EQUIVALENCE OF AUCTIONS AND POSTED PRICES*

by

Klaus Kultti

Center for Economic Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (tel.:+31-13-4663077, fax.:+31-13-4663066, e-mail: kultti@kub.nl)

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Abstract

We determine the equilibrium in two transaction mechanisms: auctions and posted prices. Agents choose whether to participate in markets where trades are consummated by auctions or in markets where sellers post prices. We show that the selling mechanisms are practically equivalent. Previous studies have shown that auction markets emerge as a unique evolutionary stable equilibrium when compared to bargaining markets. Posted price market dominate bargaining markets similarly. *Keywords:* Auctions, posted prices, random matching. *Journal of Economic Literature* Classification Numbers: C78, C73, D44.

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1.INTRODUCTION

There are three trading mechanisms or modes of trade that are commonly encountered in reality, namely auctions, bargaining, and posted prices. All of these have been analysed in economic literature auctions being perhaps the most extensively studied area. Most of auction analysis takes place in a partial equilibrium framework while bargaining theory has been applied to modelling the whole economy (Rubinstein and Wolinsky, 1985), too. Posted prices are used extensively in the literature. Search models starting from Diamond (1971) constitute a major example.

The relative performance and desirability of these selling mechanisms has been studied from the sellers' point of view in mechanism design literature where the purpose is to determine the optimal selling mechanism. In this field it is usually assumed that the seller can commit to the mechanism. Moreover the analysis is of partial equilibrium type, and static. Recently Wang (1993, 1995) has compared auctions to posted-price selling, and bargaining to posted-price selling in dynamic models. In his models the seller has a choice of the selling mechanism, and the best one depends on the associated costs. The costs are exogenous, and cannot be derived from the basics of the model. Wang also assumes that there is only one seller who meets buyers with exogenously given probabilities. This is not satisfactory, as one would expect the selling mechanism to affect the buyers' willingness to participate in the markets.

The comparison of various selling mechanisms would benefit from a dynamic setting, where the costs, reservation values and meeting probabilities are endogenously determined. A recent model by Lu and McAfee (1996) fulfills these requirements. They study the relative performance of auction markets, and bargaining markets in a setting where agents are randomly matched. Whether trades are consummated in an auction or bargaining determines the division of the surplus which determines the desirability of the markets for buyers and sellers. Lu and McAfee use evolutionary dynamics to determine stable market structures or equilibria. The equilibrium market structure is not necessarily evident since buyers tend to prefer markets where they receive a large share of the surplus while sellers do not find these markets attractive. However, when buyers go to their preferred market their number increases which makes the markets more desirable to sellers. Roughly put, Lu and McAfee find that auction markets constitute the unique stable equilibrium. In other words, of these two trading mechanisms auctions perform better.

The purpose of this article is to study posted price markets in a similar setting as Lu and McAfee (1996). It turns out that posted price markets are equivalent to auction markets which is somewhat surprising since the divison of surplus seems much different. In Lu and McAfee (1996) the seller is driven to his reservation utility if only one buyer appears while if many buyers appear the price is such that the buyers are driven to their reservation utility in auction markets. In the posted price markets there is a fixed price, and thus the sellers surplus is the same regardless of the number of buyers.

The rest of the article is organised as follows: In section 2 we present the model, and study the auction markets and posted price markets separately. In section 3 we determine the equilibria of the model. In section 4 we present conclusions.

2. THE MODEL

Let us consider markets with *B* buyers and *S* sellers where these numbers are large. Each seller has a unit of indivisible good for sale, and each buyer desires exactly one unit of this good. All sellers value the good at zero, and all buyers value the good at unity. These valuations can be regarded as reservation values in a static one period setting. In our dynamic setting the actual reservation values are determined endogenously.

We study two markets that may exist simultaneously. In both markets sellers are in fixed locations, and buyers are distributed on them randomly. In one market trades are consummated in an auction which means that before trading sellers commit to sell their goods in an auction regardless of the number of bidders. In the other market sellers post prices to which they commit. Both buyers and sellers can decide which markets to enter. Agents that manage to trade exit the markets and are replaced by identical agents that on entrance decide which markets they go to.

Time is discrete, and the agents have a common discount factor $\delta \in (0,1)$. The events within a period proceed in a fixed sequence: New sellers and buyers enter the markets, sellers post prices in the posted price market, buyers observe the prices, buyers are distributed on sellers in both markets, trading takes places, and those who trade exit the markets. Let us denote the ratio of buyers to sellers by $\theta = \frac{B}{S}$ which stays constant over time, the proportion of buyers in the posted price markets by *x*, and that of sellers by *y*. Then the proportion of buyers in the auction markets is *1-x*, and that of sellers *1-y*.

The number of buyers a seller meets is binomially distributed. Consider eg. posted price markets. There are *xB* buyers and *yS* sellers. As the buyers are, in equilibrium, distributed on the sellers independently with identical probabilities the probability that a fixed seller meets any particular buyer is 1/*yS*. Thus the number of buyers a seller meets is distributed according to Bin(*xB*, 1/*yS*). Analogously the number of buyers that a seller meets in an auction market is distributed according to Bin(*(1-x)B*, 1/*(1-y)S*). We adopt the following notation: $\alpha = \frac{1-x}{1-y}\theta$ and $\beta = \frac{x}{y}\theta$. Since binomial distributions are awkward to deal with we approximate them with Poisson distributions. In the auction market we use a Poisson distribution with rate $\alpha = \frac{1-x}{1-y}\theta$, and in the posted price market we use a Poisson distribution with rate $\beta = \frac{x}{y}\theta$.

2.1. *Auction markets*

We study first the auction markets since they are a bit simpler. When buyers are matched to a seller they submit bids for the object for sale. We do not model the auction explicitly as a game but we think that the bidders engage in a Bertrand-competition type situation where the price is such that they are driven to their reservation utility levels. Unless there happens to be only one bidder in which case it is the seller that receives his reservation utility.

A seller meets no buyers with probability $e^{-\alpha}$, exactly one buyer with probability $\alpha e^{-\alpha}$, and two or more buyers with probability $1-e^{-\alpha}-\alpha e^{-\alpha}$. In the first two cases he gets his reservation utility which is the same as his expected utility in the end of a period. Let us denote this by U_s^a where, as in the sequel, the subindex refers to the type of agent (seller or buyer), and the superindex to the type of market (auction or posted price). A buyer always meets exactly one seller, and since there are large numbers of buyers the probability that the buyer is the only buyer in a match is the same as the probability that no other buyers are matched to the seller, i.e. $e^{-\alpha}$. With probability $1-e^{-\alpha}$ there are two or more buyers in a match. In this case the buyers receive their reservation utility which is the same as their expected utility in the end of a period. Let us denote this by U_b^a. The formulae for the expected utilities of sellers and buyers are respectively

$$U_s^a = \delta\left[\left(e^{-\alpha}+\alpha e^{-\alpha}\right)U_s^a + \left(1-e^{-\alpha}-\alpha e^{-\alpha}\right)\left(1-U_b^a\right)\right] \tag{1}$$

$$U_b^a = \delta\left[e^{-\alpha}\left(1-U_s^a\right) + \left(1-e^{-\alpha}\right)U_b^a\right] \tag{2}$$

In (1) the LHS is the expected utility of a seller evaluated at the end of a period. The RHS is discounted since everything happens in the next period. The first term is the utility from meeting no buyers or one buyer in which case the seller is driven to his reservation utility. The second term is the utility of meeting two or more buyers in which case the buyers are driven to their reservation utility, and the seller receives the rest. The interpretation of (2) is analogous the first term on the RHS being the utility from being the only buyer in a match, and the second term the utility if there are more than one buyer. Notice that in this case it does not matter who receives the good since the price is such that all buyers receive their reservation utility. The expected utilities are easily solved

$$U_s^a = \frac{\delta\left(1 - e^{-\alpha} - \alpha e^{-\alpha}\right)}{1 - \delta\alpha e^{-\alpha}} \tag{3}$$

$$U_b^a = \frac{\delta e^{-\alpha}}{1 - \delta\alpha e^{-\alpha}} \tag{4}$$

2.2. *Posted price markets*

In these markets the sellers post prices that buyers take as given. This creates some problems if nothing more is postulated since clearly the optimal pricing rule from the sellers' point of view is to post price equal to unity. Given that a fixed number of buyers are in the markets and they are randomly distributed on the sellers it does not pay to lower the price. This is a highly unsatisfactory way to think of posted price markets. One would like to introduce some elements of competition by, for instance, letting the buyers choose which sellers they go to after they have received some information about prices. This is a bit tricky, and we discuss the details at the end of this section. For the moment let us denote the market price by *p*. We focus on situations in which every seller posts the same price.

The seller meets no buyer with probability $e^{-\beta}$ and one or more buyers with probability $1 - e^{-\beta}$. The buyer is the only buyer to meet the seller he is matched to with probability $e^{-\beta}$ and with probability $1 - e^{-\beta}$ there are other buyers, too. We assume that in this case all buyers have an equal probability of trading. Thus, a buyer gets to trade with probability $e^{-\beta}\left(1 + \frac{1}{2}\beta + \frac{1}{3}\frac{\beta^2}{2!} + \frac{1}{4}\frac{\beta^3}{3!} + \ldots\right) = \frac{1 - e^{-\beta}}{\beta}$. The expected utilities of sellers and buyers are respectively

$$U_s^p = \delta\left[e^{-\beta}U_s^p + \left(1 - e^{-\beta}\right)p\right] \tag{5}$$

$$U_b^p = \delta\left[\frac{\beta - \left(1 - e^{-\beta}\right)}{\beta}U_b^p + \frac{1 - e^{-\beta}}{\beta}(1 - p)\right] \tag{6}$$

From (5) and (6) we can solve the expected utilities

$$U_s^p = \frac{\delta\left(1-e^{-\beta}\right)}{1-\delta e^{-\beta}}\,p \qquad (7)$$

$$U_b^p = \frac{\delta\left(1-e^{-\beta}\right)}{\beta(1-\delta)+\delta\left(1-e^{-\beta}\right)}(1-p) \qquad (8)$$

Next we address the question about price determination. We try to capture the idea that there is competition in the markets, and consequently prices affect the number of buyers a seller meets. To this end we assume that buyers observe all prices and then decide independently which sellers they go to. If all sellers post the same price buyers are indifferent, and in equilibrium they choose a mixed strategy that puts equal weight to each seller. If buyers observe non-uniform prices they choose a mixed strategy that puts different weights to different sellers depending on the price they post. Given the distribution of prices the buyers choose the probabilities so that they constitute a Nash-equilibrium. We aim at determining a price *p* such that it constitutes a Nash-equilibrium for sellers given buyers' behaviour. In other words, *p* should be such that no seller has an incentive to change his price if all others announce *p*. There may exist asymmetric equilibria in which sellers post different prices but our focus is on symmetric equilibria. We determine *p* by considering one time deviations. Notice that one time deviations are sufficient since at the end of every period the agents leave their current partners, and by assumption they do not recognise agents whom they have previously met.

Assume for a moment that there are *B'* buyers and *S'* sellers in the market so that $\beta = \frac{B'}{S'}$, and that proportion *z* of the sellers deviate or are forced to deviate together. That more than one seller deviates simultaneously is just a modelling trick since it makes analysis easier, but it turns out that *z* can be thought to indicate the degree of competition, too.

In equilibrium all sellers post price *p*. Consider proportion *z* of sellers who deviate for one period and post price *p'*. The buyers observe the prices and choose a mixed strategy $(\sigma, 1-\sigma)$ that determines whether they go to sellers with price *p'* or *p*. The mixed strategy is such that the buyers are indifferent between the sellers

$$\frac{1-e^{-\beta'}}{\beta'}(1-p')+\frac{\beta'-1+e^{-\beta'}}{\beta'}U_b^p = \frac{1-e^{-\tilde\beta}}{\tilde\beta}(1-p)+\frac{\tilde\beta-1+e^{-\tilde\beta}}{\tilde\beta}U_b^p \qquad (9)$$

where $\beta' = \dfrac{\sigma B'}{zS'}$ and $\tilde{\beta} = \dfrac{(1-\sigma)B'}{(1-z)S'}$. In (9) the LHS is the expected utility of a buyer that goes to a seller with price *p'*. If he manages to trade he gets utility *1-p'* . If not things return to normal next period and his expected utility is given by (8). The RHS is the utility of a buyer who goes to a seller with price *p*. Notice that the meeting probabilities change as a result of the deviation. Equation (9) determines the equilibrium value of the mixed strategy $(\sigma, 1-\sigma)$.

Deviators maximise $e^{-\beta'}U_s^p + \left(1-e^{-\beta'}\right)p'$. From (9) we can solve *p'* as a function of σ which yields the following objective function for the deviators

$$e^{-\beta'}U_s^p + 1 - e^{-\beta'} + \left(\frac{\beta'}{\tilde{\beta}} - \frac{\beta'}{\tilde{\beta}}e^{-\tilde{\beta}} - 1 + e^{-\beta'}\right)U_b^p - \frac{\beta'}{\tilde{\beta}}\left(1-e^{-\tilde{\beta}}\right)(1-p) \tag{10}$$

Instead of choosing *p'* we can think that deviating sellers maximise (10) by choosing σ. The first order condition for the maximum is

$$\begin{aligned} &-e^{-\beta'}\frac{U_s^p}{z} + e^{-\beta'}\frac{1}{z} + \left(1-e^{-\tilde{\beta}}\right)\frac{\beta U_b^p}{z(1-z)\tilde{\beta}^2} - e^{-\tilde{\beta}}\frac{\beta' U_b^p}{(1-z)\tilde{\beta}} - e^{-\beta'}\frac{U_b^p}{z} + \\ &\left[\frac{e^{-\tilde{\beta}}\beta'}{(1-z)\tilde{\beta}} - \frac{\left(1-e^{-\tilde{\beta}}\right)\beta}{z(1-z)\tilde{\beta}^2}\right](1-p) = 0 \end{aligned} \tag{11}$$

In equilibrium the deviating sellers' maximising choice of price is *p*, which means that the deviators are in exactly the same situation as the non-deviators. This means that in equilibrium σ has to be such that $\beta' = \tilde{\beta} = \beta$. Inserting this into (11) gives us the equilibrium *p* as a function of *z*

$$p = \frac{\left(1-\delta e^{-\beta}\right)\left(1-e^{-\beta} - \beta e^{-\beta}\right)}{\left(1-e^{-\beta}\right)\left(1-\delta\beta e^{-\beta}\right) - ze^{-\beta}\left(\beta(1-\delta) + \delta\left(1-\delta e^{-\beta}\right)\right)} \tag{12}$$

From (12) we see that *p* is increasing in *z*. We let *z* approach zero which means that *p* attains its lowest value. This can be interpreted as a competitive environment. The sellers have to price in such a way that not even a small number of sellers find it profitable to deviate. Positive *z* would mean that deviation is possible only if many sellers do it simultaneously. In this case a deviating seller knows that he is adversely affected since other sellers deviate, too, and thus the sellers can sustain a higher equilibrium price as the costs of deviation are partly internalised. Letting z go to zero resembles the test for Nash-equilibrium where one deviating agent is considered. Alternatively, we could postulate that a proportion of buyers are able to observe prices while the rest of the buyers are distributed on the sellers randomly. This does not change the above analysis; the first order condition (11) is the same as long as the proportion of informed buyers is strictly positive.

In the limit when z approaches zero the equilibrium price becomes

$$p = \frac{\left(1-\delta e^{-\beta}\right)\left(1-e^{-\beta}-\beta e^{-\beta}\right)}{\left(1-e^{-\beta}\right)\left(1-\delta\beta e^{-\beta}\right)} \tag{13}$$

Notice that when there are very few buyers (beta is close to zero) and demand is low, the price goes towards zero, and when there are many buyers (beta grows without limit) the price tends to unity. The price also behaves well in a sense that it is always between zero and one. It is not, however, necessarily increasing in beta if the discount factor is close to unity. This may appear somewhat odd since the message of models with perfect markets is that price should increase with demand but the logic does not apply to imperfect markets. In perfect markets increase in demand raises prices and makes suppliers better-off. The posted price markets are not anomalous since increasing demand makes sellers better-off even though because of the matching 'technology' this does not necessarily translate into higher equilibrium price.

3. THE EQUILIBRIUM

As the purpose is to compare auctions and posted price markets we allow the co-existance of both types of markets. In the beginning of a period the agents decide which markets they go to. Sellers in the posted price markets announce prices, and buyers adopt a mixed strategy that determines to which sellers they go. In equilibrium none of the agents should be able to do better by changing his strategy. There are three possible equilibrium configurations in the economy: i) Only auction markets exist, ii) only posted price markets exist, and iii) auction markets and posted price markets co-exist. The test for equilibrium is of Nash-type, and we immediately see that cases i) and ii) constitute an equilibrium. If there is only one market then any deviator goes to the other inactive market, and since he is there alone he cannot do better than in the active market.

Both markets exist simultaneously if buyers and sellers are indifferent between which markets to participate in. Equating sellers' expected utilities (3) and (7) in both markets produces a condition that we call following Lu and McAfee (1996) sellers' equilibrium curve (SE). Analogously equality of (4) and (8) gives us buyers' equilibrium curve (BE).

$$\frac{1-e^{-\beta}-\beta e^{-\beta}}{1-\delta\beta e^{-\beta}}=\frac{1-e^{-\alpha}-\alpha e^{-\alpha}}{1-\delta e^{-\alpha}} \tag{14}$$

$$\frac{e^{-\beta}}{1-\delta\beta e^{-\beta}}=\frac{e^{-\alpha}}{1-\delta\alpha e^{-\alpha}} \tag{15}$$

Both (14) and (15) hold only if $\alpha=\beta$ which is equivalent to $\frac{1-x}{1-y}\theta=\frac{x}{y}\theta$. From this we see that in equilibrium the agents enter either market in equal proportions, i.e. x=y. We state these observations as

Proposition 1. Either market by itself constitutes an equilibrium. For any θ there exist a set of equilibria $\{(x,y):0<x=y<1\}$ with two active markets.

Lu and McAfee (1996) conduct similar analysis with bargaining markets and auction markets. They also study the stability of equilibria in evolutionary dynamics. Same kind of analysis with auctions and posted price markets is not particularly interesting since the two modes of trade are practically equivalent. In evolutionary sense all equilibria are unstable which is not surprising since the agents are indifferent between all equilibria with two active markets.

The equivalence of auction and posted price markets is an interesting result but one should remember that it is based on rather restrictive assumptions. First, we study situations in which all the buyers are identical, and all the sellers are identical. It is not clear that the result holds if, say, sellers' valuations are random draws from a known distribution like in auction theory. Secondly, while the modelling of auctions is straightforward there may be other reasonable ways to determine the posted prices. We deal with a simple procedure which still requires us to introduce more structure into our framework than there is in a typical random matching model.

4. CONCLUSION

In this article we study the viability of two modes of trade; posted price markets and auction markets. We use a tractable and well specified random matching model developed by Lu and McAfee (1996) who study the viability of auctions and bargaining. They find that when agents are allowed to choose which markets to participate in auctions dominate bargaining in a sense that in evolutionary dynamics auction markets are the only stable equilibrium. We show that auctions and posted price markets are practically equivalent, and it is easily seen that posted price markets dominate bargaining markets exactly the same way as auction markets dominate bargaining markets.

To render the study of various trading institutions meaningful one must introduce some frictions that make markets non-Walrasian or imperfect. We have used a model in which agents are randomly matched. This allows us to determine which institutions are likely to emerge in equilibrium. In reality one rarely sees auction markets like in this article while posted price markets seem to exist in abundance. Markets where trades are consummated by bargaining are often characterised by posted prices, too, a familiar example being the market for cars. A crucial assumption in the article and related work is that the seller is able and willing to commit to the trading mechanism. This a not an uncontroversial assumption in markets with many buyers and sellers. In a work under progress we assume that sellers announce a trading mechanism to which they are able to commit if they meet exactly one buyer. If they meet more buyers there is competition which means that the object is sold in an auction.

REFERENCES

Diamond P. A. (1971). “A Model of Price Adjustement,” *Journal of Economic Theory* **3**, 156-168.

Lu X. and McAfee R. P. (1996). “The Evolutionary Stability of Auctions Over Bargaining,” *Games and Economic Behavior* **15**, 228-254.

Rubinstein, A., and Wolinsky, A. (1985). “Equilibrium in a Market with Sequential Bargaining,” *Econometrica* **53**, 1133-1150.

Wang R. (1993). “Auctions Versus Posted-Price Selling,” *American Economic Review* **83**, 4, 838-851.

Wang R. (1995). “Bargaining Versus Posted-Price Selling,” *European Economic Review* **39**, 1747-1764.