\{i\}).$$

Case 2: $S = \{i, j\}$, with $i, j \in N$, $i \neq j$, such that $t_{ij} > t_{0i} + t_{0j}$,

$$\kappa(\{i, j\}) = u_{ij} + u_{0i} + u_{0j} = (t_{0i} + t_{0j}) + t_{0i} + t_{0j} = 2t_{0i} + 2t_{0j} = c(\{i, j\}).$$

Case 3: $S = \{i, j\}$, with $i, j \in N$, $i \neq j$, such that $t_{ij} \leq t_{0i} + t_{0j}$,

$$\kappa(\{i, j\}) = u_{ij} + u_{0i} + u_{0j} = t_{ij} + t_{0i} + t_{0j} = c(\{i, j\}).$$

□

Next we will show how MLTS games and MTS games are related.

Theorem 4.3. Let (N, v) be an MLTS game. Then there exists an MTS game (N, c) and a constant $k \in \mathbb{R}$ such that $v(S) = 2k|S| - c(S)$ for every $S \subset N$.

Analogously, let (N, c) be an MTS game. Then there exists an MLTS game (N, v) and a constant $l \in \mathbb{R}$ such that $c(S) = 2l|S| - v(S)$ for every $S \subset N$.

PROOF: We will only prove the first part of the theorem, the second part is analogous. Let (N, v) be an MLTS game corresponding to a (reward) network (N_0, T). We define a cost network (N_0, U) by $u_{0i} = k - t_{0i}$, $u_{ij} = 2k - t_{ij}$ for every $i, j \in N$, with $k \in \mathbb{R}$ such that $U \geq 0$. We will show that the MTS game, (N, c), associated to (N_0, U) satisfies $v(S) = 2k|S| - c(S)$ for all $S \subset N$. Using Theorem 2.4 and equation (3) it is sufficient to prove this for $S \subset N$ with $|S| \leq 2$.

Case 1: $S = \{i\}$, with $i \in N$.

$$v(\{i\}) = 2t_{0i} = 2k - (2k - 2t_{0i}) = 2k - 2u_{0i} = 2k - c(\{i\}).$$

Case 2: $S = \{i, j\}$, with $i, j \in N$, $i \neq j$.

$$\begin{aligned} v(\{i,j\}) &= \max\{t_{ij} + t_{0i} + t_{0j},\ 2t_{0i} + 2t_{0j}\} \\ &= 4k - \big(4k - \max\{t_{ij} + t_{0i} + t_{0j},\ 2t_{0i} + 2t_{0j}\}\big) \\ &= 4k - \min\{2k - t_{ij} + k - t_{0i} + k - t_{0j},\ 2k - 2t_{0i} + 2k - 2t_{0j}\} \\ &= 4k - \min\{u_{ij} + u_{0i} + u_{0j},\ 2u_{0i} + 2u_{0j}\} = 4k - c(\{i,j\}). \end{aligned}$$

□

PROOF THEOREM 2.2: It is now an immediate consequence of Theorem 4.2 and Theorem 4.3.

The following example provides a cost network (N_0, T) such that the entries of T take values 0 or 1 and the associated MTS game has an empty core.

Example 4.4. Let the cost network (N_0, T) be represented by the graph in Figure 6. The edges drawn have costs 0, all other edges have costs 1.

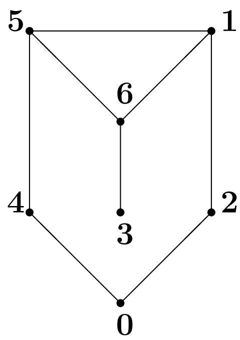

Figure 6: A graph representing the cost network (N_0, T).

It is easy to check that the optimal tour of coalition $\{1,2,3,6\}$ is $(0,2,1,6,3,0)$ and consequently $c(\{1,2,3,6\}) = 1$. Because of symmetry, $c(\{3,4,5,6\}) = 1$. An optimal tour of $\{1,2,4,5\}$ is $(0,2,1,5,4,0)$, so $c(\{1,2,4,5\}) = 0$. An optimal tour of N is $(0,2,1,6,5,4,3,0)$, so $c(N) = 2$.

If $x \in Core(c)$ then $x(\{1,2,3,6\}) \leq 1$, $x(\{3,4,5,6\}) \leq 1$ and $x(\{1,2,4,5\}) \leq 0$. So if we sum the inequalities we obtain $2x(N) \leq 2$ which contradicts $x(N) = 2$. Hence, the core is empty. □

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