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Approximating the finite-time ruin probability under interest force

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Abstract

We present an algorithm to determine both a lower and an upper bound for the finite-time probability of ruin for a risk process with constant interest force. We split the time horizon into smaller intervals of equal length and consider the probability of ruin in case premium income for a time interval is received at the beginning (resp. end) of that interval, which yields a lower (resp. upper) bound. For both bounds we present a renewal equation which depends on the distribution of the present value of the aggregate claim amount in a time interval. This distribution is determined through a generalization of Panjer's (1981) recursive method.

Classification code (JEL): G22.
Keywords: ruin probability, bounds, interest rate, renewal theory.

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1 Introduction

Although risk processes with compounding assets have recently received considerable attention (e.g. Sundt and Teugels (1995), Gjessing and Paulsen (1997), Paulsen (1998)), efficient algorithms for determining *finite-time* probabilities of ruin for a risk process with constant interest force are scarce. Dickson and Waters (1999) divide the time period under consideration into smaller intervals, and provide a recursive algorithm to determine the probability of ruin in case any claims are paid at the end of an interval. As mentioned by the authors, a drawback of their method is that it yields an underestimate of the probability of ruin, and that the accuracy of the approximations is difficult to assess as a result of the absence of exact values or good approximations of ruin probabilities for models with a strictly positive interest rate in the literature.

In this paper we present an algorithm that yields both a lower and an upper bound to the probability of ruin for the case with constant positive interest force. As in Dickson and Waters (1999) we discretize the time horizon, but instead of assuming that claims are paid at the end of an interval, we consider the cases where premium income is received at the beginning (resp. end) of an interval. We then derive renewal equations for the two resulting ruin probabilities. In order to solve these renewal equations we use the generalized version of Panjer's (1981) recursive method derived in Boogaert and De Waegenaere (1990). The combined result of lower and upper bounds that converge to the actual ruin probability makes it possible to assess the ruin probability with high accuracy.

The organization of this paper is as follows. Section 2 presents the lower and upper bounds to the probability of ruin. Section 3 describes the renewal equation that allows to determine both bounds recursively. In Section 4, numerical results are stated, and Section 5 concludes.

2 The risk process under interest force

The risk process with a constant interest rate is an extension of the classical risk process. It is assumed that claims arrive according to a Poisson process $\{N_t : t \geq 0\}$ with rate λ ($\lambda > 0$). Let T_n ($n \in \mathbb{N}$) denote the arrival time of the n-th claim. The claim sizes $\{X_n : n \in \mathbb{N}\}$ are nonnegative and i.i.d. and independent of the claim arrival process. The initial surplus of the insurance company at time $t = 0$ is k ($k \geq 0$) and premiums are received continuously over time at rate p ($p \geq 0$).

Both starting capital and premium income grow with a constant, deterministic, and continuous interest rate γ ($\gamma \geq 0$). So the present value of an amount $x \in \mathbb{R}$ at time t is given by $xe^{-\gamma t}$. The surplus Z_t of the company at time t is for the case with interest given by:

$$Z_t := ke^{\gamma t} + p(t) - \sum_{i=1}^{N_t} e^{\gamma(t-T_i)} X_i, \qquad t \geq 0, \tag{1}$$

where $p(t)$ denotes the value at time t of the aggregate premium income over $[0, t]$.

In order to approximate the probability of ruin, it is more convenient to consider the *present value* of the surplus, which is given by:

$$\begin{aligned} \tilde{Z}_t &:= k + p_c(t) - \sum_{i=1}^{N_t} e^{-\gamma T_i} X_i \\ &= k + p_c(t) - \tilde{S}_t, \qquad t \geq 0, \end{aligned}$$

where $\tilde{S}_t$ denotes the present value of the aggregate claim amount up to time t, and

$$p_c(t) := \begin{cases} pt & \text{if } \gamma = 0,\ t \geq 0, \\ \frac{p}{\gamma}(1 - e^{-\gamma t}) & \text{if } \gamma > 0,\ t \geq 0. \end{cases}$$

denotes the present value of the premiums received over $[0, t]$.

The probability of ruin in the interval $[0, T]$ for initial surplus k and given present value premium income function $p_c(.)$ is denoted by:

$$P(k, T, p_c(.)) := \Pr(\inf\{t \geq 0 : k + p_c(t) - \tilde{S}_t < 0\} < T), \qquad k, T \geq 0. \tag{2}$$

For the case where $\gamma = 0$, one has $p_c(t) = pt$, for all $t \geq 0$, and several methods exist to approximate or calculate the probability of ruin in a certain time interval, finite or infinite (e.g. De Vylder and Goovaerts, 1988).

In order to determine upper and lower bounds for $P(k, T, p_c(.))$ in case $\gamma > 0$, we consider the probability of ruin for two other present value premium income functions defined as follows: For $h > 0$

$$\begin{aligned} p_b(h, t) &:= ph \sum_{j=0}^{i-1} e^{-\gamma j h} \qquad \text{for all } t \in (\,(i-1)h, ih\,], \\ p_e(h, t) &:= ph \sum_{j=1}^{i-1} e^{-\gamma j h} \qquad \text{for all } t \in [\,(i-1)h, ih\,), \end{aligned}$$

and $p_b(h, 0) := 0$.

Hence,

- $p_b(h, t)$ equals the present value of the total premium amount at time t when an amount ph is received *just after the beginning* of each time interval of length h,

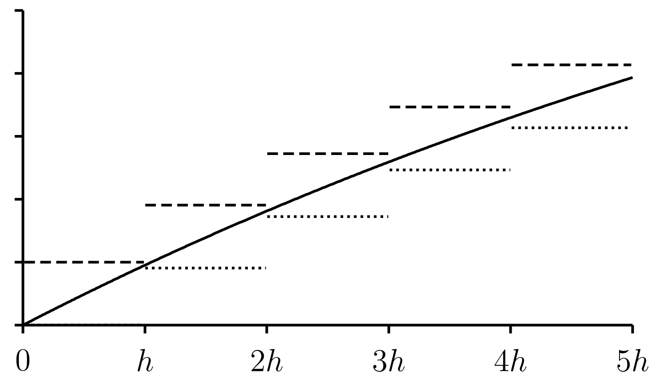

Figure 1: Present value premium income functions $p_c(t)$ (solid line), $p_b(h,t)$ (dashed line) and $p_e(h,t)$ (dotted line).

- $p_e(h,t)$ equals the present value of the total premium amount at time t when an amount ph is received *at the end* of each time interval of length h.

Since clearly, for a fixed but arbitrary $h > 0$ one has:

$$p_e(h,t) \leq p_c(t) \leq p_b(h,t), \qquad \text{for all } t \geq 0, \tag{3}$$

it follows that for every $T \geq 0$, and $h > 0$:

$$P(k,T,p_b(h,.)) \leq P(k,T,p_c(.)) \leq P(k,T,p_e(h,.)). \tag{4}$$

Therefore, the ruin probabilities for the present value premium income functions $p_b(h,.)$ and $p_e(h,.)$ can be used to determine lower and upper bounds for the probability of ruin for $p_c(.)$.

Since $p_b(h,.)$ is increasing in h, and $p_e(h,.)$ is decreasing in h, it follows that $P(k,T,p_b(h,.))$ is decreasing in h, and $P(k,T,p_e(h,.))$ is increasing in h, and consequently, one obtains better approximations as h gets smaller. Figure 1 shows the present value premium income functions $p_c(t)$, $p_b(h,t)$ and $p_e(h,t)$ as a function of the time t. We see that $p_b(h,t)$ and $p_e(h,t)$ are strictly above, respectively below, $p_c(t)$ for $t > 0$.

The following theorem shows how the bounds in (4) can be sharpened by using present value premium income functions that result from multiplying $p_b(h,.)$ and $p_e(h,.)$ by the factor

$$r_b(h) := \begin{cases} \frac{1-e^{-\gamma h}}{\gamma h} & \text{if } \gamma > 0, \\ 1 & \text{if } \gamma = 0, \end{cases}$$

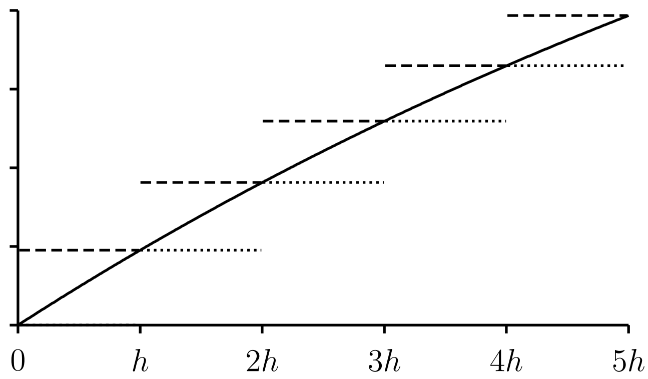

Figure 2: Present value premium income functions $p_c(t)$ (solid line), $r_b(h)\, p_b(h,t)$ (dashed line) and $r_e(h)\, p_e(h,t)$ (dotted line).

and

$$r_e(h) := \begin{cases} \frac{e^{\gamma h}-1}{\gamma h} & \text{if } \gamma > 0, \\ 1 & \text{if } \gamma = 0, \end{cases}$$

respectively. Note that $r_b(h) \leq 1$ and $r_e(h) \geq 1$. Furthermore, we have that $r_b(h)\, p_b(h, nh) = r_e(h)\, p_e(h, nh) = p_c(nh)$ for all $n \in \mathbb{N}_0$ (see Figure 2).

Theorem 1. *We have the following lower bound and upper bound for $P(k, T, p_c(.))$:*

$$P\Big(k, T, r_b(h)\, p_b(h, .)\Big) \leq P(k, T, p_c(.)) \leq P\Big(k, T, r_e(h)\, p_e(h, .)\Big), \tag{5}$$

for all $k, T \geq 0$ and $h > 0$.

Proof. It is sufficient to show that for all $\gamma \geq 0$ and $h > 0$, $p_c(.)$ satisfies

$$r_e(h)\, p_e(h,t) \leq p_c(t) \leq r_b(h)\, p_b(h,t), \qquad t \geq 0. \tag{6}$$

For $\gamma = 0$ the result immediately follows from (3). Assume that $\gamma > 0$ and let $t > 0$ be given, and take i such that $t \in ((i-1)h, ih]$. Then

$$\begin{aligned} p_c(t) \leq p_c(ih) &= \frac{p}{\gamma}(1 - e^{-\gamma i h}) = \frac{p}{\gamma}(1 - e^{-\gamma h}) \sum_{j=0}^{i-1} e^{-\gamma j h} \\ &= \frac{1 - e^{-\gamma h}}{\gamma h} p_b(h,t) = r_b(h)\, p_b(h,t). \end{aligned}$$

For $t = 0$ one has $p_c(0) = r_b(h)\, p_b(h, 0) = 0$.

Similarly, let $t \geq 0$ and i be such that $t \in [(i-1)h, ih)$. Then

$$\begin{aligned} p_c(t) &\geq p_c((i-1)h) = \frac{p}{\gamma}(1 - e^{-\gamma(i-1)h}) \\ &= \frac{p}{\gamma}(1 - e^{-\gamma h}) \sum_{j=0}^{i-2} e^{-\gamma j h} = \frac{p}{\gamma}(e^{\gamma h} - 1) \sum_{j=1}^{i-1} e^{-\gamma j h} \\ &= \frac{e^{\gamma h} - 1}{\gamma h} p_e(h,t) = r_e(h)\, p_e(h,t). \end{aligned}$$

This concludes the proof. □

In order to calculate the upper and lower bounds in (5), we take h such that exactly N ($N \in \mathbb{N}$) intervals of length h fit into the larger interval $[0, T]$, i.e.,

$$h := T/N, \qquad T > 0,\ N \in \mathbb{N}. \tag{7}$$

The following theorem shows that, due to the structure of the present value premium income functions $p_b(h, .)$ and $p_e(h, .)$, the continuous time ruin probability in (5) can be written as discrete time ruin probabilities.

Theorem 2. *For h given by (7) define*

$$\begin{aligned} Q(k, n, c) := 1 - \Pr\Big(\tilde{S}_h \leq k + c,\ \tilde{S}_{2h} \leq k + c + ce^{-\gamma h}, \ldots, \\ \tilde{S}_{nh} \leq k + c + ce^{-\gamma h} + \cdots + ce^{-(n-1)\gamma h}\Big), \end{aligned} \tag{8}$$

for $n \in \{1, 2, \ldots, N\}$, $k, c \in \mathbb{R}$ and $\gamma \geq 0$.
Then, for all $p, k \geq 0$ and $\gamma \geq 0$:

(i) $P(k, T, rp_b(h, .)) = Q(k, N, rph) \qquad r > 0,$

(ii) $P(k, T, rp_e(h, .)) = Q(k - rph, N, rph) \qquad r > 0,$

(iii) $P(k, T, p_c(.))$ *satisfies:*

$$Q\big(k, N, p_l\big) \leq P(k, T, p_c(.)) \leq Q\big(k - p_u, N, p_u\big). \tag{9}$$

where

$$p_l := ph\, r_b(h) = \begin{cases} p(1 - e^{-\gamma h})/\gamma & \text{if } \gamma > 0, \\ ph & \text{if } \gamma = 0, \end{cases}$$

and

$$p_u := ph\, r_e(h) = \begin{cases} p(e^{\gamma h} - 1)/\gamma & \text{if } \gamma > 0, \\ ph & \text{if } \gamma = 0. \end{cases}$$

Proof.

(i) This is a trivial consequence of the fact that, given $\tilde{p}(t) = r\, p_b(h,t)$ for all $t \geq 0$, one has for every $i = 1, 2, \ldots, N$:

$$\min\Big\{\tilde{Z}_t : t \in ((i-1)h, ih]\Big\} = \tilde{Z}_{ih},$$

when $\tilde{Z}_t = k + \tilde{p}(t) - \tilde{S}_t$. Consequently, ruin occurs at some $t \in [0, T]$ if and only if ruin occurs at some time $t \in \{ih : i = 1, 2, \ldots, N\}$, i.e.,

$$\begin{aligned} P(k, T, r\, p_b(h,.)) &= \Pr\big(\exists i \in \{1, 2, \ldots, N\} : \tilde{Z}_{ih} < 0\big) \\ &= 1 - \Pr(\tilde{S}_h \leq k + rph,\ \tilde{S}_{2h} \leq k + rph + rphe^{-\gamma h}, \ldots, \\ &\qquad \tilde{S}_{Nh} \leq k + rph + \cdots + rphe^{-\gamma(N-1)h}) \\ &= Q(k, N, rph). \end{aligned}$$

(ii) Given $\tilde{p}(.) = r\, p_e(h,.)$, i.e., a premium amount rph is received at the end of the period, we have for all $i = 1, \ldots, N$:

$$\tilde{Z}_t \geq \tilde{Z}_{ih} - rphe^{-\gamma ih}, \qquad \text{for all } t \in [(i-1)h, ih)\,,$$

when $\tilde{Z}_t = k + \tilde{p}(t) - \tilde{S}_t$. Hence, if the surplus becomes negative at a certain time $t \in [(i-1)h, ih)$, then it will be negative just prior to the premium income at time ih. Since the probability of a claim arrival exactly at time ih is zero, the surplus just prior to the premium income at time ih is, with probability 1, equal to $\tilde{Z}_{ih} - rphe^{-\gamma ih}$. This yields

$$\begin{aligned} P(k, T, rp_e(h,.)) &= \Pr\big(\exists i \in \{1, 2, \ldots, N\} \text{ s.t. } \tilde{Z}_{ih} - rphe^{-\gamma ih} < 0\big) \\ &= 1 - \Pr\Big(\tilde{S}_h \leq k,\ \tilde{S}_{2h} \leq k + rphe^{-\gamma h}, \ldots, \\ &\qquad \tilde{S}_{Nh} \leq k + rphe^{-\gamma h} + \cdots + rphe^{-(n-1)\gamma h}\Big) \\ &= Q(k - rph, N, rph). \end{aligned}$$

(iii) Follows immediately by combining (6) and (i) with $r = r_b(h)$, and by combining (6) and (ii) with $r = r_e(h)$.

□

In the next section, we show how, for given values of k, N and c, $Q(k, N, c)$ can be computed recursively.

3 Renewal equation for $\mathbf{Q}(.,.,.)$

An essential property of the classical risk process ($\gamma = 0$) is the fact that, since the claim number process $\{N_t : t \geq 0\}$ is an homogeneous Poisson process and the claim height process $\{X_i : i \in \mathbb{N}\}$ is an i.i.d. process independent of $\{N_t : t \geq 0\}$, the total claim height process $\{S_t = \sum_{i=1}^{N_t} X_i : t \geq 0\}$ has stationary and independent increments. Clearly, when $\gamma > 0$, the present value claim height process $\{e^{-\gamma T_i} X_i : i \in \mathbb{N}\}$ is no longer an i.i.d. process, since claims that occur at different times are discounted over different time periods. Consequently, the distribution of an increment in the total present value claim height no longer only depends on the length of this time period, i.e., the process does not have stationary increments. However, in the case of exponential inter-occurrence times, we can show a relation between the distributions of the total discounted claim height over different time periods of the same length. This will allow us to derive a recursive formula for $Q(k, N, c)$.

It is well known that the conditional distribution of the stochastic vector $(T_1, T_2, \ldots, T_n)$, conditional on the event that exactly n claims occurred in the time period $[0, T]$, equals the distribution of the vector $(U_{(1)}, U_{(2)}, \ldots, U_{(n)})$, where $U_{(i)}$ denotes the i-th order statistic in a sequence of n i.i.d. random variables with uniform distribution over $[0, T]$.

It is clear that the following generalization holds:

Lemma 3. *For all $0 \leq s < t$ and $m, n \in \mathbb{N} : m < n$, we have that*

$$(T_{m+1} - s, T_{m+2} - s, \ldots, T_n - s) \mid (N_s = m, N_t = n) \sim (U_{(1)}, U_{(2)}, \ldots, U_{(n-m)}),$$

where $U_{(i)}$ denotes the i-th order statistic in a sequence of $n - m$ i.i.d. random variables with uniform distribution over $[0, t - s]$.

Proof. Straightforward generalization of the proof of Theorem 2.3.1 in Ross (1996). □

Given this result, we can now show that the present value risk process has increments that are independent and "nearly" stationary in the following sense:

Lemma 4. *The present value risk process satisfies:*

(i) $\{\tilde{S}_t : t \geq 0\}$ *has independent increments,*

(ii) for all $0 \leq t_1 < t_2 < \cdots < t_n$ and all $u \geq 0$, the random vector $(\tilde{S}_{t_1+u} - \tilde{S}_u,\ \tilde{S}_{t_2+u} - \tilde{S}_{t_1+u},\ \ldots,\ \tilde{S}_{t_n+u} - \tilde{S}_{t_{n-1}+u})$ has the same distribution as the random vector $(e^{-\gamma u}\tilde{S}_{t_1},\ e^{-\gamma u}(\tilde{S}_{t_2} - \tilde{S}_{t_1}),\ \ldots, e^{-\gamma u}(\tilde{S}_{t_n} - \tilde{S}_{t_{n-1}}))$.

Proof.

(i) See Boogaert and Haezendonck (1989).

(ii) We first show that for all $0 \le v < w$

$$\tilde{S}_{w+u} - \tilde{S}_{v+u} \sim e^{-\gamma u}(\tilde{S}_w - \tilde{S}_v). \tag{10}$$

For simplicity of notation, we consider the case where $v = 0, u = s$, and $w = t - s$ for some $s < t$. For each $m, n \in \mathbb{N}_0 : m \le n$, we denote $(U_{(1)}, U_{(2)}, \ldots, U_{(n-m)})$ for the $n - m$ order statistics of a sequence of $n - m$ i.i.d. random variables with uniform distribution over $[0, t - s]$.

First notice that it follows from Lemma 3 that for any $m, n \in \mathbb{N}_0$ with $m \le n$, we have:

$$\begin{aligned}
&\Pr\left(e^{-\gamma s}\textstyle\sum_{i=m+1}^{n} e^{-\gamma(T_i - s)} X_i \le x \;\middle|\; N_s = m, N_t = n\right) \\
&= \Pr\left(e^{-\gamma s}\textstyle\sum_{i=m+1}^{n} e^{-\gamma U_{(i-m)}} X_i \le x\right) \\
&= \Pr\left(e^{-\gamma s}\textstyle\sum_{i=1}^{n-m} e^{-\gamma U_{(i)}} X_i \le x\right) \\
&= \Pr\left(e^{-\gamma s}\textstyle\sum_{i=1}^{n-m} e^{-\gamma T_i} X_i \le x \;\middle|\; N_{t-s} = n - m\right).
\end{aligned}$$

Consequently, we see that for any $x \in \mathbb{R}$:

$$\begin{aligned}
&\Pr(\tilde{S}_t - \tilde{S}_s \le x) \\
&= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \Pr\left(e^{-\gamma s}\textstyle\sum_{i=m+1}^{n} e^{-\gamma(T_i - s)} X_i \le x \;\middle|\; N_s = m, N_t = n\right) \\
&\qquad \cdot \Pr(N_s = m, N_t = n) \\
&= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \Pr\left(e^{-\gamma s}\textstyle\sum_{i=1}^{n-m} e^{-\gamma T_i} X_i \le x \;\middle|\; N_{t-s} = n - m\right) \\
&\qquad \cdot \Pr(N_s = m)\Pr(N_{t-s} = n - m) \\
&= \Pr\left(e^{-\gamma s}\textstyle\sum_{i=1}^{N_{t-s}} e^{-\gamma T_i} X_i \le x\right) \\
&= \Pr(e^{-\gamma s}\tilde{S}_{t-s} \le x) \\
&= \Pr\left(e^{-\gamma s}(\tilde{S}_{t-s} - \tilde{S}_0) \le x\right).
\end{aligned}$$

The proof of the more general case where $0 \le v < w$ and $u \ge 0$ goes along the same lines. From (i) we know that the increments are independent and therefore (10) can be applied to all increments separately, which yields the desired result.

□

We can now come to the recursive formula for $Q(k, N, c)$.

Theorem 5. *Let $c, \gamma \geq 0,\ k \geq -c$. Then we have:*

$$Q(k, 1, c) = 1 - G_h(k + c) \tag{11}$$

and for $n = 2, \ldots, N$

$$Q(k, n, c) = 1 - G_h(k + c) + \int_0^{k+c} Q\big((k - y + c)e^{\gamma h}, n - 1, c\big) dG_h(y), \tag{12}$$

where $G_h(.)$ denotes the cdf of $\tilde{S}_h$.

Proof. For $n = 1$ we have

$$Q(k, 1, c) = 1 - \Pr(\tilde{S}_h \leq k + c) = 1 - G_h(k + c).$$

Let us denote

$$Y_{i,h} := \tilde{S}_{ih} - \tilde{S}_{(i-1)h}.$$

Thus, $Y_{i,h}$ equals the present value of the total claim amount in the time interval $((i-1)h, ih]$, and $Y_{1,h}$ is equal to $\tilde{S}_h$ and has cdf $G_h(.)$.

Then, for all $n = 2, \ldots, N$, we have

$$\begin{aligned} Q(k, n, c) &= 1 - \Pr\big(Y_{1,h} \leq k + c, Y_{1,h} + Y_{2,h} \leq k + c + ce^{-\gamma h}, \ldots, \\ &\qquad Y_{1,h} + \cdots + Y_{n,h} \leq k + c + ce^{-\gamma h} + \cdots + ce^{-(n-1)\gamma h}\big) \\ &= 1 - \int_0^{k+c} \Pr\big(Y_{2,h} \leq k + c + ce^{-\gamma h} - y, \ldots, \\ &\qquad Y_{2,h} + \cdots + Y_{n,h} \leq k + c + ce^{-\gamma h} + \cdots + ce^{-(n-1)\gamma h} - y\big) dG_h(y). \end{aligned} \tag{13}$$

Given Lemma 4(ii), if we choose $u = h$ and $t_i = ih$ for $i = 1, 2, \ldots, n$, it follows that $(Y_{2,h}, \ldots, Y_{n,h})$ has the same distribution as $(e^{-\gamma h}Y_{1,h}, \ldots, e^{-\gamma h}Y_{n-1,h})$. Therefore, for every $y \in [0, k + c]$, we have:

$$\begin{aligned} &\Pr\big(Y_{2,h} \leq k + c + ce^{-\gamma h} - y, \ldots, \\ &\quad Y_{2,h} + \cdots + Y_{n,h} \leq k + c + ce^{-\gamma h} + \cdots + ce^{-(n-1)\gamma h} - y\big) \\ &\quad = \Pr\big(e^{-\gamma h}Y_{1,h} \leq k + c + ce^{-\gamma h} - y, \ldots, \\ &\qquad e^{-\gamma h}(Y_{1,h} + \cdots + Y_{n-1,h}) \leq k + c + ce^{-\gamma h} + \cdots + ce^{-(n-1)\gamma h} - y\big) \\ &\quad = \Pr\big(Y_{1,h} \leq (k - y + c)e^{\gamma h} + c, \ldots, \\ &\qquad Y_1 + \cdots + Y_{n-1,h} \leq (k - y + c)e^{\gamma h} + c + ce^{-\gamma h} + \cdots + ce^{-(n-2)\gamma h}\big) \\ &\quad = 1 - Q\big((k - y + c)e^{\gamma h}, n - 1, c\big). \end{aligned}$$

Substituting the expression above in (13) yields the desired result. ∎

The above recursive formula can be used to compute lower and upper bounds for the probability of ruin in the interval $[0,T]$ for given values of c,k,T and h. It therefore remains to determine $G_h(x) = \Pr(\tilde{S}_h \leq x)$. For the classical risk process, the algorithm by Panjer (1981) gives a recursive formula for the density of S_h. In the following theorem, we show how an alternative to Panjer's recursion can be used to determine the density $g_h(.)$ of the present value of the total claim height $\tilde{S}_h$.

Theorem 6. *Let the claim heights X_n ($n \in \mathbb{N}$) be nonnegative and absolute continuous distributed with pdf $f(.)$. The density $g_h(.)$ ($h > 0$) of the variable $\tilde{S}_h$ then satisfies the following integral equation:*

$$g_h(x) = \lambda h e^{-\lambda h} \tilde{f}_h(x) + \lambda h \int_0^x \frac{x-y}{x} \tilde{f}_h(x-y) g_h(y) dy, \qquad x > 0, \tag{14}$$

where $\tilde{f}_h(.)$ denotes the pdf of $X_1 e^{-\gamma T_1} \mid \{N_h = 1\}$. Hence,

$$\tilde{f}_h(x) = \frac{1}{h} \int_0^h f(xe^{\gamma u}) e^{\gamma u} du, \qquad x > 0. \tag{15}$$

Proof. See Boogaert and De Waegenaere (1990). □

Now, if $g_h(.)$ is the solution of integral equation (14), then the initializing step in the recursive algorithm can be computed as follows:

$$\begin{aligned} Q(k,1,r) &= 1 - G_h(k+r) \\ &= 1 - e^{-\lambda h} - \int_0^{k+r} g_h(y) dy. \end{aligned}$$

The combined results of Theorems 2, 5 and 6 now allow to determine the lower and upper bounds in (9) for any given $h > 0$. The following theorem shows that both the lower and upper bound converge to $P(k,t,p_c(.))$ as h tends to zero.

Theorem 7. *For all $k, T \geq 0$ and $\gamma \geq 0$,*

$$\lim_{h \downarrow 0} P\Big(k, T, r_b(h)\, p_b(h,.)\Big) = P(k,T,p_c(.)) = \lim_{h \downarrow 0} P\Big(k, T, r_e(h)\, p_e(h,.)\Big).$$

Proof. It is seen easily that, for any $\gamma, t \geq 0$, one has

$$\lim_{h \downarrow 0} \Big(r_b(h)\, p_b(h,t)\Big) = p_c(t) = \lim_{h \downarrow 0} \Big(r_e(h)\, p_e(h,t)\Big).$$

This allows to show that both the time of ruin for $r_b(h)p_b(h,.)$ and for $r_e(h)p_e(h,.)$ converge almost sure to the time of ruin for $p_c(.)$. Since convergence almost sure implies convergence in distribution (see Mittelhammer, 1996, page 258) the result follows.

□

4 Numerical results

In this section we present numerical results in cases where the claim heights have an exponential distribution or a Pareto distribution.

For the interpretation of the numerical results it is important to know that, as is the case for the classical risk process, certain normalizations can be done without loss of generalization.

Theorem 8.

(i) Let $T = \bar{T}/\alpha$, $\lambda = \alpha\bar{\lambda}$, $\gamma = \alpha\bar{\gamma}$, $p = \alpha\bar{p}$. Then the ruin probability $P(k, T, p_c(.))$ does not depend on α, for all $\alpha > 0$.

(ii) Let $k = \alpha\bar{k}$, $p = \alpha\bar{p}$ and let the pdf of the claim size distribution be given by $f(x/\alpha)/\alpha$ for some function $f(.)$. Then the ruin probability $P(k, T, p_c(.))$ does not depend on α, for all $\alpha > 0$.

Proof. Goes along the usual lines. □

Hence, Theorem 8(i) shows that the probability of ruin does not change if the time horizon is rescaled and other time-related model parameters are modified accordingly. Theorem 8(ii) says that the probability of ruin does not change if the monetary unit is re-evaluated.

4.1 Exponentially distributed claim size

We consider the case where the claim sizes $\{X_i : i \in \mathbb{N}\}$ are exponentially distributed with mean μ, i.e. $f(x) = \frac{1}{\mu}e^{-x/\mu}$. For $\gamma > 0$, it follows immediately that the function $\tilde{f}_h(.)$ in Theorem 6 is given by:

$$\tilde{f}_h(x) = \begin{cases} \frac{1}{h\gamma x}\left(e^{-\frac{x}{\mu}} - e^{-\frac{xe^{\gamma h}}{\mu}}\right) & \text{if } x > 0, \\ \frac{1}{h\gamma\mu}(e^{\gamma h} - 1), & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases}$$

As in Dickson and Waters (1999), we take $\mu = 1$ and $\lambda = 1$. The safety loading θ is equal to 0.1, and consequently the premium density $p = 1.1$.

For the case with no interest ($\gamma = 0$) the literature provides accurate ruin probabilities. Table 1 shows the ruin probabilities by Seal (1978) denoted by "S" for several values of the initial surplus k and the time horizon T, and the lower bounds and upper bounds (LB–UB) derived in this paper computed using interval length $h = 0.01$. It is clear from Table 1 that the algorithm produces good bounds when $\gamma = 0$.

T		$k=0$	$k=5$	$k=10$
1	LB–UB	0.4616–0.4649	0.0138–0.0139	0.0003–0.0003
	S	0.4634	0.0138	0.0003
5	LB–UB	0.7178–0.7204	0.1024–0.1029	0.0092–0.0093
	S	0.7196	0.1027	0.0092
10	LB–UB	0.7838–0.7859	0.1901–0.1908	0.0318–0.0320
	S	0.7854	0.1906	0.0319
20	LB–UB	0.8303–0.8320	0.2950–0.2958	0.0820–0.0822
	S	0.8318	0.2956	0.0821

Table 1: Ruin probabilities and lower and upper bounds (LB–UB) for $\gamma = 0$.

T		$k=0$	$k=5$
1	LB–UB	0.4596–0.4629	0.0126–0.0127
	AVG	0.4612	0.0127
	SIM	0.4613	0.0127
	DW	0.4598	0.0126
5	LB–UB	0.7014–0.7040	0.0778–0.0781
	AVG	0.7027	0.0780
	SIM	0.7033	0.0780
	DW	0.7019	0.0778
10	LB–UB	0.7538–0.7560	0.1259–0.1264
	AVG	0.7549	0.1262
	SIM	0.7556	0.1263
	DW	0.7544	0.1260
20	LB–UB	0.7806–0.7825	0.1627–0.1632
	AVG	0.7816	0.1630
	SIM	0.7821	0.1631
	DW	0.7812	0.1628

Table 2: Simulated ruin probabilities and lower and upper bounds for $\gamma = 0.05$.

We have computed lower bounds and upper bounds for interest rate $\gamma = 0.05$ and initial surplus $k = 0$ and $k = 5$. Table 2 shows the approximations by Dickson and Waters (1999) (DW), simulated ruin probabilities (SIM), the lower bounds (LB) and upper bounds (UB) that we discussed in this paper, as well as the average of lower and upper bound (AVG). All bounds are computed with a fixed interval length $h = 0.01$ which is slightly larger than the step size used by Dickson and Waters

T		LB–UB	AVG	SIM
1	$h = 1$	0.3248–0.6321	0.4784	0.4613
	$h = 0.5$	0.3849–0.5476	0.4663	
	$h = 0.25$	0.4211–0.5038	0.4625	
5	$h = 1$	0.5884–0.8294	0.7089	0.7033
	$h = 0.5$	0.6405–0.7685	0.7045	
	$h = 0.25$	0.6701–0.7355	0.7028	
10	$h = 1$	0.6587–0.8605	0.7596	0.7556
	$h = 0.5$	0.7025–0.8098	0.7561	
	$h = 0.25$	0.7273–0.7823	0.7548	
20	$h = 1$	0.6968–0.8750	0.7859	0.7821
	$h = 0.5$	0.7352–0.8301	0.7826	
	$h = 0.25$	0.7570–0.8057	0.7813	

Table 3: Lower and upper bounds for $k = 0$, $\gamma = 0.05$ and different values of h.

(1999) ($h = 0.009$).

The simulated ruin probabilities were computed using $250 \cdot 10^6$ simulation runs of the risk process— which results in 95% confidence intervals with length at most $1.3 \cdot 10^{-4}$.

Clearly, for small h the lower bound and upper bound are close together and the actual probability of ruin is known with relatively high precision. However, also for higher values of h a good approximation of the ruin probability can be obtained by averaging the lower and upper bound. Indeed, the simulated ruin probabilities are located remarkably close to the average of the lower and upper bound of our algorithm. This is illustrated in Table 3 which repeats the simulated ruin probabilities of Table 2 for $k = 0$ and shows the lower and upper bounds computed resulting from $h = 1$, $h = 0.5$ and $h = 0.25$.

Another illustration of this effect is shown in Figure 3 where the average of the lower and upper bound for the case with $k = 5$ and $T = 20$ is plotted against the step size h. We see that, as h approaches zero, the average quickly approaches the the actual ruin probability.

4.2 Pareto distributed claim size

We consider the same example as in Section 4.1, but now with i.i.d. Pareto$(2, 3)$ claim size distribution, i.e. $\Pr(X_i \leq x) = 1 - (a/(a + x))^b$, with $a = 2, b = 3$. To

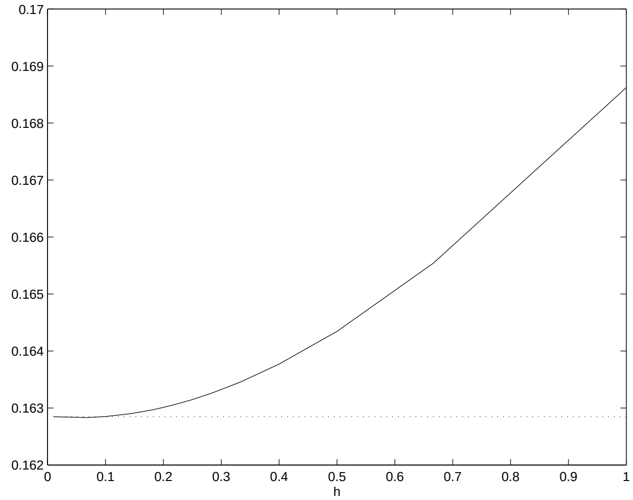

Figure 3: Average of the bounds of the probability of ruin, for $k = 5$, $\gamma = 0.05$ and $T = 20$, as a function of step size h.

γ	LB–UB	AVG	SIM	DW
0.000	0.4201–0.4237	0.4219	0.4219	0.4201
0.025	0.4182–0.4227	0.4204	0.4209	0.4191
0.050	0.4172–0.4217	0.4195	0.4199	0.4181
0.075	0.4162–0.4207	0.4185	0.4188	0.4170
0.100	0.4152–0.4196	0.4174	0.4179	0.4160

Table 4: Ruin probabilities for Pareto claim size distribution with $k = 0$, $T = 1$.

solve integral equation (14) we need $\tilde{f}_h(.)$ which, for $\gamma > 0$, is now equal to

$$\tilde{f}_h(x) = \begin{cases} \frac{a^b}{h\gamma x}\Big((a+x)^{-b} - (a+xe^{\gamma h})^{-b}\Big) & \text{if} \;\; x > 0, \\ \frac{b}{ha\gamma}(e^{\gamma h} - 1) & \text{if} \;\; x = 0, \\ 0 & \text{if} \;\; x < 0. \end{cases}$$

Note that, similarly to the case with exponential claim size distribution, the mean claim size of the Pareto$(2, 3)$ distribution is equal to 1.

Table 4 shows the lower bound and upper bound (LB–UB) for the probability of ruin in $[0, 1]$, the average of both bounds (AVG), as well as the approximations by Dickson and Waters (1999) (DW) and simulated (SIM) probabilities, for the case with zero starting capital ($k = 0$). The bounds are computed with step size $h = 0.009$. Taking the simulated values as reference, the results indicate that the bounds presented in this paper, and in particular the average of lower and upper bound, closely approximate the actual ruin probability.

5 Conclusion

In this paper lower and upper bounds for the finite-time probability of ruin of a risk process with a constant interest force are derived. The time horizon is divided into small intervals and two alternative premium income functions are considered where a fixed amount is received either at the beginning or at the end of the interval. This yields a lower bound and an upper bound for the continuous time probability of ruin when the premium income is received at a constant rate over time. In Section 3 a recursive algorithm is developed that enables the computation of the bounds.

Existing numerical results and simulations show that the recursive algorithm yields accurate lower and upper bounds for the probability of ruin for a finite time horizon for any nonnegative interest force. An alternative method to approximate finite-time ruin probabilities, which is due to Dickson and Waters (1999) yields an underestimate for the ruin probability. Since our algorithm yields both lower and upper bounds, it is possible to determine the accuracy of the bounds/approximations. Moreover, it can be shown that the bounds converge to the actual ruin probability.

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