A Nucleolus for Stochastic Cooperative Games

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Abstract

This paper extends the definition of the nucleolus to stochastic cooperative games, that is, to cooperative games with random payoffs to the coalitions. It is shown that the nucleolus is nonempty and that it belongs to the core whenever the core is nonempty. Furthermore, it is shown for a particular class of stochastic cooperative games that the nucleolus can be determined by calculating the traditional nucleolus introduced by *Schmeidler (1969)* of a specific deterministic cooperative game.

KEYWORDS: Nucleolus, cooperative game theory, random variables, preferences.

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1 Introduction

In stochastic cooperative games, the payoffs individuals can obtain by cooperating with each other are random variables instead of deterministic amounts. Moreover, the players are not allowed to await the realizations of these payoffs before they decide upon an allocation of these payoffs. These kinds of cooperative games fall outside the scope of traditional (deterministic) cooperative game theory. Models that can deal with such situations were introduced by *Charnes* and *Granot (1973)* and, more recently, by *Suijs, Borm, De Waegenaere* and *Tijs (1995)*. The major difference between these two models is that the first model assumes risk neutral behaviour of all the players while the latter model incorporates risk neutral as well as risk averse and risk loving behaviour of the players. This paper introduces a nucleolus for the games introduced by *Suijs et al. (1995)*.

The nucleolus, a solution concept for deterministic cooperative games, originates from *Schmeidler (1969)*. This solution concept yields an allocation such that the maximal excess of the coalitions is minimized. The excess describes how dissatisfied a coalition is with the proposed allocation. The larger the excess of a particular allocation, the more a coalition is dissatisfied with this allocation. For Schmeidler's nucleolus the excess is defined as the difference between the payoff a coalition can obtain when cooperating on its own and the payoff received by the proposed allocation. So, when less is allocated to a coalition, the excess of this coalition increases and the other way around.

Since the nucleolus depends mainly on the definition of the excess, other nucleoli are found when different definitions of excesses are used. Such a general approach can be found in *Potters* and *Tijs (1992)*. They introduced the general nucleolus as the solution that minimizes the maximal excess of the coalitions, using generally defined excess functions.

A similar argument holds for stochastic cooperative games. If we can specify the excesses we can define a nucleolus for these games. Unfortunately, this is not that simple. Defining excess functions for stochastic cooperative games appears to be not as straightforward as for deterministic cooperative games. Indeed, how should one quantify the difference between the random payoff a coalition can achieve on its own and the random payoff received by the proposed allocation when the behaviour towards risk can differ between the members of this coalition? Furthermore, the excess of one coalition should be comparable to the excess of another coalition.

Charnes and *Granot (1976)* introduced a nucleolus for cooperative games in stochastic characteristic function form. There, the excess was based on the probability that the payoff a coalition can obtain when cooperating on its own, exceeds the payoff they obtained in the proposed allocation. Indeed, it is quite reasonable to assume that a coalition is less satisfied with the proposed allocation if this probability increases.

For the excess defined in this paper we interpret the excess of Schmeidler's nucleolus in a slightly different way. Bearing the conditions of the core in mind, this excess can be interpreted as follows. Given an allocation of the grand coalition's payoff we distinguish two cases. In the first case, a coalition wants to leave the grand coalition. Then the excess equals the minimal amount of money a coalition needs on top of what they already get such that this coalition is willing to stay in the grand coalition. In the second case, a coalition has no incentive to leave the grand coalition. Then the excess equals minus the maximal amount of money that can be taken away from this coalition such that this coalition still has no incentive to leave the grand coalition. This interpretation is used to define the excess for stochastic cooperative games.

The paper is organized as follows. Section 2 consists mainly of preliminaries. It briefly recalls the definition of a cooperative game with stochastic payoffs. Furthermore it states the assumptions we make on the preferences of the players and it introduces the necessary definitions and notations. Then in Section 3 the excess functions are introduced and, subsequently, a nucleolus. Moreover, it is shown that this nucleolus is a well defined solution concept in the sense that it always yields a nonempty subset of allocations. Section 4 shows that the nucleolus is a subset of the core whenever the core is nonempty. Moreover, it shows that for the class of stochastic cooperative games introduced in *Suijs* and *Borm (1996)* the nucleolus can be determined by calculating Schmeidler's nucleolus of a specific deterministic cooperative game.

2 Stochastic cooperative games

A stochastic cooperative game is described by a tuple $\Gamma = (N, (X_S)_{S \subset N}, (\succsim_i)_{i \in N})$. Here, the set of players is denoted by N. The payoff a coalition $S \subset N$ can achieve by cooperating is denoted by a random variable X_S. So, there is a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ such that for each $S \subset N$ the payoff function $X_S : \Omega \to \mathbb{R}$ is measurable, that is, $X_S^{-1}(B) \in \mathcal{H}$ for each element B of $\mathcal{R}$, the Borel σ-algebra. Since coalitions are not allowed to await the realization of X_S before they decide on the allocation, the random payoff X_S has to be allocated. An allocation of the random payoff X_S among the members of S is described by a pair $(d, r) \in H^S \times \Delta^S$, where $H^S = \{d \in \mathbb{R}^S | \sum_{i \in S} d_i \leq 0\}$ and $\Delta^S = \{r \in \mathbb{R}^S | \forall_{i \in S} : r_i \geq 0, \sum_{i \in S} r_i = 1\}$. The random payoff to player $i \in S$ then equals $d_i + r_i X_S$. So, r allocates fractions of the random payoff X_S to the members of S and d denotes the transfer payments. Note that these transfer payments need not be efficient. Moreover, it should be noted that for notational reasons the definition of an allocation used in this paper differs from its original definition in *Suijs et al. (1995)*. Originally, d was an allocation of the expected payoff $E(X_S)$ and r was an allocation of the residual payoff $X_S - E(X_S)$. Finally, note that the random payoff $d_i + r_i X_S$ to player $i \in S$ is measurable with respect to the probability space $(\Omega, \mathcal{H}, \mathbb{P})$. Next, define

$$\mathcal{L}(\Gamma) = \{d + rX_S | d \in \mathbb{R}, r \in [0, 1], S \subset N\}. \tag{1}$$

Then $\mathcal{L}(\Gamma)$ is the set of all random payoffs player $i \in N$ can receive in the game Γ. Finally, $\succsim_i$ are the complete and transitive preferences of player i over the set $\mathcal{L}(\Gamma)$.

Examples of situations where this model may apply appear in insurance. Individuals facing losses that can occur to them in the future have to decide now if they want an insurance for these losses or not and, if so, which premium they want to pay for it. Furthermore, groups of individuals may benefit from taking a collective insurance instead of many individual ones. Another example appears when considering linear production games with random prices. Here, a coalition has to decide which goods to produce given the resources they posses without exactly knowing the revenues that are generated by these goods.

In the remainder of this paragraph we go through some necessary preliminaries. Therefore, consider again the set $\mathcal{L}(\Gamma)$. Denote by F_X the distribution function of the random variable $X \in \mathcal{L}(\Gamma)$. Thus, $F_X(t) = \mathbb{P}(\{\omega | X(\omega) \leq t\})$ for all $t \in \mathbb{R}$. Next, define $\mathcal{F}(\Gamma) = \{F_X | X \in \mathcal{L}(\Gamma)\}$ to be the set of distribution functions corresponding to the random payoffs in $\mathcal{L}(\Gamma)$. Now, let $(F_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}(\Gamma)$. Then the sequence $(F_k)_{k\in\mathbb{N}}$ weakly converges to $F \in \mathcal{F}(\Gamma)$, denoted by $F_k \xrightarrow{w} F$, if $\lim_{k\to\infty} F_k(t) = F(t)$ for all $t \in \{t' \in \mathbb{R} | F \text{ is continuous in } t'\}$. Subsequently, we say that a sequence $(X_k)_{k\in\mathbb{N}}$ of random variables converges to the random variable X if and only if the corresponding sequence $(F_k)_{k\in\mathbb{N}}$ of probability distribution functions weakly converges to the probability distribution function F of X. Furthermore, let $(\mathcal{F}(\Gamma), \rho)$ be a metric space with

$$\rho(F, G) = \int_{-\infty}^{\infty} |F(t) - G(t)| e^{-|t|} dt$$

for all $F, G \in \mathcal{F}(\Gamma)$. The following two results can be found in *Feller (1950)* and *Feller (1966)*.

Proposition 2.1 $F_k \xrightarrow{w} F$ if and only if $\lim_{k\to\infty} \rho(F_k, F) = 0$.

Proposition 2.2 Let $(d^k)_{k\in\mathbb{N}}$ and $(r^k)_{k\in\mathbb{N}}$ be convergent sequences in $\mathbb{R}$ with limits d and r, respectively. Take $X \in \mathcal{L}(\Gamma)$. Denote by F the distribution function of $d + rX$ and by F_k the distribution function of $d^k + r^k X$ for all $k \in \mathbb{N}$. Then $F_k \xrightarrow{w} F$.

This proposition has the following implication which will be frequently used in the remainder of this paper. Let a subset $O \subset \mathcal{F}(\Gamma)$ be called open if for each $F \in O$ there exists $\varepsilon > 0$ such that $\{G \in \mathcal{F}(\Gamma) | \rho(F, G) < \varepsilon\} \subset O$. Furthermore, let $(d^k)_{k\in\mathbb{N}}$ and $(r^k)_{k\in\mathbb{N}}$ be convergent sequences in $\mathbb{R}$ with limits d and r, respectively. Take $X \in \mathcal{L}(\Gamma)$ and denote by F and F_k the distribution function of $d + rX$ and $d^k + r^k X$, respectively. Next, let $O \subset \mathcal{F}(\Gamma)$ be an open set such that $F \in O$. Proposition 2.2 and the definition of an open subset then imply that there exists k^0 such that $F_k \in O$ for all $k > k^0$.

For the introduction of a nucleolus we focus on cooperative games with stochastic payoffs $\Gamma = (N, (X_S)_{S\subset N}, (\succsim_i)_{i\in N})$ where the preferences of each player satisfy the following additional conditions:

(C1) continuity, i.e., $\{F_X \in \mathcal{F}(\Gamma)|X \succsim_i Y\}$ and $\{F_X \in \mathcal{F}(\Gamma)|X \precsim_i Y\}$ are closed sets in $(\mathcal{F}(\Gamma), \rho)$ for all $Y \in \mathcal{L}(\Gamma)$, [1] [2]

(C2) for any $X, Y \in \mathcal{L}(\Gamma)$ there exist $\bar{d}, \underline{d} \in \mathbb{R}$ such that $X + \underline{d} \prec_i Y \prec_i X + \bar{d}$,

(C3) for all $X \in \mathcal{L}(\Gamma)$ and all $d > 0$ we have that $X + d \succ_i X$.

Example 2.3 Let the preferences $\succsim_{\alpha_i}$ with $\alpha_i \in (0,1)$ be such that $X \succsim_{\alpha_i} Y$ if and only if $u^X_{\alpha_i} := \sup\{t|F_X(t) < \alpha_i\} \geq u^Y_{\alpha_i} := \sup\{t|F_Y(t) < \alpha_i\}$, where $u^X_{\alpha_i}$ denotes the α_i-quantile of X. This type of preferences may appear in insurance problems. They are used by insurance companies if the premium is determined on the basis of the percentile principle. This type of preferences satisfies conditions (C1) - (C3). To see this, note that $u^{d_i+r_iX}_{\alpha_i} = d_i + r_i u^X_{\alpha_i}$. Then, it is clear that $\succsim_{\alpha_i}$ satisfies (C2) and (C3). For continuity, take $Y \in \mathcal{L}(\Gamma)$. We have to show that the set $\{X \in \mathcal{L}(\Gamma)|X \succsim_{\alpha_i} Y\}$ is a closed set. Therefore, let $(d^k_i + r^k_i X)_{k\in\mathbb{N}}$ be a convergent sequence in $\{X \in \mathcal{L}(\Gamma)|X \succsim_{\alpha_i} Y\}$ and denote its limit by $\bar{X}$. So, $d^k_i + r^k_i u^X_{\alpha_i} \geq u^Y_{\alpha_i}$ for all $k \in \mathbb{N}$. It is sufficient to show that $\bar{X} \succsim_{\alpha_i} Y$, i.e., $u^{\bar{X}}_{\alpha_i} \geq u^Y_{\alpha_i}$. Since the sequence converges we know from Lemma A.3 in Appendix A that there exist convergent subsequences $(d^l_i)_{l\in\mathbb{N}}$ and $(r^l_i)_{l\in\mathbb{N}}$ with limits d_i and r_i, respectively, such that $d_i + r_i X = \bar{X}$. Since $d^l_i + r^l_i X \succsim_{\alpha_i} Y$ implies $d^l_i + r^l_i u^X_{\alpha_i} \geq u^Y_{\alpha_i}$ it follows that $d_i + r_i u^X_{\alpha_i} \geq u^Y_{\alpha_i}$. Consequently, we have that $u^{\bar{X}}_{\alpha_i} \geq u^Y_{\alpha_i}$.

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[1] Since the preferences are complete, an equivalent statement is that $\{F_X \in \mathcal{F}(\Gamma)|X \succ_i Y\}$ and $\{F_X \in \mathcal{F}(\Gamma)|X \prec_i Y\}$ are open sets in $(\mathcal{F}(\Gamma), \rho)$ for all $Y \in \mathcal{L}(\Gamma)$.

[2] For ease of notation, the sets $\{F_X \in \mathcal{F}(\Gamma)|X \succsim_i Y\}$ and $\{F_X \in \mathcal{F}(\Gamma)|X \precsim_i Y\}$ are often denoted by $\{X \in \mathcal{L}(\Gamma)|X \succsim_i Y\}$ and $\{X \in \mathcal{L}(\Gamma)|X \precsim_i Y\}$, respectively.

Example 2.4 Let $\succsim^{b_i}$ with $b_i \in \mathbb{R}$ describe the following preferences. For $X, Y \in \mathcal{L}(\Gamma)$ it holds that $X \succsim^{b_i} Y$ if $E(X) + b_i\sqrt{V(X)} \geq E(Y) + b_i\sqrt{V(Y)}$, where E denotes the expectation and V the variance. This type of preferences can be found for example in portfolio decision theory, where an agent's evaluation of a portfolio depends on the expected revenue of the portfolio and the standard deviation of the revenue. These preferences satisfy conditions (C1) - (C3). To see this, note that

$$E(d_i + r_iX) + b_i\sqrt{V(d_i + r_iX)} = d_i + r_iE(X) + b_ir_i\sqrt{V(X)}$$

holds for $d_i \in \mathbb{R}$ and $r_i \in [0, 1]$. Then the same arguments as in Example 2.3 can be used to show that $\succsim^{b_i}$ satisfies conditions (C1) - (C3).

Example 2.5 Let $\succsim_i$ describe the preferences of an expected utility maximizing player. So, $X \succsim_i Y$ if $E(u_i(X)) \geq E(u_i(Y))$, where E denotes the expectation and u_i is the monotonically increasing utility function of player i. These preferences satisfy conditions (C1) - (C3) if for all $S \subset N$ either $X_S \geq 0$ or $X_S \leq 0$. So, the random payoff of a coalition cannot have both positive and negative realizations. From the fact that u_i is increasing it follows that (C2) and (C3) are satisfied. For the continuity condition (C1) we refer to Lemma A.4 in Appendix A.

In order to define a nucleolus one needs to specify for each coalition $S \subset N$ an excess function E_S. The excess function assigns to each allocation (d, r) of the grand coalition N a real number representing the complaint of coalition S. The larger the complaint of a coalition the more this coalition is dissatisfied with the proposed allocation. For the excess function introduced in this paper we need the following notation. Define

$$I_S(\Gamma) = \{(d, r) \in \mathbb{R}^S \times R^S | \forall_{i \in S} : d_i + r_iX_S \succsim_i X_{\{i\}}\},$$

as the set of possibly nonfeasible individually rational allocations for coalition S. Here, an allocation $(d, r) \in I_S(\Gamma)$ is called feasible if $\sum_{i \in S} d_i \leq 0$. Furthermore, define

$$IR_S(\Gamma) = \{(d, r) \in I_S(\Gamma) | \sum_{i \in S} d_i \leq 0\},$$

as the set of feasible individually rational allocations for coalition S and

$$PO_S(\Gamma) = \{(d,r) \in IR_S(\Gamma) |\ \nexists_{(d',r') \in IR_S(\Gamma)} \forall_{i \in S} : d'_i + r'_i X_S \succ_i d_i + r_i X_S\},$$

as the set of feasible Pareto optimal allocations for S. Note that assumption (C3) implies that $\sum_{i \in S} d_i = 0$ whenever $(d,r) \in PO_S(\Gamma)$. Finally, we make another assumption, namely,

(C4) $I_S(\Gamma) \neq \emptyset$ for all $S \subset N$.

Note that this assumption is satisfied if Γ is superadditive[3]. Moreover, it should be noted that a coalition S is unlikely to be formed when $I_S(\Gamma) = \emptyset$. Since in that case for every allocation of X_S there is at least one member of S whose payoff is not individually rational. Hence, he would be better off by leaving coalition S and form a coalition on his own. Finally, denote by $CG(N)$ the class of all cooperative games with stochastic payoffs with player set N satisfying conditions (C1) - (C4).

For gaining a clearer insight into the situation and the (forthcoming) mathematics in particular, we make use of a simplified graphical representation of the problem. At the moment this might seem a bit overdone, but for the remainder of this paper these figures might turn out to be very helpful. The notions introduced in the preceding paragraph are illustrated in Figure 1.

Figure 1 represents a cooperative game with stochastic payoffs with two expected utility maximizing players. The axes represent the utility levels of the players. For simplicity, we have assumed that payoffs are individually rational if and only if the corresponding expected utility is greater than or equal to zero. So, the set $I_S(\Gamma)$ is represented by the positive orthant. Furthermore, the set $IR_S(\Gamma)$ of individually rational allocations is depicted by the shaded area, and the set $PO_S(\Gamma)$ of Pareto optimal allocations is depicted by the

[3] A game $\Gamma = (N, (X_S)_{S \subset N}, (\succsim_i)_{i \in N})$ is called superadditive if for all disjunct $S, T \subset N$ the following statement is true. For each allocation (d^S, r^S) of X_S and each allocation (d^T, r^T) of X_T there exists an allocation (d,r) of $X_{S \cup T}$ such that $d_i + r_i X_{S \cup T} \succsim_i d_i^S + r_i^S X_S$ for all $i \in S$ and $d_i + r_i X_{S \cup T} \succsim_i d_i^T + r_i^T X_S$ for all $i \in T$. So whatever the allocation of X_S and X_T are, there is always an allocation of $X_{S \cup T}$ such that all members of $S \cup T$ are (weakly) better off.

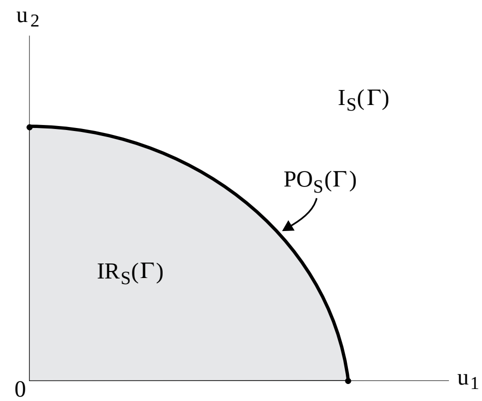

FIGURE 1: Individually rational and Pareto optimal allocations in a two-player example.

bold printed curve. Finally, note that this and the forthcoming figures do not arise from a concrete example.

In Figure 1 both $IR_S(\Gamma)$ and $PO_S(\Gamma)$ are compact subsets. The following propositions show that this holds in general for the class $CG(N)$ of cooperative games with stochastic payoffs.

Proposition 2.6 $IR_S(\Gamma)$ is a compact subset of $I_S(\Gamma)$ for each coalition $S \subset N$.

PROOF: See Appendix B. □

Proposition 2.7 The set of Pareto optimal allocations $PO_S(\Gamma)$ is a compact subset of $I_S(\Gamma)$ for each coalition $S \subset N$.

PROOF: See Appendix B. □

Furthermore, we need to consider the following sets. Define for each $S \subset N$

$$PD_S(\Gamma) = \{(d,r) \in I_S(\Gamma) | \exists_{(d',r') \in PO_S(\Gamma)} \forall_{i \in S} : d'_i + r'_i X_S \succsim_i d_i + r_i X\}$$

as the set of (possibly nonfeasible) allocations that are (weakly) dominated by a Pareto optimal allocation, and

$$NPD_S(\Gamma) = \{(d,r) \in I_S(\Gamma) | \nexists_{(d',r') \in PO_S(\Gamma)} \forall_{i \in S} : d'_i + r'_i X_S \succsim_i d_i + r_i X\}$$

as the set of (possibly nonfeasible) allocations that are not dominated by Pareto optimal allocations. Note that $IR_S(\Gamma) \subset PD_S(\Gamma)$. The reverse, however, need not be true, as the next example shows.

Example 2.8 Consider the following two player example. Let X_S be such that $-X_S$ is exponentially distributed with expectation equal to 1 for all $S \subset N$. Furthermore let players 1 and 2 be expected utility maximizers with utility functions $u_1(t) = -e^{-0.5t}$ and $u_2(t) = -e^{-0.25t}$, respectively. Then $E(u_1(d_1 + r_1 X_{\{1,2\}})) = -e^{-d_1} \frac{1}{1-0.5r_1}$ and $E(u_2(d_2 - r_2 X_{\{1,2\}})) = -e^{-d_2} \frac{1}{1-0.25r_2}$. An allocation $(d,r) \in I_{\{1,2\}}(\Gamma)$ is individually rational if $E(u_1(d_1 + r_1 X_{\{1,2\}})) \geq -2$ and $E(u_2(d_2 + r_2 X_{\{1,2\}})) \geq -1.25$. Furthermore, (d, r^*) is Pareto optimal if and only if $r_1^* = \frac{1}{3}$ and $r_2^* = \frac{2}{3}$ (see *Wilson (1968)*). Now, consider the allocation (d,r) with $d_1 = 0.1$, $d_2 = 0.1$, $r_1 = 1$ and $r_2 = 0$. Since $d_1 + d_2 > 0$ this allocation is nonfeasible. However, the Pareto optimal allocation (d^*, r^*) with $d_1^* = -0.9$, $d_2^* = 0.9$, $r_1^* = \frac{1}{3}$ and $r_2^* = \frac{2}{3}$ is feasible and preferred by both players. Indeed,

$$E(u_1(d_1^* + r_1^* X_{\{1,2\}})) = -1.8820 > -1.9025 = E(u_1(d_1 + r_1 X_{\{1,2\}}))$$

and

$$E(u_2(d_2^* + r_2^* X_{\{1,2\}})) = -0.9582 > -0.9753 = E(u_2(d_2 + r_2 X_{\{1,2\}})).$$

So even nonfeasible allocations can be Pareto dominated.

The next proposition states a very intuitive result. Namely that for every Pareto dominated allocation (d,r) and every non-Pareto dominated allocation (d',r'), which all members of S weakly prefer to the Pareto dominated allocation (d, r), there exists a Pareto optimal allocation such that for each player the Pareto optimal allocation is weakly better than (d, r) but weakly worse than (d', r').

Proposition 2.9 Let $\Gamma \in CG(N)$. Take $(d, r) \in PD_S(\Gamma)$ and $(\tilde{d}, \tilde{r}) \in NPD_S(\Gamma)$ such that $d_i + r_i X_S \precsim_i \tilde{d}_i + \tilde{r}_i X_S$ for all $i \in S$. Then there exists $(\hat{d}, \hat{r}) \in PO_S(\Gamma)$ such that

$$d_i + r_i X_S \precsim_i \hat{d}_i + \hat{r}_i X_S \precsim_i \tilde{d}_i + \tilde{r}_i X_S$$

for all $i \in S$.

PROOF: See Appendix B. □

A direct consequence of this proposition is that for each allocation $(d, r) \in IR_S(\Gamma)$ there exists a Pareto optimal allocation (d', r') such that $d'_i + r'_i X_S \succsim_i d_i + r_i X_S$ for all $i \in S$. Moreover, since $I_S(\Gamma)$ is nonempty by assumption (C4) we have that for each $(d, r) \in NPD_S(\Gamma)$ there exists $(d', r') \in PO_S(\Gamma)$ such that $d'_i + r'_i X_S \precsim_i d_i + r_i X_S$ for all $i \in S$.

Finally, we introduce three more sets. Therefore, let $(d, r) \in IR_N(\Gamma)$ be an individually rational allocation for the grand coalition N. Take $S \subset N$ and define

$$W_S((d, r)) = \{(d', r') \in IR_S(\Gamma) | \forall_{i \in S} : d'_i + r'_i X_S \precsim_i d_i + r_i X_N\}$$

as the set of individually rational allocations for coalitions S which are weakly worse than the payoff $d_i + r_i X_N$ for every member of S, and,

$$B_S((d, r)) = \{(d', r') \in IR_S(\Gamma) | \forall_{i \in S} : d'_i + r'_i X_S \succsim_i d_i + r_i X_N\}$$

as the set of individually rational allocations for coalition S which are weakly better than the payoff $d_i + r_i X_N$ for every member of S. Furthermore, define

$$PO^*_S((d, r)) = (W_S((d, r)) \cup B_S((d, r))) \cap PO_S(\Gamma),$$

as the set of Pareto optimal allocations for coalition S which are either weakly worse than $d_i + r_i X_N$ for all members of S or weakly better than $d_i + r_i X_N$ for all members of S. These three sets are illustrated in Figure 2. Note that $B_S((d, r))$ can be empty.

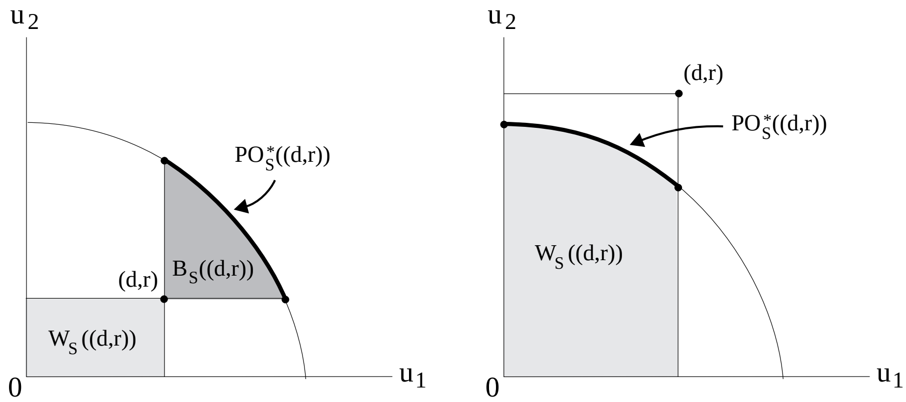

FIGURE 2: Examples of $W_S((d,r))$, $B_S((d,r))$ and $PO^*_S((d,r))$.

3 A nucleolus for stochastic cooperative games

With the definitions and notions introduced in the previous section we can now define an excess function and, consequently, a nucleolus for cooperative games with stochastic payoffs. The excess function $E_S : IR_N(\Gamma) \to \mathbb{R}$ of coaltition S is defined as follows. Take $(d,r) \in IR_N(\Gamma)$. Then the excess for coalition S is defined by

$$E_S((d,r)) = \min_{(d',r') \in PO^*_S((d,r))} \{\sum_{i \in S} \delta_i | \forall_{i \in S} : \delta_i \in \mathbb{R} \text{ and } d'_i + r'_i X_S \sim_i d_i + r_i X_N + \delta_i\}.$$

For an interpretation of the excess, let us focus on the core conditions. So, given a proposed allocation (d,r) does a coalition S have an incentive to leave the grand coalition or not.

First, consider again the excess as used in *Schmeidler (1969)*. There, the excess can be interpeted as the minimum amount of money a coalition needs on top of what they already receive from the proposed allocation such that they are indifferent between staying in the grand coalition and leaving the grand coalition. This interpretation is now applied to stochastic cooperative games. For this, note that given an allocation $(d,r) \in IR_N(\Gamma)$ a coalition S is indifferent between staying in the grand coalition N and leaving if there exists an allocation $(d',r') \in PO^*_S((d,r))$ such that each player $i \in S$ is indifferent between receiving the payoff $d'_i + r'_i X_S$ and the payoff $d_i + r_i X_N$. So, coalition cannot do strictly better by leaving the grand coalition but if they do split off they can allocate their payoff in such a way that no member is strictly worse off.

Now, suppose that a coalition S has an incentive to part company with the grand coalition N. So, there exists an allocation $(\tilde{d}, \tilde{r}) \in IR_S(\Gamma)$ such that each player $i \in S$ strictly prefers the payoff $\tilde{d}_i + \tilde{r}_i X_S$ to the payoff $d_i + r_i X_N$. To keep this coalition in the grand coalition the payoff to the members of S must increase. This can be done by giving each member $i \in S$ a deterministic amount of money δ_i. Hence, their payoff becomes $d_i + \delta_i + r_i X_N$. The excess of coalition S then equals the minimal amount of money they need so that they are just willing to stay in the grand coalition.

Next, suppose that a coalition S does not have an incentive to split off from the grand coalition. Hence, this coalition receives more than they can achieve on their own. Consequently, one can decrease the payoff of each member $i \in S$ with a deterministic amount δ_i. Then the excess equals the maximal amount of money that can be taken away from this coalition such that they are still staying in the grand coalition.

Summarizing, the excess $E_S((d, r))$ represents the minimum amount of money that coalition S needs in order to be satisfied with the allocation (d, r). Moreover, if (d, r) and (d', r') are allocations of X_N such that each player $i \in S$ prefers $d_i + r_i X_N$ to $d'_i + r'_i X_N$ then $E_S((d, r)) < E_S((d', r'))$. Hence, the excess decreases when each player $i \in S$ improves his payoff. So, in a specific way the excess $E_S((d, r))$ describes how much coalition S is satisfied with the allocation (d, r). Finally, since all players' preferences are monotonically increasing in the amount of money d they receive (see assumption (C3)) it is reasonable to say that one coalition is more satisfied with a particular allocation than another coalition if the first coalition needs less money to be satisfied than the latter one, or, in other words, if the excess of the first coalition is less than the excess of the latter. This last observation leads to the following definition of a nucleolus.

Let $\Gamma = (N, (X_S)_{S \subset N}, (\succsim_i)_{i \in N})$ be a cooperative game with stochastic payoffs and let

$$E_S((d, r)) = \min_{(d', r') \in PO^*_S((d,r))} \{\sum_{i \in S} \delta_i | \; \forall_{i \in S} : (d', r')_i \sim_i (d, r)_i + \delta_i\}. \tag{2}$$

describe the excess of coalition S at allocation $(d,r) \in IR_N(\Gamma)$. Next, denote by $E((d,r))$ the vector of excesses at allocation (d,r) and let $\theta \circ E((d,r))$ denote the vector of excesses ordered in a decreasing order. The nucleolus

$\mathcal{N}(\Gamma)$ of the game $\Gamma \in CG(N)$ is then defined by

$$\mathcal{N}(\Gamma) = \{(d,r) \in IR_N(\Gamma) | \forall_{(d',r') \in IR_N(\Gamma)} : \theta \circ E((d,r)) \leq_{lex} \theta \circ E((d',r'))\}, \quad (3)$$

where $\leq_{lex}$ is the lexicographic ordering. Next, we show that the nucleolus is a well defined solution concept for the games discussed in this paper.

In proving the nonemptiness of the nucleolus $\mathcal{N}(\Gamma)$ we make use of the results stated in *Maschler, Potters* and *Tijs (1992)*. They introduced a nucleolus for a more a general framework and showed that the nucleolus is nonempty if the domain is compact and the excess functions are continuous. Thus, we have to show that $IR_N(\Gamma)$ is compact and that $E_S((d,r))$ is continuous in (d,r) for each $(d,r) \in IR_N(\Gamma)$ and each $S \subset N$. The compactness of $IR_N(\Gamma)$ follows immediately from Proposition 2.6. The continuity proof is a bit more complicated and consists of the following parts.

First we show that $PO^*_S((d,r))$ is a nonempty compact subset of $PO_S(\Gamma)$. Then we introduce the following multifunction

$$\mathcal{E}_S((d,r)) = \{\sum_{i \in S} \delta_i | \exists_{(d',r') \in PO^*_S((d,r))} : d'_i + r'_i X_S \sim_i d_i + r_i X_N + \delta_i\}.$$

Hence, $E_S((d,r)) = \min \mathcal{E}_S((d,r))$. In the next step we show that $\mathcal{E}_S((d,r))$ is a compact subset of $\mathbb{R}$ for each allocation $(d,r) \in IR_N(\Gamma)$. This implies that the minimum in (2) exists. Subsequently, we show that this multifunction is both upper and lower semi continuous, which then implies that the excess function E_S is continuous.

Proposition 3.1 $PO^*_S((d,r))$ is a nonempty compact subset of $PO_S(\Gamma)$.

PROOF: That $PO^*_S((d,r))$ is compact follows from the facts that $W_S((d,r))$ and $B_S((d,r))$ are closed by the continuity condition (C1) and $PO_S(\Gamma)$ is compact. To show that it is nonempty let us distinguish two cases.

First, let $B_S((d,r)) \neq \emptyset$. Then there exists $(d',r') \in IR_S(\Gamma)$ such that $d'_i + r'_i X_S \succsim_i d_i + r_i X_S$ for all $i \in S$. Since $(d',r') \in IR_S(\Gamma)$ we know from Proposition 2.9 that there exists $(\bar{d},\bar{r}) \in PO_S(\Gamma)$ such that $\bar{d}_i + \bar{r}_i X_S \succsim_i d'_i + r'_i X_S$ for all $i \in S$. Hence, $(\bar{d},\bar{r}) \in PO_S(\Gamma)$ and $(\bar{d},\bar{r}) \in B_S((d,r))$. Consequently, $(\bar{d},\bar{r}) \in PO^*_S((d,r))$.

Second, let $B_S((d,r)) = \emptyset$. Take $(\tilde{d},\tilde{r}) \in I_S(\Gamma)$ such that $\tilde{d}_i + \tilde{r}_i X_S \sim_i d_i + r_i X_N$ for all $i \in S$. From $B_S((d,r)) = \emptyset$ it follows that $(\tilde{d},\tilde{r}) \in NPD_S(\Gamma)$. Proposition 2.9 then implies that there exists $(\bar{d},\bar{r}) \in PO_S(\Gamma)$ such that $\bar{d}_i + \bar{r}_i X_S \precsim_i \tilde{d}_i + \tilde{r}_i X_S$ for all $i \in S$. Hence, $(\bar{d},\bar{r}) \in W_S((d,r))$ and, consequently, $(\bar{d},\bar{r}) \in PO^*_S((d,r))$. □

Next, consider again the multifunction $\mathcal{E}_S : IR_N(\Gamma) \twoheadrightarrow \mathbb{R}$ defined by

$$\mathcal{E}_S((d,r)) = \{\sum_{i\in S} \delta_i | \exists_{(d',r')\in PO^*_S((d,r))} : d'_i + r'_i X_S \sim_i d_i + r_i X_N + \delta_i\}.$$

Proposition 3.2 Let $(d,r) \in IR_N(\Gamma)$. Then $\mathcal{E}_S((d,r))$ is a compact subset of $\mathbb{R}$.

PROOF: We have to show that $\mathcal{E}_S((d,r))$ is closed and bounded. That $\mathcal{E}_S((d,r))$ is bounded follows from the compactness of $PO^*_S((d,r))$ and the fact that for each $(d',r') \in PO^*_S((d,r))$ the number δ_i is uniquely determined by conditions (C1) - (C2). To see that $\mathcal{E}_S((d,r))$ is closed, let $(\sum_{i\in S} \delta^k_i)_{k\in\mathbb{N}}$ be a convergent sequence[4] in $\mathcal{E}_S((d,r))$ with limit $\sum_{i\in S} \delta_i$. We have to show that $\sum_{i\in S} \delta_i \in \mathcal{E}_S((d,r))$. Therefore, let $((\bar{d}^k,\bar{r}^k))_{k\in\mathbb{N}}$ be a sequence in $PO^*_S((d,r))$ such that $\bar{d}^k_i + \bar{r}^k_i X_S \sim_i d_i + \delta^k_i + r_i X_N$ for all $i \in S$. Since $PO^*_S((d,r))$ is compact there exists a convergent subsequence $((\bar{d}^l,\bar{r}^l))_{l\in\mathbb{N}}$ with limit $(\bar{d},\bar{r}) \in PO^*_S((d,r))$. Take $\bar{\delta}_i \in \mathbb{R}$ such that $\bar{d}_i + \bar{r}_i X_S \sim_i d_i + \bar{\delta}_i + r_i X_N$ for all $i \in S$. Note that $\sum_{i\in S} \bar{\delta}_i \in \mathcal{E}_S((d,r))$. The proof is finished if we can show that $\delta_i = \bar{\delta}_i$ for all $i \in S$. Therefore, let $\varepsilon > 0$ and $i \in S$. Define

$$V^\varepsilon_i = \{Y \in \mathcal{L}(\Gamma) | \bar{d}_i + \bar{r}_i X_S - \varepsilon \prec_i Y \prec_i \bar{d}_i + \bar{r}_i X_S + \varepsilon\}.$$

[4]Formally, it would be more correct to start with a convergent sequence $(a^k)_{k\in N}$ in $\mathcal{E}_S((d,r))$. Then $a^k \in \mathcal{E}_S((d,r))$ and the definition of $\mathcal{E}_S$ imply that there exist δ^k_i such that $d_i + \delta^k_i + r_i X_N \sim_i d'_i + r'_i X_S$ ($i \in S$) for some $(d',r') \in PO^*_S((d,r))$ and $\sum_{i\in S} \delta^k_i = a^k$. Consequently, the sequence $(a^k)_{k\in\mathbb{N}}$ can be replaced by a sequence $(\sum_{i\in S} \delta^k_i)_{k\in\mathbb{N}}$.

Since V_i^ε is open by the continuity of $\succsim_i$, $(\bar{d}^l, \bar{r}^l) \to (\bar{d}, \bar{r})$ and $\bar{d}_i + \bar{r}_i X_S \in V_i^\varepsilon$ there exists $L^\varepsilon \in \mathbb{N}$ such that $\bar{d}_i^l + \bar{r}_i^l X_S \in V_i^\varepsilon$ for all $l > L^\varepsilon$. This implies that $d_i + \delta_i^l + r_i X_N \in V_i^\varepsilon$ for all $l > L^\varepsilon$. Since $\varepsilon > 0$ was arbitrarily chosen it follows that

$$\lim_{l\to\infty} \left(d_i + \delta_i^l + r_i^l X_N \right) = d_i + \delta_i + r_i X_N \in \cap_{\varepsilon>0} V_i^\varepsilon.$$

Hence, $d_i + \delta_i + r_i X_N \sim_i \bar{d}_i + \bar{r}_i X_S$. Since by definition it holds that $\bar{d}_i + \bar{r}_i X_S \sim_i d_i + \bar{\delta}_i + r_i X_N$ it follows by assumption (C3) that $\delta_i = \bar{\delta}_i$. □

Lemma 3.3 $\mathcal{E}_S((d,r))$ is upper semi continuous in (d,r) for all $(d,r) \in IR_N(\Gamma)$.

PROOF: Let $((d^k, r^k))_{k\in\mathrm{N}}$ be a sequence in $IR_N(\Gamma)$ converging to (d,r). Take $\sum_{i\in S} \delta_i^k \in \mathcal{E}_S((d^k, r^k))$ such that $\sum_{i\in S} \delta_i^k$ converges to $\sum_{i\in S} \delta_i$. For upper semi continuity to be satisfied it is sufficient to show that $\sum_{i\in S} \delta_i \in \mathcal{E}_S((d,r))$.

First, take $(\bar{d}^k, \bar{r}^k) \in PO_S^*((d^k, r^k))$ such that $\bar{d}_i^k + \bar{r}_i^k X_S \sim_i d_i^k + \delta_i^k + r_i^k X_N$ for all $i \in S$. Since $((\bar{d}^k, \bar{r}^k))_{k\in\mathrm{N}}$ is a sequence in the compact set $PO_S(\Gamma)$ there exists a convergent subsequence $((\bar{d}^l, \bar{r}^l))_{l\in\mathrm{N}}$ with limit $(\bar{d}, \bar{r}) \in PO_S(\Gamma)$. Moreover, it holds that $\bar{d}_i + \bar{r}_i X_S \sim_i d_i + \delta_i + r_i X_N$ for all $i \in S$. To see this, take $\varepsilon > 0$ and $i \in S$. Define

$$V_i^\varepsilon = \{Y \in \mathcal{L}(\Gamma) | \bar{d}_i + \bar{r}_i X_S - \varepsilon \prec_i Y \prec_i \bar{d}_i + \bar{r}_i X_S + \varepsilon\}.$$

Since V_i^ε is open by the continuity of $\succsim_i$, $(\bar{d}^l, \bar{r}^l) \to (\bar{d}, \bar{r})$ and $\bar{d}_i + \bar{r}_i X_S \in V_i^\varepsilon$ there exists $L^\varepsilon \in \mathbb{N}$ such that $\bar{d}_i^l + \bar{r}_i^l X_S \in V_i^\varepsilon$ for all $l > L^\varepsilon$. This implies that $d_i^l + \delta_i^l + r_i^l X_N \in V_i^\varepsilon$ for all $l > L^\varepsilon$. Since $\varepsilon > 0$ is arbitrary we have that

$$\lim_{l\to\infty} \left(d_i^l + \delta_i^l + r_i^l X_N \right) = d_i + \delta_i + r_i X_N \sim_i \bar{d}_i + \bar{r}_i X_S.$$

The proof is finished if we can show that $(\bar{d}, \bar{r}) \in PO_S^*((d,r))$. Therefore, take $\varepsilon > 0$ and define

$$\begin{aligned}
W_S^\varepsilon((d,r)) &= \{(d',r') \in IR_S(\Gamma) | \forall_{i\in S} : d_i' + r_i' X_S \precsim_i d_i + r_i X_N + \varepsilon\}, \\
B_S^\varepsilon((d,r)) &= \{(d',r') \in IR_S(\Gamma) | \forall_{i\in S} : d_i' + r_i' X_S \succsim_i d_i + r_i X_N - \varepsilon\}, \\
PO_S^{*\varepsilon}((d,r)) &= (W_S^\varepsilon((d,r)) \cup B_S^\varepsilon((d,r))) \cap PO_S(\Gamma).
\end{aligned}$$

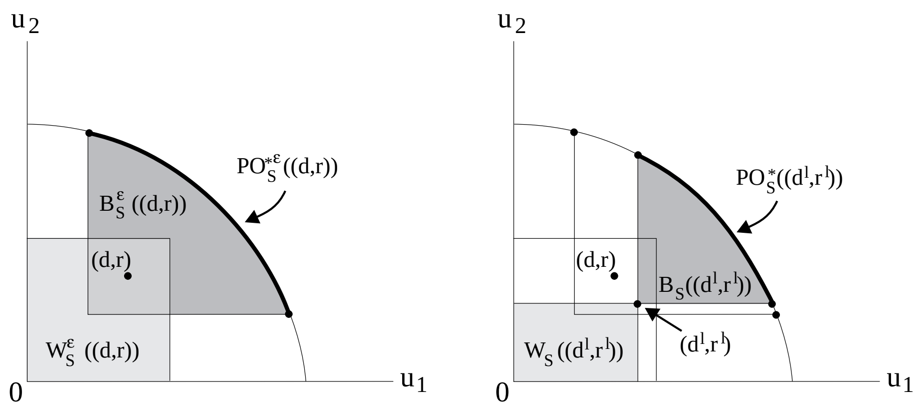

FIGURE 3: A graphical representation of $PO_S^{*\varepsilon}((d,r))$ and $PO_S^*((d^l,r^l))$.

We refer to the left figure of Figure 3 for a graphical interpretation of these sets. Note that $W_S((d,r)) = \cap_{\varepsilon>0} W_S^\varepsilon((d,r))$, $B_S((d,r)) = \cap_{\varepsilon>0} B_S^\varepsilon((d,r))$ and $PO_S^*((d,r)) = \cap_{\varepsilon>0} PO_S^{*\varepsilon}((d,r))$. Furthermore, define

$$V^\varepsilon = \{Y \in \mathcal{L}(\Gamma)^S | \forall_{i\in S} : d_i + r_i X_N - \varepsilon \prec_i Y_i \prec_i d_i + r_i X_N + \varepsilon\}.$$

Since V^ε is open by the continuity of $\succsim_i$, $(d^l,r^l) \to (d,r)$ and $(d_i + r_i X_N)_{i\in S} \in V^\varepsilon$ there exists $L^\varepsilon \in \mathbb{N}$ such that $(d_i^l + r_i^l X_N)_{i\in S} \in V^\varepsilon$ for all $l > L^\varepsilon$. This implies that $(d^l,r^l) \in W_S^\varepsilon((d,r))$ and $(d^l,r^l) \in B_S^\varepsilon((d,r))$ for all $l > L^\varepsilon$ (see also the right figure in Figure 3). Hence, $W_S((d^l,r^l)) \subset W_S^\varepsilon((d,r))$ and $B_S((d^l,r^l)) \subset B_S^\varepsilon((d,r))$ for all $l > L^\varepsilon$. Consequently, we have for all $l > L^\varepsilon$ that $PO_S^*((d^l,r^l)) \subset PO_S^{*\varepsilon}((d,r))$. In particular, we have $(\bar{d}^l,\bar{r}^l) \in PO_S^{*\varepsilon}((d,r))$ for all $l > L^\varepsilon$. Hence,

$$\lim_{l\to\infty}(\bar{d}^l,\bar{r}^l) = (\bar{d},\bar{r}) \in \cap_{\varepsilon>0} PO_S^{*\varepsilon}((d,r)) \;=\; PO_S^*((d,r)). \qquad \square$$

Lemma 3.4 $\mathcal{E}_S((d,r))$ is lower semi continuous in (d,r) for all $(d,r) \in IR_N(\Gamma)$.

PROOF: Let $((d^k, r^k))_{k\in\mathbb{N}}$ be a sequence converging to (d, r) and let $\sum_{i\in S}\delta_i \in \mathcal{E}_S((d, r))$. To prove lower semi continuity it suffices to show that there exists a sequence $(\sum_{i\in S}\delta_i^k)_{k\in\mathbb{N}}$ with $\sum_{i\in S}\delta_i^k \in \mathcal{E}_S((d^k, r^k))$ for all $k \in \mathbb{N}$ such that $\sum_{i\in S}\delta_i^k$ converges to $\sum_{i\in S}\delta_i$.

First, note that since $IR_N(\Gamma)$ is compact and $\mathcal{E}_S$ is upper semi continuous that

$$\mathcal{E}_S(IR_N(\Gamma)) = \cup_{(d,r)\in IR_N(\Gamma)} E_S((d, r))$$

is a compact subset of $\mathbb{R}$. Second, note that if there exists a sequence $(\sum_{i\in S}\delta_i^k)_{k\in\mathbb{N}}$ with $\sum_{i\in S}\delta_i^k \in \mathcal{E}_S((d^k, r^k))$ for each $k \in \mathbb{N}$ such that every convergent subsequence $(\sum_{i\in S}\delta_i^l)_{l\in\mathbb{N}}$ converges to $\sum_{i\in S}\delta_i$ then the compactness of $\mathcal{E}_S(IR_N(\Gamma))$ implies that the sequence $(\sum_{i\in S}\delta_i^k)_{k\in\mathbb{N}}$ converges to $\sum_{i\in S}\delta_i$.

Take $(\bar{d}, \bar{r}) \in PO_S^*((d, r))$ such that $\bar{d}_i + \bar{r}_i X_S \sim_i d_i + \delta_i + r_i X_N$ for all $i \in S$. Let $\varepsilon > 0$ and define

$$V^\varepsilon = \left\{ Y \in \mathcal{L}(\Gamma)^S \;\middle|\; \begin{array}{l} \forall_{i\in S} : Y_i \succ_i \bar{d}_i + \bar{r}_i X_S - \varepsilon \text{ or} \\ \forall_{i\in S} : Y_i \prec_i \bar{d}_i + \bar{r}_i X_S + \varepsilon \end{array} \right\},$$

and

$$C^\varepsilon = \left\{ Y \in \mathcal{L}(\Gamma)^S \;\middle|\; \begin{array}{l} \forall_{i\in S} : Y_i \succsim_i \bar{d}_i + \bar{r}_i X_S - \varepsilon \text{ or} \\ \forall_{i\in S} : Y_i \precsim_i \bar{d}_i + \bar{r}_i X_S + \varepsilon \end{array} \right\}.$$

Note that V^ε is open and C^ε is closed by the continuity of $\succsim_i$ for each $i \in S$. Next, we show that if $(d_i^k + r_i^k X_N)_{i\in S} \in V^\varepsilon$ then there exists $(\bar{d}^k, \bar{r}^k) \in PO_S^*((d^k, r^k))$ such that $(\bar{d}_i^k + \bar{r}_i^k X_S)_{i\in S} \in V^\varepsilon$. Therefore, let $k \in \mathbb{N}$ be such that $(d^k, r^k) \in V^\varepsilon$ and let $(\tilde{d}, \tilde{r}) \in I_S(\Gamma)$ be such that $\tilde{d}_i + \tilde{r}_i X_S \sim_i d_i^k + r_i^k X_N$ for all $i \in S$. We distinguish the following three cases.

First, suppose that $(\tilde{d}, \tilde{r}) \in NPD_S(\Gamma)$. Since $(\tilde{d}_i + \tilde{r}_i X_S)_{i\in S} \in V^\varepsilon$ it holds that $\tilde{d}_i + \tilde{r}_i X_S \succ_i \bar{d}_i + \bar{r}_i X_S - \varepsilon$ for all $i \in S$. From $(\bar{d} - \frac{1}{2}(\varepsilon, \varepsilon, \ldots, \varepsilon), \bar{r}) \in PD_S(\Gamma)$ and Proposition 2.9 it follows that there exists $(\bar{d}^k, \bar{r}^k) \in PO_S(\Gamma)$ such that

$$\bar{d}_i + \bar{r}_i X_S - \tfrac{1}{2}\varepsilon \precsim_i \bar{d}_i^k + \bar{r}_i^k X_S \precsim_i \tilde{d}_i + \tilde{r}_i X_S$$

for all $i \in S$. Thus, $(\bar{d}_i^k + \bar{r}_i^k X_S)_{i\in S} \in V^\varepsilon$. Since $\tilde{d}_i + \tilde{r}_i X_S \sim_i d_i^k + r_i^k X_N$ for all $i \in S$ it holds that $(\bar{d}^k, \bar{r}^k) \in W_S((d^k, r^k))$. Hence, $(\bar{d}^k, \bar{r}^k) \in PO_S^*((d^k, r^k))$.

Second, suppose that $(\tilde{d},\tilde{r}) \in PD_S(\Gamma)$ and that $\tilde{d}_i + \tilde{r}_i X_S \prec_i \bar{d}_i + \bar{r}_i X_S + \varepsilon$ for all $i \in S$. Since the Pareto optimality of $(\bar{d},\bar{r})$ implies that $(\bar{d}+\frac{1}{2}(\varepsilon,\varepsilon,\ldots,\varepsilon),\bar{r}) \in NPD_S(\Gamma)$ it follows from Proposition 2.9 that there exists $(\bar{d}^k,\bar{r}^k) \in PO_S(\Gamma)$ such that

$$\tilde{d}_i + \tilde{r}_i X_S \precsim_i \bar{d}_i^k + \bar{r}_i^k X_S \precsim_i \bar{d}_i + \bar{r}_i X_S + \tfrac{1}{2}\varepsilon$$

for all $i \in S$. Thus, $(\bar{d}_i^k + \bar{r}_i^k X_S)_{i\in S} \in V^\varepsilon$. Since $\tilde{d}_i + \tilde{r}_i X_S \sim_i d_i^k + r_i^k X_N$ for all $i \in S$ it holds that $(\bar{d}^k,\bar{r}^k) \in B_S((d^k,r^k))$. Hence, $(\bar{d}^k,\bar{r}^k) \in PO_S^*((d^k,r^k))$.

Finally, suppose that $(\tilde{d},\tilde{r}) \in PD_S(\Gamma)$ and that $\tilde{d}_i + \tilde{r}_i X_S \succ_i \bar{d}_i + \bar{r}_i X_S - \varepsilon$ for all $i \in S$. Then Proposition 2.9 implies that there exists $(\bar{d}^k,\bar{r}^k) \in PO_S(\Gamma)$ such that

$$\bar{d}_i + \bar{r}_i X_S - \varepsilon \precsim_i \tilde{d}_i + \tilde{r}_i X_S \precsim_i \bar{d}_i^k + \bar{r}_i^k X_S$$

for all $i \in S$. Thus, $(\bar{d}^k,\bar{r}^k) \in B_S((d^k,r^k))$. Therefore we have that $(\bar{d}^k,\bar{r}^k) \in PO_S^*((d^k,r^k))$. Moreover, $(\bar{d}_i^k + \bar{r}_i^k X_S)_{i\in S} \in V^\varepsilon$.

Now we are able to construct a sequence $(\sum_{i\in S}\delta_i^k)_{k\in\mathbb{N}}$ with $\sum_{i\in S}\delta_i^k \in PO_S^*((d^k,r^k))$ for each $k \in \mathbb{N}$ such that each convergent subsequence converges to $\sum_{i\in S}\delta_i$.

Let $(\varepsilon^m)_{m\in\mathbb{N}}$ be a strictly decreasing sequence such that $\varepsilon^m > 0$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty}\varepsilon^m = 0$. Hence, $(V^{\varepsilon^m})_{m\in\mathbb{N}}$ is a decreasing sequence in the sense that $V^{\varepsilon^m} \subset V^{\varepsilon^{m'}}$ if $m > m'$. Define $V^0 = \cap_{\varepsilon>0}V^\varepsilon$. From $(\bar{d},\bar{r}) \in PO_S^*((d,r))$ it follows that $(d_i + r_i X_N)_{i\in S} \in V^0$. Hence, $(d_i + r_i X_N)_{i\in S} \in V^\varepsilon$ for all $\varepsilon > 0$. Since (d^k,r^k) converges to (d,r) there exists $K^1 \in \mathbb{N}$ such that for all $k > K^1$ it holds that $(d_i^k + r_i^k X_S)_{i\in S} \in V^{\varepsilon^1}$. Next, take $k \in \mathbb{N}$. If $k \le K^1$ then take $\sum_{i\in S}\delta_i^k \in \mathcal{E}_S((d^k,r^k))$ arbitrary. If $k > K^1$ we distinguish the following two cases.

In the first case, suppose that $(d_i^k + r_i^k X_N)_{i\in S} \in V^0$. Then $(\bar{d},\bar{r}) \in PO_S^*((d^k,r^k))$ and $(\bar{d}_i + \bar{r}_i X_S)_{i\in S} \in V^0$. So, we can take $(\bar{d}^k,\bar{r}^k)$ equal to $(\bar{d},\bar{r})$ (See the left figure of Figure 4).

In the second case, let $(d_i^k + r_i^k X_N)_{i\in S} \notin V^0$. Then there exists $m(k) \in \mathbb{N}$ such that $(d_i^k + r_i^k X_N)_{i\in S} \in V^{\varepsilon^{m(k)}} \backslash V^{\varepsilon^{m(k)+1}}$. Subsequently, take $(\bar{d}^k,\bar{r}^k) \in PO_S^*((d^k,r^k))$ such that $(\bar{d}_i^k + \bar{r}_i^k X_S)_{i\in S} \in V^{\varepsilon^{m(k)}}$ (See the right figure of Figure 4, where the bold printed curve represents the set of allocations that belong to both $PO_S^*((d^k,r^k))$ and $V^{\varepsilon^{m(k)}}$). That

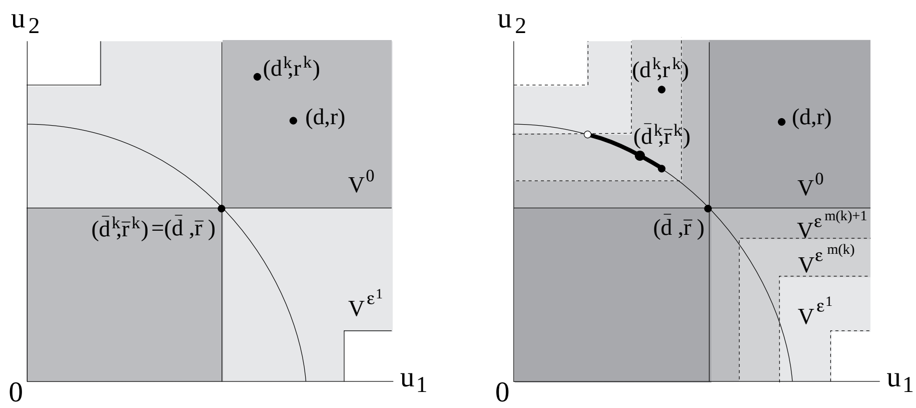

FIGURE 4: The choice of the allocation (d^k, r^k) for the two different cases.

such $(\bar{d}^k, \bar{r}^k)$ exists can be seen as follows. Let $(d', r') \in I_S(\Gamma)$ be such that $d'_i + r'_i X_S \sim_i d^k_i + r^k_i X_N$ for all $i \in S$. Then either $(d', r') \in PD_S(\Gamma)$ or $(d', r') \in NPD_S(\Gamma)$.

For the case that $(d', r') \in PD_S(\Gamma)$ then $(d^k_i + r^k_i X_N)_{i \in S} \in V^{\varepsilon^{m(k)}}$ implies that $(d'_i + r'_i X_S)_{i \in S} \in V^{\varepsilon^{m(k)}}$ and, consequently, that

$$d'_i + r'_i X_S \succsim_i \bar{d}_i + \bar{r}_i X_S - \varepsilon^{m(k)}$$

for all $i \in S$ or

$$d'_i + r'_i X_S \precsim_i \bar{d}_i + \bar{r}_i X_S + \varepsilon^{m(k)}$$

for all $i \in S$. If the first statement is true then it follows from Proposition 2.9 that there exists $(\bar{d}^k, \bar{r}^k) \in PO_S(\Gamma)$ such that $\bar{d}^k_i + \bar{r}^k_i X_S \succsim_i d'_i + r'_i X_S$ for all $i \in S$. This implies that $(\bar{d}^k_i + \bar{r}^k_i X_S)_{i \in S} \in V^{\varepsilon^{m(k)}}$ and $(\bar{d}^k, \bar{r}^k) \in PO^*_S((d^k, r^k))$. If the second statement is true then it follows from $(d', r') \in PD_S(\Gamma)$ and Proposition 2.9 that there exists $(\bar{d}^k, \bar{r}^k) \in PO_S(\Gamma)$ satisfying

$$d^k_i + r^k_i X_N \sim_i \; d'_i + r'_i X_S \precsim_i \bar{d}^k_i + \bar{r}^k_i X_S \precsim_i \bar{d}_i + \bar{r}_i X_S + \varepsilon^{m(k)}$$

for all $i \in S$. Hence, $(\bar{d}^k, \bar{r}^k) \in PO^*_S((d^k, r^k))$ and $(\bar{d}^k_i + \bar{r}^k_i X_S)_{i \in S} \in V^{\varepsilon^{m(k)}}$.

For the case that $(d', r') \in NPD_S(\Gamma)$ a similar argument holds.

Next, let δ_i^k be such that $\bar{d}_i^k + \bar{r}_i^k X_S \sim_i d_i^k + r_i^k X_N + \delta_i^k$ for all $i \in S$. Note that for $k > K^1$ we have $\sum_{i\in S} \delta_i^k \in \mathcal{E}_S((d^k, r^k))$ and $(\bar{d}_i^k + \bar{r}_i^k X_S)_{i\in S} \in V^0$ if $(d_i^k + r_i^k X_N)_{i\in S} \in V^0$ and $(\bar{d}_i^k + \bar{r}_i^k X_S)_{i\in S} \in V^{\varepsilon^{m(k)}}$ if $(d_i^k + r_i^k X_N)_{i\in S} \notin V^0$. Since $(\sum_{i\in S} \delta_i^k)_{k\in \mathbf{N}}$ is a sequence in the compact set $\mathcal{E}_S(IR_S(\Gamma))$ there exists a convergent subsequence $(\sum_{i\in S} \delta_i^l)_{l\in \mathbf{N}}$ with limit $\sum_{i\in S} \tilde{\delta}_i$. Corresponding to this convergent subsequence there is a sequence $(\bar{d}_i^l + \bar{r}_i^l X_S)_{l\in \mathbf{N}}$ such that $(\bar{d}_i^l + \bar{r}_i^l X_S)_{i\in S} \in V^0$ if $(d_i^l + r_i^l X_N)_{i\in S} \in V^0$ and $(\bar{d}_i^l + \bar{r}_i^l X_S)_{i\in S} \in V^{\varepsilon^{m(l)}}$ if $(d_i^l + r_i^l X_N)_{i\in S} \notin V^0$. Moreover, it holds that $\bar{d}_i^l + \bar{r}_i^l X_S \sim_i d_i^l + r_i^l X_N + \delta_i^l$ for all $i \in S$. This implies that $(d_i^l + r_i^l X_N + \delta_i^l)_{i\in S} \in V^0$ if $(d_i^l + r_i^l X_N)_{i\in S} \in V^0$ and $(d_i^l + r_i^l X_N + \delta_i^l)_{i\in S} \in V^{\varepsilon^{m(l)}}$ if $(d_i^l + r_i^l X_N)_{i\in S} \notin V^0$. From $V^0 = \cap_{l\in \mathbf{N}} V^{\varepsilon^{m(l)}}$ it follows that

$$\lim_{l\to\infty} (d_i^l + r_i^l X_N + \delta_i^l)_{i\in S} \;=\; (d_i + r_i X_N + \tilde{\delta}_i)_{i\in S} \;\in V^0.$$

This implies that $d_i + r_i X_N + \tilde{\delta}_i \sim_i \bar{d}_i + \bar{r}_i X_S$ for all $i \in S$. Since $\bar{d}_i + \bar{r}_i X_S \sim_i d_i + r_i X_N + \delta_i$ for all $i \in S$ assumption (C3) implies that $\tilde{\delta}_i = \delta_i$ for all $i \in S$. Consequently, it holds that

$$\lim_{l\to\infty} \sum_{i\in S} \delta_i^l = \sum_{i\in S} \tilde{\delta}_i = \sum_{i\in S} \delta_i.$$

So, each convergent subsequence $(\sum_{i\in S} \delta_i^l)_{l\in \mathbf{N}}$ converges to $\sum_{i\in S} \delta_i$. Hence, $(\sum_{i\in S} \delta_i^k)_{k\in \mathbf{N}}$ converges to $\sum_{i\in S} \delta_i$, which completes the proof. $\square$

Proposition 3.5 The excess function $E_S((d, r))$ is continuous in (d, r) for each $(d, r) \in IR_S(\Gamma)$.

PROOF: Let $((d^k, r^k))_{k\in \mathbf{N}}$ be a sequence in $IR_N(\Gamma)$ converging to $(d, r) \in IR_N(\Gamma)$. We have to show that $\lim_{k\to\infty} E_S((d^k, r^k)) = E_S((d, r))$. Since $(E_S((d^k, r^k)))_{k\in \mathbf{N}}$ is a sequence in the compact set $\mathcal{E}_S(IR_N(\Gamma))$ there exists a convergent subsequence $(E_S((d^l, r^l)))_{l\in \mathbf{N}}$ with limit η. Note that the upper semi continuity of $\mathcal{E}_S$ implies that

$\eta \in \mathcal{E}_S((d,r))$. Since $\mathcal{E}_S$ is lower semi continuous there exists a sequence $(\sum_{i\in S}\delta_i^l)_{l\in\mathrm{N}}$ such that $\sum_{i\in S}\delta_i^l \in \mathcal{E}_S((d^l,r^l))$ for all $l \in \mathbb{N}$ and $\lim_{l\to\infty}\sum_{i\in S}\delta_i^l = E_S((d,r))$. Then

$$\begin{array}{rcl} \lim_{l\to\infty} E_S((d^l,r^l)) & \leq & \lim_{l\to\infty}\sum_{i\in S}\delta_i^l \;=\; E_S((d,r)) \\ & \leq & \eta \;=\; \lim_{l\to\infty} E_S((d^l,r^l)). \end{array}$$

Hence, $\lim_{l\to\infty}\sum_{i\in S} E_S((d^l,r^l)) = E_S((d,r))$. Thus, every convergent subsequence of $(E_S((d^k,r^k)))_{k\in\mathrm{N}}$ converges to $E_S((d,r))$. The compactness of $\mathcal{E}_S(IR_N(\Gamma))$ then implies that $\lim_{k\to\infty} E_S((d^k,r^k)) = E_S((d,r))$. □

Summarizing, it is shown that the domain $IR_N(\Gamma)$ is compact and that the excess function E_S is continuous for each coalition $S \subset N$. From the results stated in *Maschler et al. (1992)* it then follows that the nucleolus $\mathcal{N}(\Gamma)$ as defined in (3) is a nonempty subset of $IR_N(\Gamma)$ for each stochastic cooperative game $\Gamma \in CG(N)$.

4 The nucleolus, the core and deterministic equivalents

For deterministic cooperative games it is known that the nucleolus as defined in *Schmeidler (1969)* is a core allocation whenever the core is nonempty. A similar result can be derived for the nucleolus $\mathcal{N}(\Gamma)$ introduced in this paper. For this, recall that an allocation $(d,r) \in IR_N(\Gamma)$ is a core allocation for the game Γ if for each coalition $S \subset N$ there exists no allocation $(\bar{d},\bar{r}) \in IR_S(\Gamma)$ such that $\bar{d}_i + \bar{r}_i X_S \succ_i d_i + r_i X_N$ for all $i \in S$. The set of all core allocations is denoted by $Core(\Gamma)$.

Theorem 4.1 Let $\Gamma \in CG(N)$. If $Core(\Gamma) \neq \emptyset$ then $\mathcal{N}(\Gamma) \subset Core(\Gamma)$.

PROOF: Take $(d,r) \in IR_N(\Gamma)$ and $S \subset N$. Let $(\tilde{d}^S,\tilde{r}^S) \in I_S(\Gamma)$ be such that $\tilde{d}_i^S + \tilde{r}_i^S X_S \sim_i d_i + r_i X_N$ for all $i \in S$. Moreover, let $(\bar{d},\bar{r}) \in PO_S^*((d,r))$ and $\delta \in \mathbb{R}^S$ be such that $\bar{d}_i + \bar{r}_i X_S \sim_i d_i + r_i X_N + \delta_i$ for all $i \in S$ and $\sum_{i\in S}\delta_i = E_S((d,r))$. Regarding the sign of the excess, we distinguish three cases.

First, suppose $(\tilde{d}^S, \tilde{r}^S) \in PD_S(\Gamma)\backslash PO_S(\Gamma)$. Then $\bar{d}_i + \bar{r}_i X_S \succsim_i \tilde{d}_i^S + \tilde{r}_i^S X_S$ for all $i \in S$. Hence, $\delta_i \geq 0$ for all $i \in S$. Since $(\tilde{d}^S, \tilde{r}^S)$ is not Pareto optimal there exists $j \in S$ such that $\bar{d}_j + \bar{r}_j X_S \succ_j \tilde{d}_j^S + \tilde{r}_j^S X_S \sim_j d_j + r_j X_N$. Then $\delta_j > 0$ and, consequently, $E_S((d,r)) = \sum_{i \in S} \delta_i > 0$.

Second, suppose $(\tilde{d}^S, \tilde{r}^S) \in PO_S(\Gamma)$. This implies that $0 \in \mathcal{E}_S((d,r))$. Hence, $E_S((d,r)) \leq 0$.

Third, suppose $(\tilde{d}^S, \tilde{r}^S) \in NPD_S(\Gamma)\backslash PO_S(\Gamma)$. Then $\bar{d}_i + \bar{r}_i X_S \precsim_i \tilde{d}_i^S + \tilde{r}_i^S X_S$ for all $i \in S$. Hence, $\delta_i \leq 0$ for all $i \in S$. Moreover, since $(\tilde{d}^S, \tilde{r}^S)$ is not Pareto optimal there exists $j \in S$ such that $\bar{d}_j + \bar{r}_j X_S \prec_j \tilde{d}_i^S + \tilde{r}_j^S X_S \sim_j d_j + r_j X_N$. So, $\delta_j < 0$ and, consequently, $E_S((d,r)) = \sum_{i \in S} \delta_i < 0$.

Now we show that the excess vector corresponding to a core allocation is lexicographically smaller then the excess vector corresponding to an allocation that does not belong to the core. This implies that the latter allocation cannot belong to the nucleolus of the game whenever core allocations exist. Hence, the nucleolus must be a subset of the core.

Take $(d,r) \in Core(\Gamma)$ and $(d',r') \notin Core(\Gamma)$. Since $(d,r) \in Core(\Gamma)$ it follows from the core conditions that $(\tilde{d}^S, \tilde{r}^S) \in NPD_S(\Gamma)$ for all $S \subset N$. Hence, $E_S((d,r)) \leq 0$ for all $S \subset N$. Since $(d',r') \notin Core(\Gamma)$ there exists a coalition $S \subset N$ and an allocation $(\hat{d}, \hat{r}) \in IR_S(\Gamma)$ for S such that $\hat{d}_i + \hat{r}_i X_S \succ_i d'_i + r'_i X_N$ for all $i \in S$. Hence, $(\tilde{d}'^S, \tilde{r}'^S) \in PD_S(\Gamma)\backslash PO_S(\Gamma)$ and, consequently, $E_S((d',r')) > 0$. This implies that $\theta \circ E_S((d,r)) <_{lex} \theta \circ E_S((d',r'))$. Thus $(d',r') \notin \mathcal{N}(\Gamma)$. □

Next, consider the class $MG(N)$ of cooperative games with stochastic payoffs introduced in *Suijs* and *Borm (1996)*. For this particular class of games it was shown that the core of a game $\Gamma = (N, (X_S)_{S \subset N}, (\succsim_i)_{i \in N})$ is nonempty if and only if the core of a corresponding deterministic game $\Delta_\Gamma = (N, (x_S)_{S \subset N}, (\succsim_i)_{i \in N})$ is nonempty. This deterministic game Δ_Γ is called the deterministic equivalent of Γ. The preferences $\succsim_i$ of a game $\Gamma \in MG(N)$ are such that there exists a function $m_i : L^1(\mathbb{R}) \to \mathbb{R}$ satisfying

(M1) for all $X \in L^1(\mathbb{R}) : X \sim_i m_i(X)$,

(M2) for all $X, Y \in L^1(\mathbb{R}) : X \succsim_i Y$ if and only if $m_i(X) \geq m_i(Y)$, (M3) for all $d \in \mathbb{R}$: $m_i(d) = d$,

(M4) for all $X \in L^1(\mathbb{R})$ and all $d \in \mathbb{R}$: $m_i(X + d) = d + m_i(X)$,

with $L^1(\mathbb{R})$ the set of all random variables with finite expectation. Here, $m_i(X)$ represents the deterministic equivalent of the random payoff X according to player i. So, player i is indifferent between receiving the random payoff X and receiving the amount $m_i(X)$ with certainty. Furthermore, the payoff x_S of coalition S in the game Δ_Γ is defined by

$$x_S = \max_{(d,r)\in IR_S(\Gamma)} \sum_{i\in S} m_i(d_i + r_i X_S),$$

for all $S \subset N$. Moreover, *Suijs* and *Borm (1996)* also showed that an allocation $(d, r) \in IR_S(\Gamma)$ is Pareto optimal if and only if for the corresponding allocation $(m_i(d_i+r_iX_S))_{i\in S}$ in Δ_Γ it holds that $\sum_{i\in S} m_i(d_i + r_i X_S) = x_S$. Finally, note that $\succsim_i$ satisfies conditions (C2) and (C3) for all $i \in N$ and that the preferences discussed in Example 2.3 and Example 2.4 satisfy (M1) - (M4). Moreover, if the utility functions discussed in Example 2.5 are exponential then conditions (M1) - (M4) are also satisfied.

In the remainder of this section we show that the nucleolus $\mathcal{N}(\Delta_\Gamma)$ of the deterministic equivalent coincides with the nucleolus introduced by *Schmeidler (1969)*. Moreover, we show that an allocation $(d, r) \in IR_N(G)$ belongs to the nucleolus $\mathcal{N}(\Gamma)$ if and only if the corresponding allocation $(m_i(d_i + r_i X_N))_{i\in N}$ in the deterministic equivalent Δ_Γ belongs to the nucleolus $\mathcal{N}(\Delta_\Gamma)$ of Δ_Γ.

Let $\Gamma \in CG(N)$ be such that conditions (M1)-(M4) are satisfied. For the deterministic equivalent Δ_Γ of Γ it holds that

$$I_S(\Delta_\Gamma) = \{y \in \mathbb{R}^S | \forall_{i\in S} : y_i \geq x_{\{i\}}\}$$

is the set of (non feasible) individually rational allocations for coalition S,

$$IR_S(\Delta_\Gamma) = \{y \in I_S(\Gamma) | \sum_{i\in S} y_i \leq x_S\}$$

is the set of feasible individually rational allocations for S and

$$PO_S(\Delta_\Gamma) = \{y \in IR_S(\Delta_\Gamma) | \nexists_{y'\in IR_S(\Delta_\Gamma)} \forall_{i\in S} : y'_i > y_i\},$$

is the set of Pareto optimal allocations of S. Obviously, $IR_S(\Delta_\Gamma)$ and $PO_S(\Delta_\Gamma)$ are compact. Next, note that for each allocation $y \in IR_N(\Delta_\Gamma)$ and each $S \subset N$ we have that

$$\begin{aligned} W_S(y) &= \{y' \in IR_S(\Delta_\Gamma) | \forall_{i\in S} : y'_i \leq y_i\}, \\ B_S(y) &= \{y' \in IR_S(\Delta_\Gamma) | \forall_{i\in S} : y'_i \geq y_i\}, \\ PO^*_S(y) &= (W_S(y) \cup B_S(y)) \cap PO_S(\Delta_\Gamma). \end{aligned}$$

So $PO^*_S(y)$ is the set of Pareto optimal allocations for S such that all members of S prefer the allocation y to such a Pareto optimal allocation or all members of S prefer the Pareto optimal allocation to y. The excess function $E_S : IR_N(\Delta_\Gamma) \to \mathbb{R}$ can now be rewritten as

$$E_S(y) = \min_{y' \in PO^*_S(y)} \{\sum_{i\in S} \delta_i | \forall_{i\in S} : y'_i = y_i + \delta_i\}.$$

Since $\delta_i = y'_i - y_i$ and $\sum_{i\in S} y'_i = x_S$ it follows that $E_S(y) = x_S - \sum_{i\in S} y_i$. Hence, $\mathcal{N}(\Delta_G)$ coincides with the traditional nucleolus for the game Δ_Γ.

Now, take $(d, r) \in IR_N(\Gamma)$. The excess of coalition S then equals

$$\begin{aligned} E_S((d,r)) &= \min_{(d',r') \in PO^*_S((d,r))} \{\sum_{i\in S} \delta_i | \forall_{i\in S} : d'_i + r'_i X_S \sim_i d_i + r_i X_N + \delta_i\} \\ &= \min_{(d',r') \in PO^*_S((d,r))} \{\sum_{i\in S} \delta_i | \forall_{i\in S} : m_i(d'_i + r'_i X_S) = m_i(d_i + r_i X_N + \delta_i)\} \\ &= \min_{(d',r') \in PO^*_S((d,r))} \{\sum_{i\in S} \delta_i | \forall_{i\in S} : m_i(d'_i + r'_i X_S) = \delta_i + m_i(d_i + r_i X_N)\} \\ &= \min_{(d',r') \in PO^*_S((d,r))} \sum_{i\in S} (m_i(d'_i + r'_i X_S) - m_i(d_i + r_i X_N)) \\ &= x_S - \sum_{i\in S} m_i(d_i + r_i X_N) \ = \ E_S((m_i(d_i + r_i X_N))_{i\in N}). \end{aligned}$$

So, the excess of coalition S at allocation (d, r) equals the excess introduced by *Schmeidler (1969)* of coalition S at the corresponding allocation $(m_i(d_i+r_i X_N))_{i\in N}$ in the deterministic equivalent Δ_Γ. Moreover, for each allocation $(d, r) \in PO_N(\Gamma)$ in Γ the vector $(m_i(d_i + r_i X_N))_{i\in N}$ is an allocation of x_N in Δ_Γ and, vice versa, for each allocation y of x_N in Δ_Γ there exists an allocation $(d, r) \in PO_N(\Gamma)$ in Γ such that $m_i(d_i + r_i X_N) = y_i$ for all $i \in N$. This result has the following three implications.

First, since the deterministic equivalent of a deterministic cooperative game is the deterministic game itself it follows that the nucleolus $\mathcal{N}$ coincides with Scmeidler's nucleolus on the class of deterministic cooperative games.

Second, an allocation (d, r) belongs to the nucleolus $\mathcal{N}(\Gamma)$ of the game Γ if and only if the corresponding allocation $(m_i(d_i + r_i X_N))_{i\in N}$ belongs to the nucleolus $\mathcal{N}(\Delta_\Gamma)$ of the corresponding deterministic equivalent Δ_Γ.

Third, the nucleolus $\mathcal{N}(\Delta_\Gamma)$ is nonempty if $IR_N(\Delta_\Gamma) \neq \emptyset$. Hence, for all games $\Gamma \in MG(N)$ the nucleolus $\mathcal{N}(\Gamma)$ is nonempty if $IR_N(\Delta_\Gamma) \neq \emptyset$. This is in particular interesting since *Suijs* and *Borm (1996)* also showed that the relation between stochastic cooperative games $\Gamma \in MG(N)$ and their deterministic equivalents Δ_Γ also holds if the following more general definition of an allocation is used. Instead of a pair $(d, r) \in H^S \times \Delta^S$ an allocation of the random payoff X_S is described by a pair $(d, Y) \in H^S \times L^1(\mathbb{R})^S$, where Y is an S-dimensional vector of random variables such that $\sum_{i\in S} Y_i = X_S$. Furthermore, note that the preferences discussed in Example 2.3, Example 2.4 and Example 2.5 are not continuous on the set $L^1(\mathbb{R})$ of all random variables with finite expectation. Hence, condition (C1) is not satisfied in case this definition of an allocation is used. For a stochastic cooperative game $\Gamma \in MG(N)$, however, the nucleolus still exists.

Example 4.2 Consider the following three player game Γ. Let $-X_{\{i\}} \sim \text{Exp}(1)$ for $i = 1, 2, 3$ and let $X_S = \sum_{i\in S} X_{\{i\}}$ if $|S| \geq 2$. So, each player individually faces a random cost which is exponentially distributed with expectation equal to 1. The cost of a coalition then equals the sum of the cost of the members of this coalition. Furthermore, all players are expected utility maximizers with utility functions $u_1(t) = -e^{-0.5t}$, $u_2(t) = -e^{-0.33t}$ and $u_3(t) = -e^{-0.25t}$, respectively. For the deterministic equivalent m_i it holds that $m_i(d_i + r_i X_S) = u_i^{-1}(E(u_i(d_i + r_i X_S)))$. For the deterministic equivalent Δ_Γ of Γ we then get $x_{\{1\}} = -1.3863$, $x_{\{2\}} = -1.2164$, $x_{\{3\}} = -1.1507$, $x_{\{1,2\}} = -2.2314$, $x_{\{1,3\}} = -2.1878$, $x_{\{2,3\}} = -2.1582$ and $x_{\{1,2,3\}} = -3.1800$. The nucleolus $\mathcal{N}(\Delta_\Gamma)$ of this game is equal to $(-1.0933, -1.0633, -1.0234)$. To determine the nucleolus $\mathcal{N}(\Gamma)$ note that an allocation (d, r) is Pareto optimal if and only if $r = \frac{1}{9}(2, 3, 4)$. Then the only allocation (d, r) for which $(m_i(d_i + r_i X_N))_{i\in N} = \mathcal{N}(\Delta_\Gamma)$ is the allocation (d^*, r^*) with $d^* = (-0.3865, -0.0034, 0.3899)$ and $r^* = \frac{1}{9}(2, 3, 4)$. Hence, $\mathcal{N}(\Gamma) = \{(d^*, r^*)\}$.

Appendix A

For the following lemma stated in this appendix we use the following notation. Let $X, Y \in \mathcal{L}(\Gamma)$ and let $(d^k)_{k\in\mathbb{N}}$ and $(r^k)_{k\in\mathbb{N}}$ be sequences in $\mathbb{R}$ and $[0,1]$, respectively. Denote by F, F^k and G the probability distribution functions of X, $d^k + r^k X$ and Y, respectively. Moreover, note that

$$F^k(t) = \mathbb{P}(\{\omega | d^k + r^k X(\omega) \leq t\}) = \mathbb{P}(\{\omega | X(\omega) \leq \tfrac{t-\delta^k}{r^k}\}) = F(\tfrac{t-d^k}{r^k}),$$

if $r^k \neq 0$.

Lemma A.3 If $F^k \xrightarrow{w} G$ then there exists $d \in \mathbb{R}$ and $r \in [0,1]$ such that $Y = d + rX$.

PROOF: First, since $(r^k)_{k\in\mathbb{N}}$ is a sequence in $[0,1]$ we may assume without loss of generality that $(r^k)_{k\in\mathbb{N}}$ converges to $r \in [0,1]$. Second, note that $F^k \xrightarrow{w} G$ implies that $\lim_{k\to\infty} F^k(t) = G(t)$ for all $t \in C_G = \{t \in \mathbb{R} | G \text{ is continuous in } t\}$. Note that $\mathbb{R}\backslash C_G$ is a countable set.

Consider the following two cases.

I: $r = 0$. In this case we have that Y is a degenerate random variable, i.e, $\mathbb{P}(\{\omega | Y(\omega) = d\}) = 1$ for some $d \in \mathbb{R}$. Hence, $F^k \xrightarrow{w} G$ implies that $\lim_{k\to\infty} d^k = d$.

II: $r > 0$. In this case we show that $\lim_{k\to\infty} d^k = d$ for some $d \in \mathbb{R}$. Suppose that the sequence $(d^k)_{k\in\mathbb{N}}$ does not converge. Then there are three possibilities.

First, it holds that $\lim_{k\to\infty} d^k = +\infty$. Then $\lim_{k\to\infty} F^k(t) = \lim_{k\to\infty} F(\frac{t-d^k}{r^k}) = 0$ for all $t \in C_G$. Consequently, it must hold that $G(t) = 0$ for all $t \in C_G$. Clearly, this is a contradiction.

Second, it holds that $\lim_{k\to\infty} d^k = -\infty$. Then $\lim_{k\to\infty} F^k(t) = \lim_{k\to\infty} F(\frac{t-d^k}{r^k}) = 1$ for all $t \in C_G$. Consequently, it must be true that $G(t) = 1$ for all $t \in C_G$. Again, this is a contradiction.

Third, there exist convergent subsequences $(d^l)_{l\in\mathbb{N}}$ and $(d^m)_{m\in\mathbb{N}}$ with limits d and $\bar{d}$, respectively, such that $d > \bar{d}$. Let $t_1 \in C_G$. Since $\lim_{k\to\infty} F^k(t_1) = G(t_1)$

it follows that $\lim_{l\to\infty} F^l(t_1) = \lim_{l\to\infty} F(\frac{t_1-d^l}{r^l}) = G(t_1)$ and $\lim_{m\to\infty} F^m(t_1) = \lim_{m\to\infty} F(\frac{t_1-d^m}{r^m}) = G(t_1)$. So, $\lim_{l\to\infty} F(\frac{t_1-d^l}{r^l}) = \lim_{m\to\infty} F(\frac{t_1-d^m}{r^m})$. From the fact that probability distribution functions are nondecreasing and continuous from the right it follows that F is constant on the interval $[\frac{t_1-d}{r}, \frac{t_1-\bar{d}}{r})$. To be more precisely, $F(t) = \lim_{l\to\infty} F(\frac{t_1-d^l}{r^l})$ for all $t \in [\frac{t_1-d}{r}, \frac{t_1-\bar{d}}{r})$. This implies that G is constant on the interval $[t_1, t_1+d-\bar{d})$. To see this, take $\tau \in [t_1, t_1+d-\bar{d})$. If G is continuous in τ then it follows from $F^k \xrightarrow{w} G$ that $G(\tau) = \lim_{l\to\infty} F(\frac{\tau-d^l}{r^l}) = \lim_{m\to\infty} F(\frac{\tau-d^m}{r^m})$. Since either $\frac{\tau-d}{r} \in [\frac{t_1-d}{r}, \frac{t_1-\bar{d}}{r})$ or $\frac{\tau-\bar{d}}{r} \in [\frac{t_1-d}{r}, \frac{t_1-\bar{d}}{r})$ it holds that $G(\tau) = \lim_{l\to\infty} F(\frac{t_1-d^l}{r^l})$.

If G is not continuous in τ then there exists $\tau_1, \tau_2 \in C_G$ such that $t_1 \leq \tau_1 < \tau < \tau_2 < t_1 + d - \bar{d}$. Hence, by the same argument as above we have that $G(\tau_1) = G(\tau_2) = \lim_{l\to\infty} F(\frac{t_1-d^l}{r^l})$. Since G is nondecreasing it holds that $G(\tau_1) \leq G(\tau) \leq G(\tau_2)$. Thus $G(\tau) = \lim_{l\to\infty} F(\frac{t_1-d^l}{r^l})$. Consequently, G is constant on the interval $[t_1, t_1 + d - \bar{d})$.

Next, take $t \in [t_1, t_1 + d - \bar{d})$. By the same argument as above it follows that F is constant on $[\frac{t-d}{r}, \frac{t-\bar{d}}{r})$ and that G is constant on $[t, t + d - \bar{d})$. Hence, G is constant on the interval $[t_1, t_1 + 2(d - \bar{d}))$. Repeating this argument yields that G is constant on (t_1, ∞). Finally, since this holds for all $t_1 \in C_G$ it follows that G is constant on $\mathbb{R}$. Obviously, this is a contradiction.

Next, let F_d denote the probability distribution function of $d+rX$. Since $\lim_{k\to\infty} d^k = d$ and $\lim_{k\to\infty} r^k = r$ it follows that $F^k \xrightarrow{w} F_d$. Hence, $F_d(t) = G(t)$ for all $t \in C_G$. Since F_d and G are continuous from the right it follows that $F_d(t) = G(t)$ for all $t \in \mathbb{R}$. Consequently, $Y = d + rX$. □

Lemma A.4 The preference relation $\succsim_i$ arising from an expected utility maximizing player satisfies the continuity condition (C1).

PROOF: Let X be random variable and let $(d_i^k + r_i^k X)_{k\in\mathbb{N}}$ be a convergent sequence with limit $\bar{X}$. From Lemma A.3 we know that there exists d_i and r_i such that $\lim_{k\to\infty} d_i^k = d_i$,

$\lim_{k\to\infty} r_i^k = r_i$ and $\bar{X} = d_i + r_i X_S$. It is sufficient to show that $\lim_{k\to\infty} E(u_i(d_i^k + r_i^k X)) = E(u_i(d_i + r_i X))$.

Suppose that $(E(u_i(d_i^k + r_i^k X))$ does not converge to $E(u_i(d_i + r_i X))$. Then there are three posssibilities

(i) there exists $\varepsilon > 0$ and $K^\varepsilon \in \mathbb{N}$ such that $E(u_i(d_i^k + r_i^k X)) < E(u_i(d_i + r_i X)) - \varepsilon$ for all $k \geq K^\varepsilon$,

(ii) there exists $\varepsilon > 0$ and $K^\varepsilon \in \mathbb{N}$ such that $E(u_i(d_i^k + r_i^k X)) > E(u_i(d_i + r_i X)) + \varepsilon$ for all $k \geq K^\varepsilon$.

(iii) there exists subsequences $(E(u_i(d_i^m + r_i^m X)))_{m\in\mathbb{N}}$ and $(E(u_i(d_i^l + r_i^l X)))_{l\in\mathbb{N}}$ such that either $E(u_i(d_i^m + r_i^m X)) < E(u_i(d_i + r_i X)) - \varepsilon$ for some $\varepsilon > 0$ and all $m > M^\varepsilon$ or $E(u_i(d_i^l + r_i^l X)) > E(u_i(d_i + r_i X)) + \varepsilon$ for some $\varepsilon > 0$ and all $l > L^\varepsilon$ or both.

In the first case, define $\bar{d}_i^k = \inf_{l\geq k} d_i^k$ and $\bar{r}_i^k = \inf_{l\geq k} r_i^k$ if $X \geq 0$ and $\bar{r}_i^k = \sup_{l\geq k} r_i^k$ if $X \leq 0$ for all $k \in \mathbb{N}$. Then $(\bar{d}_i^k + \bar{r}_i^k X(\omega))_{k\in\mathbb{N}}$ is an increasing sequence with limit $d + rX(\omega)$ for all $\omega \in \Omega$. Moreover, $\bar{d}_i^k + \bar{r}_i^k X(\omega) \leq d_i^k + r_i^k X(\omega)$ for all $k \in \mathbb{N}$ and all $\omega \in \Omega$. Hence, $E(u_i(\bar{d}_i^k + \bar{r}_i^k X)) \leq E(u_i(d_i^k + r_i^k X))$ for all $k \in \mathbb{N}$. Next, let $V \subset \mathbb{R}$ be a compact set and $K^\varepsilon \in \mathbb{N}$ such that

$$\left| \int_{V^c} u_i(d_i + r_i t) dF(t) \right| < \tfrac{1}{4}\varepsilon$$

and

$$\left| \int_{V^c} u_i(\bar{d}_i^{K^\varepsilon} + \bar{r}_i^{K^\varepsilon} t) dF(t) \right| < \tfrac{1}{4}\varepsilon,$$

where V^c denotes the complement of V and F the probability distribution function of X. Then

$$\begin{aligned} 0 \quad &\leq \quad \int_{V^c} u_i(d_i + r_i t) dF(t) - \int_{V^c} u_i(\bar{d}_i^{K^\varepsilon} + \bar{r}_i^{K^\varepsilon} t) dF(t) = \\ &\left| \int_{V^c} u_i(d_i + r_i t) dF(t) - \int_{V^c} u_i(\bar{d}_i^{K^\varepsilon} + \bar{r}_i^{K^\varepsilon} t) dF(t) \right| < \tfrac{1}{2}\varepsilon. \end{aligned}$$

Since $(\bar{d}_i^k)_{k\in\mathbb{N}}$ and $(\bar{r}_i^k)_{k\in\mathbb{N}}$ are increasing sequences and u_i is an increasing function we have that

$$0 \leq \int_{V^c} u_i(d_i + r_i t) dF(t) - \int_{V^c} u_i(\bar{d}_i^k + \bar{r}_i^k t) dF(t) < \tfrac{1}{2}\varepsilon,$$

for all $k \geq K^{\varepsilon}$. The compactness of V implies that there exists $L^{\varepsilon} \geq K^{\varepsilon}$ such that

$$0 \leq \int_V u_i(d_i + r_i t) dF(t) - \int_V u_i(\bar{d}_i^k + \bar{r}_i^k t) dF(t) < \tfrac{1}{2}\varepsilon,$$

for all $k \geq L^{\varepsilon}$. Hence,

$$0 \leq \int_{-\infty}^{\infty} u_i(d_i + r_i t) dF(t) - \int_{-\infty}^{\infty} u_i(\bar{d}_i^k + \bar{r}_i^k t) dF(t) < \varepsilon,$$

for all $k \geq L^{\varepsilon}$. This implies that

$$\int_{-\infty}^{\infty} u_i(d_i + r_i t) dF(t) - \varepsilon < \int_{-\infty}^{\infty} u_i(\bar{d}_i^k + \bar{r}_i^k t) dF(t) \leq \int_{-\infty}^{\infty} u_i(d_i^k + r_i^k t) dF(t),$$

for all $k \geq L^{\varepsilon}$. This contradicts the fact that

$$\int_{-\infty}^{\infty} u_i(d_i^k + r_i^k t) dF(t) < \int_{-\infty}^{\infty} u_i(d_i + r_i t) dF(t) - \varepsilon$$

for all $k \geq L^{\varepsilon}$.

For the second case, one can derive a contradiction in a similar way as for the first case. Finally, in the third case a contradiction can be derived by applying the argument of the first two cases to the appropriate subsequences. Hence, $\lim_{k \to \infty} E(u_i(d_i^k + r_i^k X)) = E(u_i(d_i + r_i X))$. $\square$

Appendix B

PROOF OF PROPOSITION 2.6: Since $IR_S(\Gamma) \subset I_S(\Gamma) \subset \mathbb{R}^S \times \mathbb{R}^S$ it is sufficient to prove that $IR_S(\Gamma)$ is closed and bounded in $\mathbb{R}^S \times \mathbb{R}^S$. Since

$$IR_S(\Gamma) = \{(d,r) \in I_S(\Gamma) | \sum_{i \in S} d_i \leq 0\}$$

and $I_S(\Gamma)$ is closed by the continuity of $\succsim_i$ for all $i \in S$ it follows that $IR_S(\Gamma)$ is closed. To see that $IR_S(\Gamma)$ is bounded, define for each $i \in S$ and each $r_i \in [0,1]$

$$\underline{d}_i(r_i) = \min\{d_i | d_i + r_i X_S \succsim_i X_{\{i\}}\}.$$

Note that $\underline{d}_i(r_i)$ exists by assumptions (C1) and (C2) and that $\underline{d}_i(r_i) + r_i X_S \sim_i X_{\{i\}}$. To show that $\min_{r_i \in [0,1]} \underline{d}_i(r_i)$ exists it suffices to show that $\underline{d}_i(r_i)$ is continuous in r_i. Therefore, consider the sequence $(r_i^k)_{k \in \mathbb{N}}$ with $r_i^k \in [0,1]$ and $\lim_{k \to \infty} r_i^k = r_i$. By definition we have for all $k \in \mathbb{N}$ that $\underline{d}_i(r_i^k) + r_i^k X_S \sim_i X_{\{i\}}$. Hence, $\underline{d}_i(r_i^k) + r_i^k X_S \sim_i \underline{d}_i(r_i) + r_i X_S$ for all $i \in S$. Since $\succsim_i$ is continuous it follows that

$$\lim_{k \to \infty} \left(\underline{d}_i(r_i^k) + r_i^k X_S \right) = \lim_{k \to \infty} \underline{d}_i(r_i^k) + r_i X_S \sim_i \underline{d}_i(r_i) + r_i X_S$$

Then assumption (C3) implies that $\lim_{k \to \infty} \underline{d}_i(r_i^k) = \underline{d}_i(r_i)$. Consequently, $\underline{d}_i(r_i)$ is continuous in r_i and

$$\underline{d}_i = \min_{r_i \in [0,1]} \underline{d}_i(r_i)$$

exists and is finite for all $i \in S$.

Since $(d,r) \in IR_S(\Gamma)$ implies that $d_i + r_i X_S \succsim_i X_{\{i\}}$ for all $i \in S$ it follows by condition (C3) that $d_i \geq \underline{d}_i$ for all $i \in S$. Hence, $(d,r) \in IR_S(\Gamma)$ implies that

$$d \in \{\tilde{d} \in \mathbb{R}^S | \forall_{i \in S} : \tilde{d}_i \geq \underline{d}_i, \sum_{i \in S} \tilde{d}_i \leq 0\}$$

and $r \in \Delta^S$. Since both sets are bounded, we have that $IR_S(\Gamma)$ is bounded. □

PROOF OF PROPOSITION 2.7: Since $PO_S(\Gamma) \subset IR_S(\Gamma)$ and $IR_S(\Gamma)$ is compact it is sufficient to show that $PO_S(\Gamma)$ is closed in $IR_S(\Gamma)$. Let $(d,r) \in IR_S(\Gamma)$ be such that $(d,r) \notin PO_S(\Gamma)$. Then there exists $(\bar{d},\bar{r}) \in IR_S(\Gamma)$ such that $\bar{d}_i + \bar{r}_i X_S \succ_i d_i + r_i X_S$

for all $i \in S$. Next, consider the set $\{(d',r') \in IR_S(\Gamma) | d'_i + r'_i X_S \prec_i \bar{d}_i + \bar{r}_i X_S\}$. By the continuity of $\succsim_i$ this set is open in $IR_S(\Gamma)$. Indeed, by the continuity of $\succsim_i$ we have that $\{Y \in \mathcal{L}(\Gamma)^S | \exists_{i \in S} : Y_i \succsim_i \bar{d}_i + \bar{r}_i X_S\}$ is closed. Hence, Proposition 2.6 implies that $\{(d',r') \in IR_S(\Gamma) | \exists_{i \in S} : d'_i + r'_i X_S \succsim_i \bar{d}_i + \bar{r}_i X_S\}$ is closed in $I_S(\Gamma)$. Hence, it is also closed in $IR_S(\Gamma)$. Consequently, $\{(d',r') \in IR_S(\Gamma) | \forall_{i \in S} : d'_i + r'_i X_S \prec_i \bar{d}_i + \bar{r}_i X_S\}$ must be open in $IR_S(\Gamma)$. Since (d,r) belongs to the latter set there exists an open neighbourhood O of (d,r) in $IR_S(\Gamma)$ such that $O \subset \{(d',r') \in IR_S(\Gamma) | \forall_{i \in S} : d'_i + r'_i X_S \prec_i \bar{d}_i + \bar{r}_i X_S\}$. This implies that $(\tilde{d},\tilde{r}) \notin PO_S(\Gamma)$ whenever $(\tilde{d},\tilde{r}) \in O$. Hence, $IR_S(\Gamma) \backslash PO_S(\Gamma)$ is open in $IR_S(\Gamma)$ and, consequently, $PO_S(\Gamma)$ is closed in $IR_S(\Gamma)$. □

PROOF OF PROPOSITION 2.9: Let $(d,r) \in PD_S(\Gamma)$ and $(\tilde{d},\tilde{r}) \in NPD_S(\Gamma)$. Without loss of generality we may assume that $(d,r) \in IR_S(\Gamma)$.[5] Take $\delta_i \in \mathbb{R}$ be such that $d_i + \delta_i + r_i X_S \sim_i \tilde{d}_i + \tilde{r}_i X_S$. Note that $\delta_i \geq 0$ by condition (C3). Next, take $\bar{r} \in \Delta^S$ and $t \in [0,1]$. Let $\bar{d}_i(\bar{r},t)$ be such that $\bar{d}_i(\bar{r},t) + \bar{r}_i X_S \sim_i d_i + t\delta_i + r_i X_S$. Note that the allocation $(\bar{d}(\bar{r},t),\bar{r})$ is feasible if and only if $\sum_{i \in S} \bar{d}_i(\bar{r},t) \leq 0$. First, we show that $\bar{d}_i(\bar{r},t)$ is continuous in $(\bar{r},t)$.

Let $((\bar{r}^k,t^k))_{k \in \mathbb{N}}$ be a convergent sequence with limit $(\bar{r},t)$. We have to show that $\lim_{k\to\infty} \bar{d}_i(\bar{r}^k,t^k) = \bar{d}_i(\bar{r},t)$. Note that $\bar{d}_i(\bar{r}^k,t^k) + \bar{r}^k_i X_S \sim_i d_i + t^k\delta_i + r_i X_S$ for all $k \in \mathbb{N}$. Define for $\varepsilon > 0$

$$V_i^\varepsilon = \{Y \in \mathcal{L}(\Gamma) | d_i + t\delta_i + r_i X_S - \varepsilon \prec_i Y \prec_i d_i + t\delta_i + r_i X_S + \varepsilon\}.$$

Since $t^k \to t$ there exists $K^\varepsilon \in \mathbb{N}$ such that $d_i + t^k\delta_i + r_i X_S \in V_i^\varepsilon$ for all $k > K^\varepsilon$. Consequently, we have that $\bar{d}_i(\bar{r}^k,t^k) + \bar{r}^k_i X_S \in V_i^\varepsilon$ for all $k > K^\varepsilon$. This implies that $\lim_{k\to\infty} \left(\bar{d}_i(\bar{r}^k,t^k) + \bar{r}^k_i X_S\right) \in \cap_{\varepsilon>0} V_i^\varepsilon$. So,

$$\lim_{k\to\infty} \left(\bar{d}_i(\bar{r}^k,t^k) + \bar{r}^k_i X_S\right) = \lim_{k\to\infty} \bar{d}_i(\bar{r}^k,t^k) + \bar{r}_i X_S \sim_i d_i + t\delta_i + r_i X_S.$$

Since $\bar{d}_i(\bar{r},t) + \bar{r}_i X_S \sim_i d_i + t\delta_i + r_i X_S$ it follows from condition (C3) that $\lim_{k\to\infty} \bar{d}_i(\bar{r}^k,t^k) = \bar{d}_i(\bar{r},t)$.

[5] If $(d,r) \notin IR_S(\Gamma)$ then there exists $(d',r') \in IR_S(\Gamma)$ such that $d'_i + r'_i X_S \succ_i d_i + r_i X_S$ for all $i \in S$. If $\delta'_i < 0$ is such that $d'_i + \delta'_i + r'_i X_S \sim_i d_i + r_i X_S$ for all $i \in S$ then $(d' + \delta', r')$ is still a feasible allocation. Thus, $(d' + \delta', r') \in IR_S(\Gamma)$. Continuing the proof with the allocation (d,r) replaced by $(d' + \delta', r')$ would yould the same result.

Next, define $f(t) = \min_{\bar{r}\in\Delta^S} \sum_{i\in S} \bar{d}_i(\bar{r}, t)$ for all $t \in [0,1]$. Then f is a continuous function. Moreover, since $(d,r) \in IR_S(\Gamma)$ and $\bar{d}_i(r,0) = d_i$ for all $i \in S$ it follows from the feasibility of (d,r) that $f(0) \leq \sum_{i\in S} \bar{d}_i(r,0) = \sum_{i\in S} d_i \leq 0$. Furthermore, since $d_i + \delta_i + r_i X_S \sim_i \tilde{d}_i + \tilde{r}_i X_S$ for all $i \in S$ and $(\tilde{d}, \tilde{r}) \in NPD_S(\Gamma)$ it follows that $(d+\delta, r) \in NPD_S(\Gamma)$. This implies that $f(1) \geq 0$. For, if $f(1) < 0$ then there exists $r^* \in \Delta^S$ such that $\sum_{i\in S} \bar{d}_i(r^*, 1) < 0$ and $\bar{d}_i(r^*, 1) + r^*_i X_S \sim_i d_i + \delta_i + r_i X_S$ for all $i \in S$. Consequently, the allocation yielding the payoffs

$$\bar{d}_i(r^*, 1) - \tfrac{1}{|S|} \sum_{i\in S} \bar{d}_i(r^*, 1) + r^*_i X_S$$

for each $i \in S$ is feasible and preferred to $d_i + \delta_i + r_i X_S$ by all players $i \in S$. Clearly, this contradicts the fact that $(d+\delta, r) \in NPD_S(\Gamma)$. Thus, $f(0) \leq 0 \leq f(1)$. The continuity of f then implies that there exists $\hat{t}$ such that $f(\hat{t}) = 0$.

Let $\hat{r} \in \Delta^S$ be such that $\sum_{i\in S} \bar{d}_i(\hat{r}, \hat{t}) = 0$. Then the allocation $(\bar{d}(\hat{r}, \hat{t}), \hat{r})$ is Pareto optimal. To see this, first note that $\sum_{i\in S} \bar{d}_i(\bar{r}, \hat{t}) \geq 0$ for all $\bar{r} \in \Delta^S$. Second, note that the definition of $\bar{d}_i(\bar{r}, t)$ implies that

$$\bar{d}_i(\hat{r}, \hat{t}) + \hat{r}_i X_S \sim_i \bar{d}_i(\bar{r}, \hat{t}) + \bar{r}_i X_S \tag{4}$$

for all $i \in S$ and all $\bar{r} \in \Delta^S$. Next, take $\bar{r} \in \Delta^S$. If $\sum_{i\in S} \bar{d}_i(\bar{r}, \hat{t}) > 0$ then the allocation $(\bar{d}(\bar{r}, \hat{t}), \bar{r})$ is not feasible. From expression (4) it then follows that there exists no feasible allocation $(\bar{d}, \bar{r})$ which all players $i \in S$ prefer to the allocation $(\bar{d}(\hat{r}, \hat{t}), \hat{r})$.

If $\sum_{i\in S} \bar{d}_i(\bar{r}, \hat{t}) = 0$ then the allocation $(\bar{d}(\bar{r}, \hat{t}), \bar{r})$ is feasible. Moreover, an allocation $(\bar{d}, \bar{r})$ that all players $i \in S$ prefer to $(\bar{d}(\bar{r}, \hat{t}), \bar{r})$ must be infeasible by condition (C3) and expression (4). Hence, there exists no feasible allocation $(\bar{d}, \bar{r})$ which all players $i \in S$ prefer to $(\bar{d}(\hat{r}, \hat{t}), \hat{r})$. Consequently, $(\bar{d}(\hat{r}, \hat{t}), \hat{r})$ is Pareto optimal. $0 \leq \hat{t} \leq 1$ then implies that

$$d_i + r_i X_S \precsim_i \bar{d}_i(\hat{r}, \hat{t}) + \hat{r}_i X_S \precsim_i d_i + \delta_i + r_i X_S \sim_i \tilde{d}_i + \tilde{r}_i X_S$$

for all $i \in S$. $\square$

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