Games Arising from Infinite Production Situations

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Abstract

OWEN (1975) introduced linear production (LP) situations and TIMMER, BORM and SUIJS (1998) introduced more general situations involving the linear transformation of products (LTP). They showed that the corresponding LTP games are totally balanced. In this paper we look at LTP situations with an infinite number of transformation techniques. The linear program that calculates the maximal profit, is a semi-infinite program. We show that an optimal solution of the dual program exists and that it is a core-element of the corresponding LTP game.

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1 Introduction

OWEN (1975) introduced linear production (LP) situations. These are production situations where there is a finite set of producers, each of them owns a bundle of resources and all producers can use the same finite set of linear production techniques. The products can be sold on the market at given prices and all producers are price takers. This model has two restrictions. First, each production process can only have one output good while many production processes have by-products. Second, all producers can use the same production techniques while in reality some producers have a production technique that nobody else has. To overcome these restrictions, TIMMER, BORM and SUIJS (1998) introduced situations involving the linear transformation of products (LTP). In these situations, each linear transformation technique has at least one output good and different producers may have different production techniques. More precise, in an LTP situation there is a finite set of producers and each of them controls a finite number of transformation techniques. We define the set of goods to be the set of products and resources. Each producer owns a bundle of goods that he can use (like resources)

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in his transformation process or that he can sell directly on the market (like products). The outcome of the transformation process, the produced goods, will also be sold on the market. The goal of each producer is to maximize his profit given his transformation techniques, bundle of goods and the exogenous market prices.

In this paper, we consider semi-infinite LTP situations, which are LTP situations with a countable, infinite number of transformation techniques. Something similar for LP situations and corresponding games has been studied by FRAGNELLI, PATRONE, SIDERI and TIJS (1999) using duality results by TIJS (1979). We will also work with linear semi-infinite programs. One of the first papers in this area was written by CHARNES, COOPER and KORTANEK (1962). Many results on this subject can be found in GLASHOFF and GUSTAFSON (1983) and in the recent book by GOBERNA and LÓPEZ (1998).

This paper is organized as follows. Section 2 starts with a formal introduction of finite LTP situations and corresponding games. In section 3 we extend this to semi-infinite LTP situations where we consider a countable, infinite number of transformation techniques. Some examples show what problems we may encounter. Therefore, in each of the sections 4 and 5 a set of conditions will be presented that ensures the existence of an optimal dual solution and the existence of a core-element of the corresponding semi-infinite LTP game.

2 Finite LTP Situations and Games

Situations involving the linear transformation of products were introduced by TIMMER et al. (1998). To illustrate these situations, consider the following example. A tailor uses a large piece of silk fabric, thread and buttons to produce full dresses. From the left-overs he makes some doll's dresses. More precise, assume that this tailor needs a piece of silk fabric of 10 by 1.50 meters, 100 meters of thread, 70 buttons and 24 hours of labour to produce 6 full dresses and 2 doll's dresses. Assuming that the production process is linear, this production or *transformation technique* is represented by the following vector a,

$$a = \begin{bmatrix} 6 \\ 2 \\ -1 \\ -100 \\ -70 \\ -24 \end{bmatrix}$$

where the rows correspond from the top downward to respectively full dresses, doll's dresses, pieces of silk fabric, meters of thread, buttons and hours of labour. So, silk fabric, thread, buttons and labour are the input goods in this transformation process while full dresses and doll's dresses are the output goods. Since the production technique is linear, any nonnegative multiple of a is a possible production technique. The value of this nonnegative multiple is called the *activity level*. The activity level of this tailor is restricted by the amount of input goods at his disposal.

LTP situations[4] are production situations in which each producer controls some transformation techniques and a bundle of resource goods. Denote by M the finite set of goods and by N the finite set of producers. Each producer $i \in N$ owns a bundle of goods $\omega(i) \in \mathbb{R}_+^M$. A transformation technique is denoted by a vector $a \in \mathbb{R}^M$ and it says that a producer needs $-a_j$ units of the goods j with $a_j < 0$ to produce a_l units of the goods l with $a_l > 0$. If $a_j = 0$ for some good j, then this good is not used in the transformation technique. We assume that each technique needs at least one input good to produce at least one output good. Hence every vector a contains at least one positive and one negative element. Denote by D_i the set of transformation techniques controlled by producer i, that is, $D_i = \{k | \text{ producer } i \text{ controls } a^k\}$, and denote by y_k the activity level of transformation technique a^k. Using transformation technique a^k, a producer needs $\{-a_j^k y_k | \, j \in M : \; a_j^k < 0\}$ to produce $\{a_j^k y_k | \, j \in M : \; a_j^k > 0\}$. We assume that the production process cannot be reversed, so $y_k \geq 0$ for all k, and for any two players $i, j \in N$ the sets D_i and D_j are disjoint, $D_i \cap D_j = \emptyset$.

Denote by $D = \cup_{i \in N} D_i$ the finite set of all available transformation techniques. Let $y = (y_k)_{k \in D}$ be the vector in $\mathbb{R}_+^D$ of all activity levels and let A be the technology matrix in $\mathbb{R}^{M \times D}$ with k^{th} column a^k. Define the related matrix $G \in \mathbb{R}_+^{M \times D}$ with k^{th} column g^k by $g_j^k = \max\{0, -a_j^k\}$ for all $j \in M$. This matrix states which of the goods and how much of them are needed as inputs in the various transformation processes when all activity levels equal one. If the activity level of a^k equals y_k then a producer needs the bundle $g^k y_k$ of goods to produce the bundle $(a^k + g^k) y^k$ since $a_j^k + g_j^k = \max\{a_j^k, 0\}$.

Combining all his knowledge, producer $i \in N$ can use the bundle Gy of goods to produce the bundle $(A + G)y$. Here $y_k = 0$ if $k \notin D_i$ because producer i can only use his own transformation techniques. The amount Gy of goods he uses, should not exceed the amount $\omega(i)$ of goods at his disposal, so $Gy \leq \omega(i)$. Producer i starts with the bundle $\omega(i)$ from which he uses Gy to produce $(A+G)y$. Therefore, after the transformation, the producer is left with the bundle $\omega(i) - Gy + (A + G)y = \omega(i) + Ay$ which he can sell on the market at exogenous prices $p \in \mathbb{R}_+^M \setminus \{0\}$. We assume that the market is insatiable, so, all goods can be sold. Furthermore, all producers are pricetakers. The goal of each producer is to maximize his profit from the sale of his remaining goods:

$$
\begin{array}{ll}
\max & p^T(\omega(i) + Ay) \\
\text{s.t.} & Gy \leq \omega(i) \\
 & y_k = 0 \text{ if } k \notin D_i \\
 & y \geq 0.
\end{array}
$$

Producers can also cooperate by pooling their transformation techniques and resources. The coalition $S \subset N$, $S \neq \emptyset$, of producers then acts like one big producer with resource bundle $\omega(S) = \sum_{i \in S} \omega(i)$ and $D(S) = \cup_{i \in S} D_i$ is its set of available transformation techniques. The profit maximization

[4] In TIMMER et al. (1998) these situations are called extended LTP situations.

problem for this coalition looks as follows:

$$\begin{aligned} \max \quad & p^T(\omega(S) + Ay) \\ \text{s.t.} \quad & Gy \le \omega(S) \\ & y_k = 0 \text{ if } k \notin D(S) \\ & y \ge 0. \end{aligned} \tag{1}$$

In short, an *LTP situation* is described by a 5-tuple $\langle N, A, D, \omega, p\rangle$ where $\omega = (\omega(i))_{i\in N}$. The corresponding *LTP game* (N, v) is such that the characteristic function v assigns to each coalition $S \subset N$ the maximal profit it can obtain as given in (1) and $v(\emptyset) = 0$.

One of the main issues in cooperative game theory is how to divide the benefits from cooperation. In LTP games we would like to know how to divide the joint profit among the cooperating producers. The core $C(v)$,

$$C(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \ge v(S) \text{ for all } S \subset N \right\},$$

is the set of allocations x of $v(N)$ upon which no coalition S of producers can improve. If an allocation $x \in C(v)$ is proposed as a distribution of the total profit $v(N)$, where producer i gets the amount x_i, then coalition S will get at least as much as it can obtain on its own since $\sum_{i\in S} x_i \ge v(S)$. Therefore, no coalition S has an incentive to leave the grand coalition N. A game is called *balanced* if it has a nonempty core and it is called *totally balanced* if each subgame $(S, v_{|S})$ has a nonempty core, where $v_{|S}$ is the game v restricted to coalition S with $v_{|S}(T) = v(T)$ for all $T \subset S$. The following theorem, based on a theorem in TIMMER et al. (1998), shows that LTP games are totally balanced.

Theorem 2.1 *Let $\langle N, A, D, \omega, p\rangle$ be an LTP situation. Then the corresponding LTP game is totally balanced.*

Proof. Since each subgame $(S, v_{|S})$ is an LTP game, we only have to prove that the LTP game (N, v) is balanced. For this, recall that the value $v(N)$ for coalition N equals

$$\begin{aligned} v(N) = \quad \max \quad & p^T(\omega(N) + Ay) \\ \text{s.t.} \quad & Gy \le \omega(N) \\ & y \ge 0. \end{aligned}$$

Note that $p^T\omega(N)$ is a constant. Therefore the corresponding dual minimization program is

$$\begin{aligned} \min \quad & (z + p)^T\omega(N) \\ \text{s.t.} \quad & G^T z \ge A^T p \\ & z \ge 0. \end{aligned} \tag{2}$$

Since the set of feasible solutions of this program, $\{z \in \mathbb{R}^M | G^T z \ge A^T p, z \ge 0\}$, is nonempty, closed, convex and bounded from below, this minimization problem can be solved and a minimum exists. Let the minimum of (2) be taken in $\underline{z}$. Define $x \in \mathbb{R}^N$ by $x_i = (\underline{z} + p)^T\omega(i)$ for all $i \in N$. Then $\sum_{i\in N} x_i = \sum_{i\in N}(\underline{z} + p)^T\omega(i) = (\underline{z} + p)^T\omega(N) = v(N)$ where the last equality follows from duality theory. Notice that $\underline{z}$ is also a feasible solution of the problem $\min\{(z+p)^T\omega(S)|G^Tz \geq \omega(S),\, z \geq 0\}$ for all coalitions S. Therefore,

$$\begin{aligned}
(\underline{z}+p)^T\omega(S) &\geq \min\{(z+p)^T\omega(S)|G^Tz \geq \omega(S),\, z \geq 0\} \\
&= \max\{p^T(\omega(S)+Ay)|Gy \leq \omega(S),\, y \geq 0\} \\
&\geq \max\{p^T(\omega(S)+Ay)|Gy \leq \omega(S),\, y_k = 0 \text{ if } k \notin D(S),\, y \geq 0\} \\
&= v(S)
\end{aligned}$$

and $\sum_{i\in S} x_i = \sum_{i\in S}(\underline{z}+p)^T\omega(i) = (\underline{z}+p)^T\omega(S) \geq v(S)$. We conclude that $x \in C(v)$. $\square$

This proof shows that we can find a core-element of the LTP game (N, v) via the dual program corresponding to the profit maximization problem. The set of core-elements we can find in this way has been thoroughly studied for linear production situations by VAN GELLEKOM, POTTERS, REIJNIERSE, TIJS and ENGEL (1998). In the next sections we will use this method to find a core-element for the LTP game corresponding to an LTP situation with an infinite number of transformation techniques.

3 Semi-Infinite LTP Situations

In many production situations, there are an infinite number of techniques available to the producer. For example, a firm may have a finite number of transformation techniques on the short run, but when we think of the long run, this firm can choose from an infinite number of techniques. It can continue its current production process, it can expand its activities, it can produce some extra goods or it can switch to the use of some completely different transformation techniques. A second example concerns cooking. If you have a recipe for baking pancakes from flour, milk, eggs, butter and sugar, then you can get an infinite number of recipes for pancakes by changing the amounts of the ingredients slightly. Each recipe then gives a slightly different pancake.

We define a *semi-infinite LTP situation* as a 5-tuple $\langle N, A, D, \omega, p\rangle$ where the set D contains a countable, infinite number of transformation techniques. All other variables are as defined in the previous section. The following examples show some problems we may encounter in semi-infinite LTP situations.

Example 3.1 Consider the semi-infinite LTP situation with a single producer, two goods, bundle of goods $\omega = (3, 0)^T$, market prices $p = (1, 3)^T$ and technology matrix

$$A = \begin{bmatrix} -4 & -3\frac{1}{2} & -3\frac{1}{3} & -3\frac{1}{4} & \cdots & -3-\frac{1}{k} & \cdots \\ 2 & 2 & 2 & 2 & \cdots & 2 & \cdots \end{bmatrix}.$$

The primal profit 'maximization' problem is

$$\begin{aligned}
\sup \quad & p^T(\omega + Ay) \\
\text{s.t.} \quad & Gy \leq \omega \\
& y \geq 0.
\end{aligned}$$

Note that we have replaced the maximum by a supremum since there is an infinite number of activity levels and an optimal solution may not exist. This problem is equal to

$$\sup\{3 + p^T Ay | \sum_{k=1}^{\infty}(3 + 1/k)y_k \leq 3,\ y \geq 0\} = \lim_{k\to\infty}(3 + 3 - 1/k) = 6.$$

There is no optimal solution for this problem, that is, there exists no vector $\hat{y}$ of activity levels such that $p^T(\omega + A\hat{y}) = 6$. The corresponding dual problem is

$$\begin{aligned} &\inf\{(z+p)^T\omega |\, G^T z \geq A^T p,\ z \geq 0\} \\ &= \inf\{3z_1 + 3 | (3 + 1/k)z_1 \geq 3 - 1/k,\ k = 1, 2, \ldots,\ z \geq 0\} \\ &= 3 \cdot 1 + 3 = 6. \end{aligned}$$

The set of optimal solutions $\{z \in \mathbb{R}^2 | z_1 = 1,\ z_2 \geq 0\}$ is nonempty. In this example we see that the primal problem may have no optimal solution while the dual problem has optimal solutions.

Example 3.2 Consider the following semi-infinite LTP situation with a single producer, two goods, bundle of goods $\omega = (0,1)^T$, prices $p = (1,1)^T$ and technology matrix

$$A = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & \cdots & -\frac{1}{k} & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & \cdots \end{bmatrix}.$$

Then

$$\begin{aligned} v &= \sup\{p^T(\omega + Ay) |\, Gy \leq \omega,\ y \geq 0\} \\ &= \sup\{1 + p^T Ay |\, \textstyle\sum_{k=1}^{\infty} y_k/k \leq 0,\ y \geq 0\} \\ &= 1 + 0 = 1 \end{aligned}$$

with optimal activity vector $y = 0$. The dual problem equals

$$\begin{aligned} &\inf\{z_2 + 1 |\, z_1/k \geq 1 - 1/k,\ k = 1, 2, \ldots,\ z \geq 0\} \\ &= \inf\{z_2 + 1 |\, z_1 \geq k - 1,\ k = 1, 2, \ldots,\ z \geq 0\} = +\infty \end{aligned}$$

since there exists no feasible solution z. Therefore there are no optimal solutions to the dual program of this example while there exists an optimal solution to the primal problem.

Example 3.3 We have a semi-infinite LTP situation with a single producer, five goods and

$$A = \begin{bmatrix} -2 & -2 & -2 & -2 & & -2 & \\ 0 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & & -\frac{1}{k} & \\ 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 1 & 1 & 1 & 1 & & 1 & \\ 0 & 1 & 1 & 1 & & 1 & \end{bmatrix},\ \omega = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ p = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 4 \end{bmatrix}.$$

The profit maximization problem gives

$$v = \sup\{2 + p^T Ay |\, 0 \leq y_1 \leq 1,\ y_k = 0,\ k = 2, 3, \ldots\} = 4.$$

The corresponding dual problem gives

$$\inf\{2+2z_1 |\, 2z_1 \geq 2,\ 2z_1 + z_2/k \geq 3,\ k = 2, 3, \ldots,\ z \geq 0\} = 5.$$

Here we have a duality gap: the primal maximization program does not have the same optimal value as the dual problem.

These examples show that semi-infinite LTP situations may deal with duality gaps and the absence of optimal solutions for both the primal and the dual program. We would like to have conditions on semi-infinite LTP situations such that there is no duality gap, the dual problem has an optimal solution and the primal problem has a feasible solution. Then we can find a core-element of the game via the dual problem. We do not need the existence of an optimal solution of the primal problem to attain this core-element.

In the following two sections we present two sets of conditions that ensure we can find a core-element of the LTP game corresponding to a semi-infinite LTP situation via the dual problem.

4 Conditions Involving Cones

In this section we will present a first set of conditions on semi-infinite LTP situations and we show that this guarantees that the corresponding LTP games have a nonempty core.

Denote by 0_M the M-dimensional zero-vector and by e^j the j^{th} unit vector in $\mathbb{R}^M$ with $e^j_m = 1$ if $m = j$ and $e^j_m = 0$ if $m \neq j$. If B is an (infinite) set of vectors in $\mathbb{R}^q$ for some integer number q then we obtain the convex cone generated by B, denoted by $CC(B)$, by taking all nonnegative multiples of finite convex combinations of elements in B. Thus,

$$CC(B) = \left\{ x \,\middle|\, x = \sum_{i=1}^{t} \lambda_i b^i,\ \lambda_i \geq 0,\ b^i \in B,\ i = 1, 2, \ldots, t,\ t \geq 1 \right\}$$

Define the sets K_1 and K_2 as follows.

$$K_1 = CC\left(\left\{(g^k)_{k\in D},\ (e^j)_{j\in M}\right\}\right) = \mathbb{R}^M_+$$

$$K_2 = CC\left(\left\{\begin{pmatrix} g^k \\ p^T a^k \end{pmatrix}_{k\in D},\ \begin{pmatrix} e^j \\ 0 \end{pmatrix}_{j\in M}\right\}\right)$$

The last equality for K_1 follows from $g^k \in \mathbb{R}^M_+$ for all $k \in D$. In the literature, see for example GLASHOFF and GUSTAFSON (1983) and GOBERNA and LÓPEZ (1998), the convex cones K_1 and K_2 are usually called the first and second moment cone and denoted by M and N respectively. We renamed these cones since we already use M and N to denote respectively the set of goods and the set of producers. Denote by $int(K_1)$ the interior of K_1 and by $cl(K_2)$ the closure of K_2. Consider the following two conditions.

Condition 4.1

$$\omega(N) \in int(K_1) = \mathbb{R}^M_{++}$$

This condition states that the coalition N of all producers should own some positive amount of all goods in M.

Condition 4.2

$$\begin{pmatrix} 0_M \\ 1 \end{pmatrix} \notin cl(K_2)$$

An interpretation of this condition is that doing nothing, which is equivalent to activity level $y_k = 0$ for all $k \in D$, cannot result in a positive profit. The following theorem shows the nonemptiness of the core under these conditions.

Theorem 4.3 *Let $\langle N, A, D, \omega, p\rangle$ be a semi-infinite LTP situation. If conditions 4.1 and 4.2 are satisfied then the corresponding LTP game is balanced.*

Proof. Conditions 4.1 and 4.2 are satisfied and therefore it follows from respectively theorems $8.1.(v), (vi)$ and 4.4.(i) in GOBERNA and LÓPEZ (1998) that the dual problem for coalition N,

$$\begin{array}{ll} \inf & (z+p)^T\omega(N) \\ \text{s.t.} & G^T z \geq A^T p \\ & z \geq 0, \end{array}$$

is feasible, there exists an optimal dual solution and there is no duality gap. Let $\underline{z}$ be an optimal dual solution. We can show in a similar way as in the proof of theorem 2.1 that $x \in \mathbb{R}^N$, defined by $x_i = (\underline{z}+p)^T\omega(i)$ for all $i \in N$, is a core-element of the corresponding LTP game. $\square$

We will now return to our examples in the previous section. In the first example we have that $g^k = (3+\frac{1}{k}, 0)^T$ and $p^T a^k = 3 - \frac{1}{k}$ for all $k \in D$. Thus

$$K_1 = CC\left\{ \begin{pmatrix} 3+\frac{1}{k} \\ 0 \end{pmatrix}_{k\in D}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2_+$$

and

$$K_2 = CC\left\{ \begin{pmatrix} 3+\frac{1}{k} \\ 0 \\ 3-\frac{1}{k} \end{pmatrix}_{k\in D}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

We see that condition 4.2 is satisfied since $(0,0,1)^T \notin cl(K_2)$ but condition 4.1 is not satisfied because $\omega_2 = 0$. However, there is no duality gap and there exists an optimal dual solution.

In the second example we see that $g^k = (\frac{1}{k}, 0)^T$ and $p^T a^k = 1 - \frac{1}{k}$ for all $k \in D$. Therefore $K_1 = \mathbb{R}^2_+$ and

$$K_2 = CC\left\{ \begin{pmatrix} \frac{1}{k} \\ 0 \\ 1-\frac{1}{k} \end{pmatrix}_{k\in D}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Here condition 4.1 is not satisfied since $\omega_1 = 0$ and the same holds for condition 4.2 since $(0,0,1)^T \in cl(K_2)$. The dual problem has no feasible solutions.

Finally, in the third example we have that $g^1 = (2,0,0,0,0)^T$, $g^k = (2, \frac{1}{k}, 0, 0, 0)^T$, $k \geq 2$, $p^T a^1 = 2$ and $p^T a^k = 3$, $k \geq 2$. So $K_1 = \mathbb{R}^5_+$ and

$$K_2 = CC \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ \frac{1}{k} \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}_{k \geq 2}, \begin{pmatrix} \\ e^j \\ \\ 0 \end{pmatrix}_{j \in M} \right\}.$$

In this example, condition 4.1 is not satisfied but condition 4.2 is and there is a duality gap.

From these examples we may conclude that conditions 4.1 and 4.2 are sufficient but not necessary conditions in theorem 4.3.

5 Economic Conditions

In this section a second set of conditions on semi-infinite LTP situations will be presented. These conditions also guarantee total balancedness of the corresponding LTP games. Similar conditions for linear production (LP) situations can be found in FRAGNELLI, PATRONE, SIDERI and TIJS (1999).

Condition 5.1

$$\sup_{k \in D} p^T a^k = \gamma < +\infty$$

All transformation techniques a^k should generate a finite profit of at most γ when $y_k = 1$, that is, the techniques are operated at the unit activity level.

Condition 5.2

$$\max_{j \in M} g_j^k \geq \alpha > 0 \text{ for all } k \in D$$

This condition states that for each transformation technique there is always some positive amount α of a resource needed at the unit activity level.

Recall that

$$\begin{aligned} v(N) = \quad & \sup \quad p^T(\omega(N) + Ay) \\ & \text{s.t.} \quad Gy \leq \omega(N) \\ & \qquad\quad y \geq 0. \end{aligned}$$

We will use the following result by KARLIN and STUDDEN (1966), which we translated to semi-infinite LTP situations for coalition N.

Theorem 5.3 *Suppose that $v(N)$ is finite and that $\omega(N) \in \mathbb{R}^M_{++}$. Then there is no duality gap and the dual program has an optimal solution.*

We can now prove the following result.

Theorem 5.4 *Let $\langle N, A, D, \omega, p \rangle$ be a semi-infinite LTP situation. If conditions 5.1 and 5.2 are satisfied then the corresponding LTP game is totally balanced.*

Proof. Since each subgame $(S, v_{|S})$ of an LTP game is another LTP game, we only have to prove that the core of (N, v) is nonempty.

By conditions 5.1 and 5.2 it follows that the dual feasible region $\{z \in \mathbb{R}^M_+ \,|\, G^T z \geq A^T p\}$ is nonempty since $z^T = \gamma^{-1}\alpha(1, 1, \ldots, 1)$ is a feasible dual solution. It also follows that the primal profit maximization problem has a finite optimal profit. From the result by KARLIN and STUDDEN, theorem 5.3, it follows that if $\omega(N) \in \mathbb{R}^M_{++}$ then there is no duality gap and there exists an optimal dual solution $\underline{z}$. As we have shown before, the vector $x \in \mathbb{R}^N$ with $x_i = (\underline{z} + p)^T \omega(i)$ for all $i \in N$ is an element of $C(v)$.

If $\omega(N) \notin \mathbb{R}^M_{++}$ then one or more goods in M are not available, that is, there exists at least one good $j \in M$ such that $\omega_j(N) = 0$. We may eliminate these goods and all techniques that need a positive amount of them since it is impossible to use these transformation techniques. This reduced problem satisfies $\omega_j(N) > 0$ for all non-eliminated goods j. Again by the result of KARLIN and STUDDEN it follows that there is no duality gap in this reduced problem and there exists an optimal solution $\hat{z}$. To obtain an element of $C(v)$ we define $\underline{z}_j = \hat{z}_j$ for all non-eliminated goods j and $\underline{z}_j = 0$ for all eliminated goods j. Then we can show in a similar way as in the proof of theorem 2.1 that $x \in \mathbb{R}^N$, $x_i = (\underline{z} + p)^T \omega(i)$, is a core-element of the corresponding LTP game. □

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