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Cost monotonic 'construct and charge' rules for connection situations

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Abstract: The special class of conservative charge systems for minimum cost spanning tree (mcst) situations is introduced. These conservative charge systems lead to single-valued rules for mcst situations, which can also be described with the aid of obligation functions and are, consequently, cost monotonic. A value-theoretic interpretation of these rules is also provided.

Key-words: cost allocation, minimum cost spanning tree situations, cost monotonicity, sharing values.

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1 Introduction

A minimum cost spanning tree (mcst) situation arises when there is a group of agents $N = \{1, 2, \ldots, n\}$ who all want to be connected with a source 0,

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directly or via other agents, and where connections are costly. To such mcst situations correspond two problems: to construct a minimum cost spanning tree (mcst) which connects all the agents with the source and to divide the cost of constructing an mcst among the agents.

To *construct* an mcst two methods are mainly used: the Prim algorithm (Prim (1957)) and the Kruskal algorithm (Kruskal (1956)). Both algorithms determine an mcst forming, in every step of the algorithm, exactly one edge, for a total number of steps equal to n. To divide the cost of an mcst among the agents, both algorithms are suitable to define cost allocation protocols which *charge* the agents with "fractions" of the cost of each edge constructed in each step of the procedure.

Construct and *Charge* rules, formally introduced in Section 4, rely on this idea of allocation protocol.

In this paper the Kruskal algorithm is central. This algorithm works in the following way: in the first step an edge between nodes in $N \cup \{0\}$ of minimal cost is formed. In every subsequent step, a new edge of minimal cost is formed, under the constraint that no cycles are formed. In summary, given an ordering of the edges with respect to their increasing costs, a sequence of edges is produced and after n steps an mcst appears.

In Feltkamp et al. (1994a,b), Norde et al. (2004), Branzei et al. (2004) and Tijs et al. (2004) particular allocation protocols based on the Kruskal algorithm are studied. Recently, we have discovered that we can embed all such allocation protocols on mcst situations in a larger class of Construct and Charge rules.

An interesting property of Construct and Charge rules is that some of them are independent from the ordering of the edges with respect to their increasing costs and some others not. For example, the Proportional rule introduced in Feltkamp et al. (1994b) is dependent on the feasible orderings of the edges with respect to increasing costs. The ERO-rule introduced in Feltkamp et al. (1994a,b), which has been rebaptized as the P-value (Branzei et al.(2004)), the P^τ-values (Norde et al. (2004), Branzei et al. (2004)) and the Obligation rules (Tijs et al. (2004)) do not depend on the orderings of the edges with respect to increasing costs.

The aim of this paper is to introduce and characterize the class of Construct and Charge rules whose particular elements are "conservative". For such conservative Construct and Charge rules it turns out that different feasible orders of the edges (w.r.t. increasing costs) lead to the same cost allocations. Moreover, it turns out that conservative Construct and Charge rules are Obligation rules (Tijs et al. (2004)).

We start introducing some basic notions in the next section. In Section 3 the definition of a charge system is introduced, specific examples are given and some basic properties, like the conservativeness property for charge systems and a related concept of potential, are studied. Based on charge systems and orderings of the edges with respect to increasing costs, the definition of a Construct and Charge rule for mcst situations is given in Section 4, together with some examples and properties for such rules. In Section 5 the connection with Obligation rules is studied. A value-theoretic approach is provided in Section 6 using sharing values for cost games. Section 7 concludes.

2 Preliminaries and notations

An (undirected) *graph* is a pair $< V, E >$, where V is a set of vertices or nodes and E is a set of edges e of the form $\{i, j\}$ with $i, j \in V$, $i \neq j$. The *complete graph* on a set V of vertices is the graph $< V, E_V >$, where $E_V = \{\{i, j\} | i, j \in V \text{ and } i \neq j\}$. A *path* between i and j in a graph $< V, E >$ is a sequence of nodes $i = i_0, i_1, \ldots, i_k = j$, $k \geq 1$, such that $\{i_s, i_{s+1}\} \in E$ for each $s \in \{0, \ldots, k-1\}$. A *cycle* in $< V, E >$ is a path with all distinct edges from i to i for some $i \in V$. Two nodes $i, j \in V$ are connected in $< V, E >$ if $i = j$ or if there exists a path between i and j in E. A *connected component* of V in $< V, E >$ is a maximal subset of V with the property that any two nodes in this subset are connected in $< V, E >$.

Now, we consider *minimum cost spanning tree* (mcst) *situations*. In an mcst situation a set $N = \{1, \ldots, n\}$ of agents is involved willing to be connected as cheap as possible to a source (i.e. a supplier of a service) denoted by 0. In the sequel we use the notation $N' = N \cup \{0\}$. An mcst situation can be represented by a tuple $< N', E_{N'}, w >$, where $< N', E_{N'} >$ is the complete graph on the set N' of nodes or vertices, and $w : E_{N'} \to I\!R_+$ is a map which assigns to each edge $e \in E_{N'}$ a nonnegative number $w(e)$ representing the weight or cost of edge e. We call w a *weight function*.

Since in our paper the graph of possible edges is always the complete graph, we simply denote an mcst situation with set of users N, source 0, and weight function w by $< N', w >$. Often we identify an mcst situation $< N', w >$ with the corresponding weight function w. We denote by $\mathcal{W}^{N'}$ the set of all mcst situations $< N', w >$ (or w) with node set N'. For each $S \subseteq N$, one can consider the mcst subsituation $< S', w_{|S'} >$, where $S' = S \cup \{0\}$ and $w_{|S'} : E_{S'} \to I\!R_+$ is the restriction of the weight function w to $E_{S'} \subseteq E_{N'}$, i.e. $w_{|S'}(e) = w(e)$ for each $e \in E_{S'}$.

Let $< N', w >$ be an mcst situation. Two nodes i and j are called (w, N')-connected if $i = j$ or if there exists a sequence of nodes $i = i_0, \ldots, i_k = j$ in N', $k \geq 1$, with $w(\{i_s, i_{s+1}\}) = 0$ for every $s \in \{0, \ldots, k-1\}$. A (w, N')-*component* of N' is a maximal subset of N' with the property that any two nodes in this subset are (w, N')-connected. We denote by $M_i(w)$ the (w, N')-component to which i belongs and by $\mathcal{M}(w)$ the set of all (w, N')-components of N'. Clearly, the collection of (w, N')-components forms a partition of N'.

We define the set $\Sigma_{E_{N'}}$ of *linear orders* on $E_{N'}$ as the set of all bijections $\sigma : \{1, \ldots, |E_{N'}|\} \to E_{N'}$, where $|E_{N'}|$ is the cardinality of the set $E_{N'}$. For each mcst situation $< N', w >$ there exists at least one linear order $\sigma \in \Sigma_{E_{N'}}$ such that $w(\sigma(1)) \leq w(\sigma(2)) \leq \ldots \leq w(\sigma(|E_{N'}|))$. We denote by w^σ the column vector $\big(w(\sigma(1)), w(\sigma(2)), \ldots, w(\sigma(|E_{N'}|))\big)^t$.

For any $\sigma \in \Sigma_{E_{N'}}$ we define the set

$$K^\sigma = \{w \in I\!R_+^{E_{N'}} \mid w(\sigma(1)) \leq w(\sigma(2)) \leq \ldots \leq w(\sigma(|E_{N'}|))\}.$$

The set K^σ is a cone in $I\!R_+^{E_{N'}}$, which we call the *Kruskal cone with respect to* σ. One can easily see that $\bigcup_{\sigma \in \Sigma_{E_{N'}}} K^\sigma = I\!R_+^{E_{N'}}$. For each $\sigma \in \Sigma_{E_{N'}}$ the cone K^σ is a simplicial cone with generators $e^{\sigma,k} \in K^\sigma$, $k \in \{1, 2, \ldots, |E_{N'}|\}$, where

$$
\begin{array}{c}
e^{\sigma,k}(\sigma(1)) = e^{\sigma,k}(\sigma(2)) = \ldots = e^{\sigma,k}(\sigma(k-1)) = 0 \\
\text{and} \\
e^{\sigma,k}(\sigma(k)) = e^{\sigma,k}(\sigma(k+1)) = \ldots = e^{\sigma,k}(\sigma(|E_{N'}|)) = 1
\end{array}
\tag{1}
$$

[Note that $e^{\sigma,1}(\sigma(k)) = 1$ for all $k \in \{1, 2, \ldots, |E_{N'}|\}$].

This implies that each $w \in K^\sigma$ can be written in a unique way as non-negative linear combination of these generators. To be more concrete, for $w \in K^\sigma$ we have

$$
w = w(\sigma(1))e^{\sigma,1} + \sum_{k=2}^{|E_{N'}|} \big(w(\sigma(k)) - w(\sigma(k-1))\big)\, e^{\sigma,k}. \tag{2}
$$

Any mcst situation gives rise to two problems: the construction of a network $\Gamma \subseteq E_{N'}$ of minimal cost connecting all users to the source, and a cost sharing problem of distributing this cost in a fair way among users. The cost of a network Γ is $w(\Gamma) = \sum_{e \in \Gamma} w(e)$. A network Γ is a *spanning network* on $S' \subseteq N'$ if for every $e \in \Gamma$ we have $e \in E_{S'}$ and for every $i \in S$ there is a path in Γ from i to the source.

Now, we recall the definition of a *minimal mcst situation* given in Tijs et al. (2005). Let $w \in \mathcal{W}^{N'}$. For each path $P = (i_0, i_1, \ldots, i_k)$ from i to j in the graph $< N', E_{N'} >$ we denote the set of its edges by $E(P)$, that is $E(P) = \{\{i_0, i_1\}, \{i_1, i_2\}, \ldots, \{i_{k-1}, i_k\}\}$. Moreover, we call $\max_{e \in E(P)} w(e)$ the *top of the path* P and denote it by $t(P)$. We denote by $\mathcal{P}_{ij}^{N'}$ the set of all paths without cycles from i to j in the graph $< N', E_{N'} >$. The *minimal mcst situation* $\overline{w}$ corresponding to w (cf. Bird (1976)) is given by

$$
\overline{w}(\{i,j\}) = \min_{P \in \mathcal{P}_{ij}^{N'}} \max_{e \in E(P)} w(e) = \min_{P \in \mathcal{P}_{ij}^{N'}} t(P) \tag{3}
$$

for each $i, j \in N'$, $i \neq j$.

Next we introduce some basic game theoretical notations. A *cooperative cost game* or cost game is a pair (N, c), where N denotes the finite set of *players* and $c : 2^N \to I\!R$ the *characteristic function*, with $c(\emptyset) = 0$ (here 2^N

denotes the power set of player set N). Often we identify a cost game (N, c) with the corresponding characteristic function c. A group of players $T \subseteq N$ is called a *coalition* and $c(T)$ is called the *cost* of this coalition. The class of all cost games with N as set of players is denoted by $\mathcal{G}^N$

Let $< N', w >$ be an mcst situation. The *minimum cost spanning tree game* (N, c_w) (or simply c_w), corresponding to $< N', w >$, is defined by

$$c_w(S) = \min\{w(\Gamma) | \Gamma \text{ is a spanning network on } S'\}$$

for every $S \in 2^N \backslash \{\emptyset\}$, with the convention that $c_w(\emptyset) = 0$.

The *dual unanimity game* (N, u_R^*) on $R \subseteq N$ is the game described by $u_R(T) = 1$ if $R \cap T \neq \emptyset$ and $u_R(T) = 0$, otherwise. Every cost game (N, c) can be written as a linear combination of dual unanimity games in a unique way, i.e. $c = \sum_{S \subseteq N, S \neq \emptyset} \alpha_S(c) u_S^*$. So, these dual unanimity games form a basis of the linear space $\mathcal{G}^N$. The coefficients $(\alpha_S(v))_{S \in 2^N \backslash \{\emptyset\}}$ are called *dual unanimity coefficients* of the cost game (N, c).

Let $\mathcal{H}^N \subseteq \mathcal{G}^N$. We call a map $\psi : \mathcal{H}^N \to \mathbb{R}^N$ assigning to every cost game $(N, c) \in \mathcal{H}^N$ a unique *cost allocation* in $\mathbb{R}^N$ a *value.* A value ψ is *efficient* if we have $\sum_{i \in N} \psi_i(c) = c(N)$ for each $c \in \mathcal{G}^N$. A value $\psi : \mathcal{H}^N \to \mathbb{R}^N$ is called *linear* if $\psi(\beta v + \gamma u) = \beta\psi(v) + \gamma\psi(u)$ for all games $v, u \in \mathcal{H}^N$ and real numbers $\beta, \gamma \in \mathbb{R}$ such that $\beta v + \gamma u \in \mathcal{H}^N$.

We call a map $F : \mathcal{W}^{N'} \to \mathbb{R}^N$ assigning to every mcst situation w a unique cost allocation in $\mathbb{R}^N$ a *solution.* A solution F is a *cost monotonic* solution if for all mcst situations $w, \bar{w} \in \mathcal{W}^{N'}$ such that $w(\bar{e}) \leq \bar{w}(\bar{e})$ for one edge $\bar{e} \in E_{N'}$ and $w(e) = \bar{w}(e)$ for each $e \in E_{N'} \setminus \{\bar{e}\}$, it holds that $F(w) \leq F(\bar{w})$.

The most famous value in the theory of cost games is the *Shapley value*, introduced by Shapley (1953). This value can be described in several ways. In view of the considerations presented in Section 7, we introduce the Shapley value ϕ applied to game $(N, c) \in \mathcal{G}^N$ in terms of the dual unanimity coefficients $(\alpha_S(c))_{S \in 2^N \backslash \{\emptyset\}}$, that is for each $i \in N$

$$\phi_i(c) = \sum_{S \subseteq N : i \in S} \frac{\alpha_S(c)}{|S|}. \tag{4}$$

Finally, a particular set, possibly empty, of allocations of a cost game (N, c) is the *core*, which is defined as follows:

$$core(c) = \{x \in I\!R^N | \sum_{i \in S} x_i \leq c(S) \ \forall S \in 2^N \setminus \{\emptyset\}; \sum_{i \in N} x_i = c(N)\}.$$

3 Conservative charge systems

To introduce the definition of a charge system we need some additional notations. Let $N = \{1, \ldots, |N|\}$ and $\Delta(N) = \{x \in I\!R^N_+ | \sum_{i \in N} x_i = 1\}$. We denote by $\mathcal{E}_{N'}$ the set of $|N|$-vectors of edges which form a spanning tree on N', i.e.

$$\mathcal{E}^{|N|}_{N'} = \{(a_1, \ldots, a_{|N|}) \in \prod_{i=1}^{|N|} E_{N'} | \{a_1, \ldots, a_{|N|}\} \text{ is a spanning network}\}.$$

Given an element $\mathbf{a} = (a_1, \ldots, a_{|N|}) \in \prod_{i=1}^{|N|} E_{N'}$, we denote by $\mathbf{a}_{|j}$ the restriction of $\mathbf{a}$ to the first j components, that is $\mathbf{a}_{|j} = (a_1, \ldots, a_j)$ for each $j \in \{1, \ldots, |N|\}$.

Further, for each $j \in \{1, \ldots, |N|\}$, we denote by $\Pi(\mathbf{a}_{|j})$ the partition of N' such that

$$\Pi(\mathbf{a}_{|j}) = \{T \subseteq N' | T \text{ is a connected component in } < N', \{a_1, \ldots, a_j\} >\},$$

for each $\mathbf{a} = (a_1, \ldots, a_{|N|}) \in \mathcal{E}^{|N|}_{N'}$.

Now, let $\theta \in \Theta(N')$, where $\Theta(N')$ is the family of partitions of N' and let $T \subseteq N'$. We denote by $S(\theta, T)$ the unique element of θ, if any, of which T is a subset.

Definition 1 A *charge system* $\mathcal{C}$ on N is a set of functions $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ with $C^j : \{\mathbf{a}_{|j} : \mathbf{a} \in \mathcal{E}^{|N|}_{N'}\} \to \Delta(N)$ for each $j \in \{1, \ldots, |N|\}$ satisfying the following properties:

(Connection property):	$C_i^j(\mathbf{a}_{\vert j}) = 0$ for each $i \in S(\Pi(\mathbf{a}_{\vert j-1}), \{0\})$, each $j \in \{1, \dots, \vert N\vert\}$, and each $\mathbf{a} = (a_1, \dots, a_{\vert N\vert}) \in \mathcal{E}_{N'}^{\vert N\vert}$;
(Involvement property):	$C_i^j(\mathbf{a}_{\vert j}) = 0$ for each $i \in N \setminus S(\Pi(\mathbf{a}_{\vert j}), a_j)$ each $j \in \{1, \dots, \vert N\vert\}$, and each $\mathbf{a} = (a_1, \dots, a_{\vert N\vert}) \in \mathcal{E}_{N'}^{\vert N\vert}$;
(Total aggregation property):	$\sum_{j=1}^{\vert N\vert} C_i^j(\mathbf{a}_{\vert j}) = 1$ for each $i \in N$, and each $\mathbf{a} = (a_1, \dots, a_{\vert N\vert}) \in \mathcal{E}_{N'}^{\vert N\vert}$.

Summing up, each element $\mathbf{a} \in \mathcal{E}_{N'}^{|N|}$ tells the "history" of the spanning network formation, that is adding the edge a_j to the already formed graph $\mathbf{a}_{|j-1}$, for each $j \in \{1, \dots, |N|\}$ (note that when the first edge a_1 is formed, the already formed graph is $< N', \emptyset >$. So $\Pi(\mathbf{a}_{|0})$ is the singleton partition of N'.).

As it will be explained in more detail in the next section, a charge system specifies how to allocate fractions of the edge a_j, for each $j \in \{1, \dots, |N|\}$, according to the three properties in Definition 1 in the spanning network corresponding to $\mathbf{a} \in \mathcal{E}_{N'}^{|N|}$. The connection property says that agents already connected to the source in $\mathbf{a}_{|j-1}$ should not be charged anymore. The involvement property says that only agents who are connected to nodes in a_j in the graph $\mathbf{a}_{|j}$ (i.e. *involved agents* in forming a_j) should be charged with fractions of a_j. The total aggregation property says that when the construction of the spanning network corresponding to $\mathbf{a}$ is completed, each agent has been charged for a total amount of fractions equal to 1.

The charge systems in Examples 1-4 will play a role in Section 4 to define special construct and charge rules. Let $\mathbf{a} \in \mathcal{E}_{N'}^{|N|}$. Briefly, the charge system of Example 1 charges the involved agents in forming the edge a_j, for each $j \in \{1, \dots, |N|\}$, taking into account the cardinality of their connected components in the graphs $\mathbf{a}_{|j-1}$ and $\mathbf{a}_{|j}$; the charge system of Example 2 charges the involved agents in forming the edge a_j, for each $j \in \{1, \dots, |N|\}$, proportionally to the fractions charged for some previously formed edges; the charge system of Example 3 charges uniquely one involved agent each time an edge a_j is formed, for each $j \in \{1, \dots, |N|\}$; the intuition behind the charge system of Example 4 is to charge equally the involved agents in the same connected component in $\mathbf{a}_{|j}$, for each $j \in \{1, \dots, |N|\}$, keeping into account the constraint given by the total aggregation property .

Example 1 *Consider the charge system $\hat{\mathcal{C}} = \{\hat{C}^1, \dots, \hat{C}^{|N|}\}$ on N such that for each $\boldsymbol{a} = (a_1, \dots, a_{|N|}) \in \mathcal{E}_{N'}^{|N|}$ and for each $j \in \{1, \dots, |N|\}$*

$$\hat{C}_i^j(\boldsymbol{a}_{|j}) = \begin{cases} \frac{1}{|S(\Pi(\boldsymbol{a}_{|j-1}), \{i\})|} - \frac{1}{|S(\Pi(\boldsymbol{a}_{|j}), \{i\})|} & \text{if } \{0,i\} \cap S(\Pi(\boldsymbol{a}_{|j}), a_j) = \{i\}, \\ \frac{1}{|S(\Pi(\boldsymbol{a}_{|j-1}), \{i\})|} & \text{if } \{0,i\} \cap S(\Pi(\boldsymbol{a}_{|j}), a_j) = \{0,i\} \\ & \text{and } \{0\} \cap S(\Pi(\boldsymbol{a}_{|j-1}), \{i\}) = \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

for each $i \in N$. One can easily check that the functions $\hat{C}^1, \dots, \hat{C}^{|N|}$ take values in $\Delta(N)$.

Example 2 *Consider the charge system $\tilde{\mathcal{C}} = \{\tilde{C}^1, \dots, \tilde{C}^{|N|}\}$ on N such that for each $\boldsymbol{a} = (a_1, \dots, a_{|N|}) \in \mathcal{E}_{N'}^{|N|}$ and for each $i \in N$*

$$\tilde{C}_i^1(\boldsymbol{a}_{|1}) = \begin{cases} \frac{1}{2} & \text{if } \{0,i\} \cap S(\Pi(\boldsymbol{a}_{|1}), a_1) = \{i\}, \\ 1 & \text{if } \{0,i\} \cap S(\Pi(\boldsymbol{a}_{|1}), a_1) = \{0,i\}, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

and for each $j \in \{2, \dots, |N|\}$

$$\tilde{C}_i^j(\boldsymbol{a}_{|j}) = \begin{cases} \frac{1}{2} m_i^j & \text{if } \{0,i\} \cap S(\Pi(\boldsymbol{a}_{|j}), a_j) = \{i\}, \\ 1 - \sum_{k=1}^{j-1} \tilde{C}_i^k(\boldsymbol{a}_{|k}) & \text{if } \{0,i\} \cap S(\Pi(\boldsymbol{a}_{|j}), a_j) = \{0,i\}, \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

where

$$m_i^j = \begin{cases} \min_{l\in\{1,\dots,j-1\}:\tilde{C}_i^l(\boldsymbol{a}_{|l})\neq 0} \tilde{C}_i^l(\boldsymbol{a}_{|l}) & \text{if } S(\Pi(\boldsymbol{a}_{|j-1}),\{i\})\neq\{i\}, \\ \\ 1 & \text{otherwise.} \end{cases} \tag{7}$$

We prove by induction to j that the function $\tilde{C}^j$, $j\in\{1,\dots,|N|\}$, takes values in $\Delta(N)$.

If $j=1$ it is easy to check in relation (5) that $\tilde{C}^1$ takes values in $\Delta(N)$.

Now let $j\in\{2,\dots,|N|\}$ and suppose that $\tilde{C}^l$ takes values in $\Delta(N)$ for every $l\in\{1,\dots,j-1\}$. Let $\boldsymbol{a}=(a_1,\dots,a_{|N|})\in\mathcal{E}_{N'}^{|N|}$. We distinguish two cases.

The first case is $0\notin S(\Pi(\boldsymbol{a}_{|j}),a_j)$. Then there exist $s,t\in\{1,\dots,j-1\}$ such that $S(\Pi(\boldsymbol{a}_{|s}),a_s)\cup S(\Pi(\boldsymbol{a}_{|t}),a_t) = S(\Pi(\boldsymbol{a}_{|j}),a_j)$ and $S(\Pi(\boldsymbol{a}_{|s}),a_s)\cap S(\Pi(\boldsymbol{a}_{|t}),a_t)=\emptyset$. Moreover, by relation (6) $\tilde{C}_i^v(\boldsymbol{a}_{|v})=0$ for each $i\in S(\Pi(\boldsymbol{a}_{|s}),a_s)$ and each $v\in\{s+1,\dots,j-1\}$; and $\tilde{C}_i^w(\boldsymbol{a}_{|w})=0$ for each $i\in S(\Pi(\boldsymbol{a}_{|t}),a_t)$ and each $w\in\{t+1,\dots,j-1\}$. So,

$$\sum_{i\in S(\Pi(\boldsymbol{a}_{|s}),a_s)} m_i^j = \sum_{i\in S(\Pi(\boldsymbol{a}_{|s}),a_s)} \tilde{C}_i^s(\boldsymbol{a}_{|s}) = 1 \tag{8}$$

where the first equality follows from relation (7) and the second equality from the induction hypothesis. Analogously,

$$\sum_{i\in S(\Pi(\boldsymbol{a}_{|t}),a_t)} m_i^j = \sum_{i\in S(\Pi(\boldsymbol{a}_{|t}),a_t)} \tilde{C}_i^t(\boldsymbol{a}_{|t}) = 1 \tag{9}$$

By relation (6), (8) and (9)

$$\sum_{i\in N}\tilde{C}_i^j(\boldsymbol{a}_{|j}) = \sum_{i\in S(\Pi(\boldsymbol{a}_{|j}),a_j)} \tilde{C}_i^j(\boldsymbol{a}_{|j}) = \sum_{i\in S(\Pi(\boldsymbol{a}_{|s}),a_s)} \frac{1}{2}m_i^j + \sum_{i\in S(\Pi(\boldsymbol{a}_{|t}),a_t)} \frac{1}{2}m_i^j = 1. \tag{10}$$

The last case is $0\in S(\Pi(\boldsymbol{a}_{|j}),a_j)$. Then

$$\begin{aligned} &\textstyle\sum_{i\in N}\tilde{C}_i^j(\boldsymbol{a}_{|j}) = \sum_{i\in S(\Pi(\boldsymbol{a}_{|j}),a_j)\setminus\{0\}}\tilde{C}_i^j(\boldsymbol{a}_{|j}) \\ &\textstyle= \sum_{i\in S(\Pi(\boldsymbol{a}_{|j}),a_j)\setminus\{0\}}\left(1-\sum_{k=1}^{j-1}\tilde{C}_i^k(\boldsymbol{a}_{|k})\right) \\ &\textstyle= |S(\Pi(\boldsymbol{a}_{|j}),a_j)\setminus\{0\}\}| - \sum_{k=1}^{j-1}\sum_{i\in S(\Pi(\boldsymbol{a}_{|j}),a_j)\setminus\{0\}}\tilde{C}_i^k(\boldsymbol{a}_{|k}) \\ &= |S(\Pi(\boldsymbol{a}_{|j}),a_j)\setminus\{0\}| - \big(|S(\Pi(\boldsymbol{a}_{|j}),a_j)\setminus\{0\}|-1\big) = 1, \end{aligned}$$

where the first equality follows from the involvement property of $\tilde{\mathcal{C}}$, the second equality from relation (6) and the fourth equality follows from the fact that to connect nodes in $S(\Pi(\boldsymbol{a}_{|j}), a_j)$ are needed $|S(\Pi(\boldsymbol{a}_{|j}), a_j)|$ edges and on stages from 1 to $j-1$ precisely $|S(\Pi(\boldsymbol{a}_{|j}), a_j)|-1$ edges have been already constructed and, by the induction hypothesis and relation (6), totally divided among some nodes in $S(\Pi(\boldsymbol{a}_{|j}), a_j)$.

Since in both cases it is evident that $\tilde{C}^j$ takes values in $I\!R^N_+$, for each $j \in \{1, \ldots, |N|\}$, we conclude that $\tilde{C}^1, \ldots, \tilde{C}^{|N|}$ take values in $\Delta(N)$.

Example 3 *Given a bijection $\tau : N \rightarrow \{1, 2, \ldots, |N|\}$, let the charge system $\mathcal{C}^\tau = \{C^{\tau,1}, \ldots, C^{\tau,|N|}\}$ on N be such that for each $\boldsymbol{a} = (a_1, \ldots, a_{|N|}) \in \mathcal{E}^{|N|}_{N'}$ and for each $i \in N$*

$$C^{\tau,1}_i(\boldsymbol{a}_{|1}) = \begin{cases} 1 & \text{if } \tau(i) = \max\{\tau(k) | k \in S(\Pi(\boldsymbol{a}_{|1}), a_1) \setminus \{0\}\}, \\ \\ 0 & \text{otherwise,} \end{cases}$$

and for each $j \in \{2, \ldots, |N|\}$

$$C^{\tau,j}_i(\boldsymbol{a}_{|j}) = \begin{cases} 1 & \text{if } \tau(i) = \max\{\tau(k) | k \in S(\Pi(\boldsymbol{a}_{|j}), a_j) \\ & \text{and } \sum_{l=1}^{j-1} C^{\tau,l}_k(\boldsymbol{a}_{|l}) \neq 1\}, \\ \\ 0 & \text{otherwise.} \end{cases}$$

To prove that the functions $C^{\tau,1}, \ldots, C^{\tau,|N|}$ take values in $\Delta(N)$ is left to the reader.

Example 4 *Consider the charge system $\check{\mathcal{C}} = \{\check{C}^1, \ldots, \check{C}^{|N|}\}$ on N such that for each $\boldsymbol{a} = (a_1, \ldots, a_{|N|}) \in \mathcal{E}^{|N|}_{N'}$ and each $i \in N$*

$$\check{C}^1_i(\boldsymbol{a}_{|1}) = \begin{cases} \frac{1}{2} & \text{if } \{0, i\} \cap S(\Pi(\boldsymbol{a}_{|1}), a_1) = \{i\}, \\ \\ 1 & \text{if } \{0, i\} \cap S(\Pi(\boldsymbol{a}_{|1}), a_1) = \{0, i\}, \\ \\ 0 & \text{otherwise,} \end{cases}$$

and for each $j \in \{2, \ldots, |N|\}$

$$\check{C}_i^j(\boldsymbol{a}_{|j}) = \begin{cases} \min\{1 - \sum_{k=1}^{j-1} \check{C}_i^k(\boldsymbol{a}_{|k}), \alpha\} & \textit{if } i \in S(\Pi(\boldsymbol{a}_{|j}), a_j), \\ \\ 0 & \textit{otherwise.} \end{cases}$$

where $\alpha \in I\!R^+$ *is the unique real number such that*

$$\sum_{i \in S(\Pi(\boldsymbol{a}_{|j}), a_j) \setminus \{0\}} \min\{1 - \sum_{k=1}^{j-1} \check{C}_i^k(\boldsymbol{a}_{|k}), \alpha\} = 1. \tag{11}$$

From relation (11) it directly follows that the functions $\check{C}^1, \ldots, \check{C}^{|N|}$ *take values in* $\Delta(N)$*.*

Remark 1 *We leave for the reader the straightforwardly exercise to prove that* $\hat{\mathcal{C}}, \tilde{\mathcal{C}}, \mathcal{C}^\tau$*, where* $\tau \in \Sigma_N$ *(*Σ_N *is the set of all bijections* $\tau : N \to \{1, \ldots, |N|\}$*), and* $\check{\mathcal{C}}$ *on* N *are indeed charge systems, i.e. all satisfy the connection property, the involvement property and the total aggregation property.*

In this paper, special charge systems, which we call *conservative*, will play a role. Consider a charge system $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ on N. We define the *aggregate contribution* of the charge system $\mathcal{C}$ on $\mathbf{a}_{|j}$, for each $j \in \{1, \ldots, |N|\}$ and for each $\mathbf{a} = (a_1, \ldots, a_{|N|}) \in \mathcal{E}_{N'}^{|N|}$, as the $|N|$-vector $\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j})$ calculated via the following formula

$$\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{k=1}^{j} C^k(\mathbf{a}_{|k}). \tag{12}$$

Definition 2 Let $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ be a charge system on N. We call $\mathcal{C}$ a *conservative charge system* if for all $j \in \{1, \ldots, |N|\}$ and for each pair $\mathbf{a}, \mathbf{b} \in \mathcal{E}_{N'}^{|N|}$, with $\Pi(\mathbf{a}_{|j}) = \Pi(\mathbf{b}_{|j})$ we have that

$$\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j}). \tag{13}$$

The peculiarity of conservative charge systems is that they preserve the aggregate contribution from the network construction history, i.e. the aggregate contribution corresponding to $\mathbf{a}_{|j}$, for $\mathbf{a} \in \mathcal{E}_{N'}^{|N|}$ and $j \in \{1, \ldots, |N|\}$, is only dependent on the partition of N' induced by the connected components in $< N', \{a_1, \ldots, a_j\} >$.

The proof of the following lemma is straightforward.

Lemma 1 *Let $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ be a conservative charge system on N and let $S \subseteq N'$. Let $\boldsymbol{a} = (a_1, \ldots, a_{|N|}), \boldsymbol{b} = (b_1, \ldots, b_{|N|}) \in \mathcal{E}_{N'}^{|N|}$ be such that $\Pi(\boldsymbol{a}_{|j}) = \Pi(\boldsymbol{b}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$, with $j \in \{1, \ldots, |N|\}$. Then*

$$\mathrm{A}^{\mathcal{C}}(\boldsymbol{a}_{|j}) = \mathrm{A}^{\mathcal{C}}(\boldsymbol{b}_{|j}).$$

We denote by $P^{\mathcal{C}}(S) \in I\!R_+^N$ the unique aggregate charge corresponding to the partition $\{S, \{i\}_{i \in N' \setminus S}\}$ for some $S \in 2^{N'} \setminus \{\emptyset\}$ and call it the *potential of* S w.r.t. the conservative charge system $\mathcal{C}$. The name potential is inspired by physics where each conservative vector field has a potential. In a connection situation, an intuitive interpretation of the potential $P^{\mathcal{C}}(S)$, $S \in 2^{N'} \setminus \{\emptyset\}$, is as the level of "connection work" done by nodes in N when $\{S, \{i\}_{i \in N' \setminus S}\}$ is the current set of connected components and the conservative charge system $\mathcal{C}$ is used. At the beginning of the connection process, when no edges are formed and all the connected components are singletons, the level of connection work performed by nodes should be zero. From this the convention that $P_i^{\mathcal{C}}(\{j\}) = P_i^{\mathcal{C}}(\{0\}) = 0$ for all $i, j \in N$. Other elementary properties of $P^{\mathcal{C}} : 2^{N'} \setminus \{\emptyset\} \to I\!R_+^N$ are collected in the following lemma:

Lemma 2 *Let $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ be a conservative charge system on N and let $S \in 2^{N'} \setminus \{\emptyset\}$. Let $P^{\mathcal{C}}$ be the potential w.r.t. $\mathcal{C}$. Then*

(c.1) if $0 \in S$ then $P^{\mathcal{C}}(S) = e^{S \setminus \{0\}}$;

(c.2) $P^{\mathcal{C}}(S) \in I\!R_+^N$ and $\sum_{i \in S} P_i^{\mathcal{C}}(S) = \sum_{i \in N} P_i^{\mathcal{C}}(S) = |S| - 1$;

(c.3) if $S \subseteq T \subseteq N'$, then $P^{\mathcal{C}}(S) \leq P^{\mathcal{C}}(T)$.

[Here $e^{S \setminus \{0\}} \in I\!R_+^N$ is such that $e_i^{S \setminus \{0\}} = 1$ for each $i \in S \setminus \{0\}$ and $e_i^{S \setminus \{0\}} = 0$ for each $i \in N \setminus S$.]

Proof

(c.1) Let $\mathbf{a} = (a_1, \ldots, a_{|N|}) \in \mathcal{E}_{N'}^{|N|}$ and $j \in \{1, \ldots, |N|\}$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$. Then, for each $i \in N \cap S$

$$P_i^{\mathcal{C}}(S) = \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}) = \textstyle\sum_{k=1}^{j} C_i^k(\mathbf{a}_{|k}) = 1 - \sum_{k=j+1}^{|N|} C_i^k(\mathbf{a}_{|k}) = 1,$$

where the third equality follows from the total aggregation property of $\mathcal{C}$ and the fourth equality follows from the connection property of $\mathcal{C}$. From the involvement property, we have $P_i^{\mathcal{C}}(S) = 0$ for each $i \in N \setminus S$, which finally proves property (c.1).

(c.2) If $0 \in S$ then condition (c.2) follows directly from condition (c.1).

Now consider the case $0 \notin S$. Let $\mathbf{a} = (a_1, \ldots, a_{|N|}) \in \mathcal{E}_{N'}^{|N|}$ and $j \in \{1, \ldots, |N|\}$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$. First note that since $0 \notin S$, $j = |S| - 1$. Then,

$$\begin{aligned} &\textstyle\sum_{i \in S} P_i^{\mathcal{C}}(S) = \sum_{i \in S} \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}) = \sum_{i \in S} \sum_{k=1}^{j} C_i^k(\mathbf{a}_{|k}) \\ &= \textstyle\sum_{k=1}^{j} \sum_{i \in S} C_i^k(\mathbf{a}_{|k}) = \sum_{k=1}^{j} 1 = |S| - 1, \end{aligned}$$

where the fourth equality follows from Definition 1. By the involvement property it follows that $P_i^{\mathcal{C}}(S) = 0$ for each $i \in N \setminus S$, which finally proves property (c.2).

(c.3) Let $\mathbf{a} = (a_1, \ldots, a_{|N|}) \in \mathcal{E}_{N'}^{|N|}$ and $j, l \in \{1, \ldots, |N|\}$ with $l \geq j$ be such that $\Pi(\mathbf{a}_{|j}) = \{S, \{i\}_{i \in N' \setminus S}\}$ and $\Pi(\mathbf{a}_{|l}) = \{T, \{i\}_{i \in N' \setminus T}\}$. Then,

$$\begin{aligned} &P^{\mathcal{C}}(S) = \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) = \textstyle\sum_{k=1}^{j} C^k(\mathbf{a}_{|k}) \\ &\leq \textstyle\sum_{k=1}^{j} C^k(\mathbf{a}_{|k}) + \sum_{k=j+1}^{l} C^k(\mathbf{a}_{|k}) \\ &= \textstyle\sum_{k=1}^{l} C^k(\mathbf{a}_{|k}) = \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|l}) = P^{\mathcal{C}}(T), \end{aligned}$$

which concludes the proof of property (c.3).

■

Proposition 1 *Let $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ be a conservative charge system on N. Let $\boldsymbol{a} = (a_1, \ldots, a_{|N|}) \in \mathcal{E}_{N'}^{|N|}$ and $j \in \{1, \ldots, |N|\}$ be such that $\Pi(\boldsymbol{a}_{|j}) = \{S_1, S_2, \ldots, S_m\}$, with $S_1, S_2, \ldots, S_m \subset N'$ and $m \leq n$. Then*

$$\mathrm{A}^{\mathcal{C}}(\boldsymbol{a}_{|j}) = \sum_{r=1}^{m} P^{\mathcal{C}}(S_r).$$

Proof Let $r \in \{1, 2, \ldots, m\}$. Determine $b^r(1), \ldots, b^r(p^r) \in \{1, \ldots, j\}$ such that $\Pi(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) = \{S_r, \{i\}_{i \in N' \setminus S_r}\}$ where $p^r = |S_r| - 1$.

Then for each $i \in N \setminus S_r$, by the involvement property of $\mathcal{C}$

$$P_i^{\mathcal{C}}(S_r) = \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) = 0$$

whereas for each $i \in N \cap S_r$

$$\begin{aligned} &P_i^{\mathcal{C}}(S_r) = \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(p^r)}) \\ &= \mathrm{A}_i^{\mathcal{C}}(a_{b^r(1)}, \ldots, a_{b^r(p^r)}, (a_s)_{s \in \{1, \ldots, j\} \setminus \{b^r(1), \ldots, b^r(p^r)\}}) \\ &= \mathrm{A}_i^{\mathcal{C}}(a_1, a_2, \ldots, a_j) = \mathrm{A}_i^{\mathcal{C}}(\mathbf{a}_{|j}), \end{aligned}$$

where the second equality follows from the involvement property in the edge sequence $(a_{b^r(1)}, a_{b^r(2)}, \ldots, a_{b^r(j)})$ and the third equality follows from the fact that $\mathcal{C}$ is conservative. Consequently, $\sum_{r=1}^{m} P^{\mathcal{C}}(S_r) = \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j})$. ■

4 Construct and charge rules for mcst situations

Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^{\sigma}$. We can consider a sequence of precisely $|E_{N'}| + 1$ graphs $< N', F^{\sigma,0} >, < N', F^{\sigma,1} >, \ldots,$ $< N', F^{\sigma,|E_{N'}|} >$ such that $F^{\sigma,0} = \emptyset$, $F^{\sigma,k} = F^{\sigma,k-1} \cup \{\sigma(k)\}$ for each $k \in \{1, \ldots, |E_{N'}|\}$. For each graph $< N', F^{\sigma,k} >$, with $k \in \{0, 1, \ldots, |E_{N'}|\}$, let $\pi^{\sigma,k}$ be the partition of $N \cup \{0\}$ consisting of the connected components of N' in $< N', F^{\sigma,k} >$.

Remark 2 *For each $k \in \{1, \ldots, |E_{N'}|\}$, $\pi^{\sigma,k}$ is either equal to $\pi^{\sigma,k-1}$ or obtained from $\pi^{\sigma,k-1}$ by forming the union of two elements of $\pi^{\sigma,k-1}$.*

Now we define recursively a function $\rho^\sigma : \{0, 1, \ldots, |N|\} \rightarrow \{0, 1, \ldots, |E_{N'}|\}$ by

- $\rho^\sigma(0) = 0$
- $\rho^\sigma(j) = \min\{k \in \{\rho^\sigma(j-1)+1, \ldots, |E_{N'}|\} | \pi^{\sigma,k} \neq \pi^{\sigma,\rho^\sigma(j-1)}\}$

for each $j \in \{1, \ldots, |N|\}$.

Note that $\pi^{\sigma,\rho^\sigma(i)} \neq \pi^{\sigma,\rho^\sigma(j)}$ for each $i, j \in \{0, 1, \ldots, |N|\}$ with $i \neq j$, and $\sigma(\rho^\sigma(1)), \ldots, \sigma(\rho^\sigma(|N|))$ corresponds to the $|N|$ accepted edges in the Kruskal procedure based on the ordering σ.

Example 5 *Consider the mcst situation* $< N', w >$ *with* $N' = \{0, 1, 2, 3\}$ *and* w *as depicted in Figure 1. Note that* $w \in K^\sigma$*, with* $\sigma(1) = \{1, 2\}$, $\sigma(2) = \{1, 3\}$, $\sigma(3) = \{2, 3\}$, $\sigma(4) = \{1, 0\}$, $\sigma(5) = \{2, 0\}$, $\sigma(6) = \{3, 0\}$.

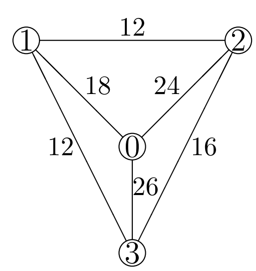

Figure 1: An mcst situation with three agents.

The sequence of seven graphs $< N', F^{\sigma,k} >$ *and the corresponding sequence of partitions* $\pi^{\sigma,k}$ *are shown in the following table*

k	$F^{\sigma,k}$	$\pi^{\sigma,k}$
0	$\{\emptyset\}$	$\{\{0\}, \{1\}, \{2\}, \{3\}\}$
1	$\{\{1, 2\}\}$	$\{\{0\}, \{1, 2\}, \{3\}\}$
2	$\{\{1, 2\}, \{1, 3\}\}$	$\{\{0\}, \{1, 2, 3\}\}$
3	$\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$	$\{\{0\}, \{1, 2, 3\}\}$
4	$\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 0\}\}$	$\{N \cup \{0\}\}$
5	$\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 0\}, \{2, 0\}\}$	$\{N \cup \{0\}\}$
6	$\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 0\}, \{2, 0\}, \{3, 0\}\}$	$\{N \cup \{0\}\}$

Then $\rho^\sigma(0) = 0,\ \rho^\sigma(1) = 1,\ \rho^\sigma(2) = 2,\ \rho^\sigma(3) = 4.$

Definition 3 Let $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ be a charge system on N. Let $\sigma \in \Sigma_{E_{N'}}$. The *Construct & Charge* (*CC*-)*rule* w.r.t. $\mathcal{C}$ and σ is the map $F^{\mathcal{C},\sigma} : K^\sigma \to I\!R^N$ given by

$$F^{\mathcal{C},\sigma}(w) = \sum_{r=1}^{|N|} w(\sigma(\rho^\sigma(r)))C^r(\sigma(\rho^\sigma(1)), \ldots, \sigma(\rho^\sigma(r))). \tag{14}$$

for each mcst situation w in the cone K^σ.

Remark 3 CC-rules $F^{\mathcal{C},\sigma}$ where $\mathcal{C}$ is a conservative charge system are called *conservative* CC*-rules.*

Definition 4 Let $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ be a charge system on N. We say that $\mathcal{C}$ has the *patch property* if for all $\sigma_1, \sigma_2 \in \Sigma_{E_{N'}}$:

$$F^{\mathcal{C},\sigma_1}(w) = F^{\mathcal{C},\sigma_2}(w)$$

for each w in the cone $K^{\sigma_1} \cap K^{\sigma_2}$.

If $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ has the patch property, we can define the map $F^{\mathcal{C}}$ on $\mathcal{W}^{N'}$ by

$$F^{\mathcal{C}}(w) = F^{\mathcal{C},\sigma}(w)$$

where $w \in \mathcal{W}^{N'}$ and $\sigma \in \Sigma_{E_{N'}}$ is such that $w \in K^\sigma$.

Remark 4 *The* P*-value (Branzei et al. (2004), Feltkamp et al. (1994b)) and the* P^τ*-values, with* $\tau \in \Sigma_N$*, introduced in Norde et al. (2004) and studied in Branzei et al. (2004), are* CC*-rules whose charge systems have the patch property, as proved in Tijs et al.(2005). In fact* $F^{\hat{\mathcal{C}}}(w) = P(w)$*, where* $\hat{\mathcal{C}}$ *is the charge system of Example 1, and* $F^{\mathcal{C}^\tau}(w) = P^\tau(w)$ *for each* $\tau \in \Sigma_N$*, where* $\mathcal{C}^\tau$ *is the charge system of Example 3. Moreover, for all* $\sigma \in \Sigma_{E_{N'}}$*, the* CC*-rule* $F^{\tilde{\mathcal{C}},\sigma}$*, where* $\tilde{\mathcal{C}}$ *is the charge system of Example 2, corresponds to the* Proportional rule *introduced in Feltkamp et al.(1994a).*

Example 6 *Consider the mcst situation* $< N', w >$ *with* $N' = \{0, 1, 2, 3\}$ *and* w *as depicted in Figure 1. Let* σ *be as in Example 5 and* $\sigma'(1) = \{1, 3\}$, $\sigma'(2) = \{1, 2\}$, $\sigma'(3) = \{2, 3\}$, $\sigma'(4) = \{1, 0\}$, $\sigma'(5) = \{2, 0\}$, $\sigma'(6) = \{3, 0\}$.

- *The charge system* $\hat{\mathcal{C}}$ *of Example 1 leads to*

$$F^{\hat{\mathcal{C}},\sigma}(w) = F^{\hat{\mathcal{C}},\sigma'}(w) = (14, 14, 14)^t.$$

- *The charge system* $\tilde{\mathcal{C}}$ *of Example 2 leads to*

$$F^{\tilde{\mathcal{C}},\sigma}(w) = (\frac{27}{2}, \frac{27}{2}, 15)^t$$

and

$$F^{\tilde{\mathcal{C}},\sigma'}(w) = (\frac{27}{2}, 15, \frac{27}{2})^t.$$

- *The charge system* $\mathcal{C}^\tau$ *of Example 3 with* $\tau(1) = 1, \tau(2) = 2, \tau(3) = 3$ *leads to*

$$F^{\mathcal{C}^\tau,\sigma}(w) = F^{\mathcal{C}^\tau,\sigma'}(w) = (18, 12, 12)^t.$$

- *The charge system* $\check{\mathcal{C}}$ *of Example 4 leads to*

$$F^{\check{\mathcal{C}},\sigma} = (13, 13, 16)^t$$

and

$$F^{\check{\mathcal{C}},\sigma'}(w) = (13, 16, 13)^t.$$

Theorem 1 *Let* $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ *be a charge system on* N*. If* $\mathcal{C}$ *has the patch property, then* $\mathcal{C}$ *is conservative.*

Proof Suppose $\mathcal{C}$ is not conservative. Then we can find a $j \in \{1, \ldots, |N|\}$ and a pair $\mathbf{a} = (a_1, \ldots, a_{|N|}), \mathbf{b} = (b_1, \ldots, b_{|N|}) \in \mathcal{E}_{N'}^{|N|}$, with $\Pi(\mathbf{a}_{|j}) = \Pi(\mathbf{b}_{|j})$ and $\mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}) \neq \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j})$.

Suppose $\Pi(\mathbf{a}_{|j}) = \{S_1, S_2, \ldots, S_m\}$ and take $w \in \mathcal{W}^{N'}$ such that

$$w(\{i, j\}) = \begin{cases} 0 & \text{if there exists } r \in \{1, \ldots, m\} \text{ s.t. } i, j \in S_r, \\ 1 & \text{otherwise,} \end{cases}$$

for each $\{i,j\} \in E_{N'}$. Let $\sigma_1 \in \Sigma_{E_{N'}}$ be such that $\sigma_1(\rho^{\sigma_1}(k)) = a_k$ for each $k \in \{1,\ldots,j\}$ and $\sigma_1(\rho^{\sigma_1}(l)) = d_l$ for each $l \in \{j+1,\ldots,|N|\}$, with $(a_1,\ldots,a_j,d_{j+1},\ldots,d_{|N|}) \in \mathcal{E}_{N'}^{|N|}$.

Let $\sigma_2 \in \Sigma_{E_{N'}}$ be such that $\sigma_2(\rho^{\sigma_2}(k)) = b_k$ for each $k \in \{1,\ldots,j\}$ and $\sigma_2(\rho^{\sigma_2}(l)) = d_l$ for each $l \in \{j+1,\ldots,|N|\}$, with $(b_1,\ldots,b_j,d_{j+1},\ldots,d_{|N|}) \in \mathcal{E}_{N'}^{|N|}$.

Then $w \in K^{\sigma_1} \cap K^{\sigma_2}$. Further,

$$\begin{aligned}
&F^{\mathcal{C},\sigma_1}(w) = \\
&= \textstyle\sum_{r=1}^{j} w(a_r)C^r(\mathbf{a}_{|r}) + \sum_{r=j+1}^{|N|} w(d_r)C^r(a_1,\ldots,a_j,d_{j+1},\ldots,d_r) \\
&= \textstyle\sum_{r=j+1}^{|N|} C^r(a_1,\ldots,a_j,d_{j+1},\ldots,d_r) \\
&= e^N - \textstyle\sum_{r=1}^{j} C^r(\mathbf{a}_{|r}) \\
&= e^N - \mathrm{A}^{\mathcal{C}}(\mathbf{a}_{|j}).
\end{aligned}$$

where the fourth equality follows from the total aggregation property.

Similarly,

$$F^{\mathcal{C},\sigma_2}(w) = e^N - \sum_{r=1}^{j} C^r(\mathbf{b}_{|r}) = e^N - \mathrm{A}^{\mathcal{C}}(\mathbf{b}_{|j}).$$

So, $F^{\mathcal{C},\sigma_1}(w) \neq F^{\mathcal{C},\sigma_2}(w)$, which yields a contradiction with the fact that $\mathcal{C}$ has the patch property. ■

Remark 5 From Theorem 1 and Remark 3 we conclude that the P-value and the P^τ-values with $\tau \in \Sigma_N$ are conservative CC-rules.

5 Conservative CC-rules are cost monotonic solutions

The main result in this section is derived from the relation between Obligation rules (Tijs et al. (2004)) and conservative CC-rules.

We first recall some definitions from Tijs et al. (2004). A function $o : 2^N \setminus \{\emptyset\} \to I\!R_+^N$ is called an *obligation function* if the following two properties hold for each $S \in 2^N \setminus \{\emptyset\}$:

o.1) $o(S) \in \Delta(S)$,

o.2) for each $T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$: $o_i(S) \geq o_i(T)$ for all $i \in S$,

where the sub-simplex $\Delta(S)$ of $\Delta(N) = \{x \in I\!R^N_+ | \sum_{i \in N} x_i = 1\}$ is given by $\Delta(S) = \{x \in \Delta(N) | \sum_{i \in S} x_i = 1\}$.

Given an obligation function o, the *obligation map* $\hat{o} : \Theta(N') \rightarrow I\!R^N$ is defined by $\hat{o}(\theta) = \sum_{S \in \theta, 0 \notin S} o(S)$ for each $\theta \in \Theta(N')$.

Let $\hat{o}$ be an obligation map on $\Theta(N')$ and let $\sigma \in \Sigma_{E_{N'}}$. The map $\phi^{\sigma,\hat{o}} : K^\sigma \rightarrow I\!R^N$ defined for each $w \in K^\sigma$ by

$$\phi^{\sigma,\hat{o}}(w) = \sum_{r=1}^{|E_{N'}|} w(\sigma(r)) \left(\hat{o}(\pi^{\sigma,r-1}) - \hat{o}(\pi^{\sigma,r})\right) \tag{15}$$

or, alternatively,

$$\phi^{\sigma,\hat{o}}(w) = \sum_{r=1}^{|N|} w(\sigma(\rho^\sigma(r))) \left(\hat{o}(\pi^{\sigma,\rho^\sigma(r-1)}) - \hat{o}(\pi^{\sigma,\rho^\sigma(r)})\right) \tag{16}$$

is used in Tijs et al. (2004) to prove that

$$\phi^{\hat{o}}(w) := \phi^{\sigma,\hat{o}}(w) = \phi^{\sigma',\hat{o}}(w) \tag{17}$$

for all $w \in K^\sigma \cap K^{\sigma'}$ (*patch property*), leading to the definition of *Obligation rule* as the map $\phi^{\hat{o}} : \mathcal{W}^{N'} \rightarrow I\!R^N$.

Other interesting properties for such maps, proved in Tijs et al. (2004), are collected in the next theorem.

Theorem 2 *Let $w \in \mathcal{W}^{N'}$. Let $\hat{o}$ be an obligation map on $\Theta(N')$. The following properties hold for the Obligation rule $\phi^{\hat{o}} : \mathcal{W}^{N'} \rightarrow I\!R^N$*

*i) (*cost monotonicity*) $\phi^{\hat{o}}$ is a cost monotonic solution for mcst situations;*

*ii) (*stability*) $\phi^{\hat{o}}(w)$ belongs to the core of the cost game (N, c_w) for every $w \in \mathcal{W}^{N'}$.*

In the following theorem, we relate conservative charge systems and CC-rules with obligation functions and Obligation rules.

Theorem 3 *Let* $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ *be a conservative charge system on* N *and let* $P^{\mathcal{C}}(S)$ *be the potential of* S *with respect to the conservative charge system* $\mathcal{C}$ *for each* $S \in 2^N \setminus \{\emptyset\}$. *Consider the map* $o^{\mathcal{C}} : 2^N \setminus \{\emptyset\} \rightarrow I\!R^N_+$ *defined by*

$$o_i^{\mathcal{C}}(S) = \begin{cases} 1 - P_i^{\mathcal{C}}(S) & \text{if } i \in S, \\ \\ 0 & \text{if } i \notin S, \end{cases} \tag{18}$$

for each $i \in N$ *and for each* $S \in 2^N \setminus \{\emptyset\}$. *Then,*

i) $o^{\mathcal{C}}$ *is an obligation function;*

ii) $\phi^{\hat{o}^{\mathcal{C}}}(w) = F^{\mathcal{C},\sigma}(w) = F^{\mathcal{C},\sigma'}(w)$ *for all* $\sigma, \sigma' \in \Sigma_{E_{N'}}$ *and* $w \in K^\sigma \cap K^{\sigma'}$, *i.e.* $\mathcal{C}$ *has the patch property.*

Proof

i) We have to prove that for $o^{\mathcal{C}}$ the properties $o.1$ and $o.2$ hold.

By definition it follows directly that $o_i^{\mathcal{C}}(S) = 0$ for each $i \in N \setminus S$ and $o_i^{\mathcal{C}}(S) \geq 0$ for each $i \in S$ and for each $S \in 2^N \setminus \{\emptyset\}$. Moreover, from condition (c.2) it follows that

$$\sum_{i \in N} o_i^{\mathcal{C}}(S) = \sum_{i \in S} 1 - P_i^{\mathcal{C}}(S) = |S| - (|S| - 1) = 1,$$

for each $S \in 2^N \setminus \{\emptyset\}$, implying that condition (o.1) holds.

Finally, by condition (c.3), we have that for each $i \in S \subseteq T \subseteq N$

$$o_i^{\mathcal{C}}(S) = 1 - P_i^{\mathcal{C}}(S) \geq 1 - P_i^{\mathcal{C}}(T) = o_i^{\mathcal{C}}(T) \tag{19}$$

for each $S \in 2^N \setminus \{\emptyset\}$ and for each $i \in S$, which proves that condition (o.2) holds too.

ii) First note that by relation (16)

$$\begin{aligned}
&\phi^{\sigma,\hat{o}^{\mathcal{C}}}(w) = \textstyle\sum_{r=1}^{|N|} w(\sigma(\rho^\sigma(r)))\ \big(\hat{o}^{\mathcal{C}}(\pi^{\sigma,\rho^\sigma(r-1)}) - \hat{o}^{\mathcal{C}}(\pi^{\sigma,\rho^\sigma(r)})\big)\\
&= \textstyle\sum_{r=1}^{|N|} w(\sigma(\rho^\sigma(r)))\Big(\big(\sum_{S\in\pi^{\sigma,\rho^\sigma(r-1)}} o^{\mathcal{C}}(S)\big) - \big(\sum_{S\in\pi^{\sigma,\rho^\sigma(r)}} o^{\mathcal{C}}(S)\big)\Big)\\
&= \textstyle\sum_{r=1}^{|N|} w(\sigma(\rho^\sigma(r)))\Big(\big(e^N - \sum_{S\in\pi^{\sigma,\rho^\sigma(r-1)}} P^{\mathcal{C}}(S)\big)\\
&-\big(e^N - \textstyle\sum_{S\in\pi^{\sigma,\rho^\sigma(r)}} P^{\mathcal{C}}(S)\big)\Big)\\
&= \textstyle\sum_{r=1}^{|N|} w(\sigma(\rho^\sigma(r)))\big(- A^{\mathcal{C}}(\sigma(\rho^\sigma(1)),\ldots,\sigma(\rho^\sigma(r-1)))\\
&+A^{\mathcal{C}}(\sigma(\rho^\sigma(1)),\ldots,\sigma(\rho^\sigma(r)))\big)\\
&= \textstyle\sum_{r=1}^{|N|} w(\sigma(\rho^\sigma(r)))C^r(\rho^\sigma(1)),\ldots,\sigma(\rho^\sigma(r)) = F^{\mathcal{C},\sigma}(w).
\end{aligned} \tag{20}$$

where the third equality follows from relation (18) and the fourth one from the definition of potential. Now the proof of ii) follows directly from i) and relations (17) and (20) on the obligation rule $\phi^{\hat{o}^{\mathcal{C}}}$.

■

The next theorem which follows from Theorems 1 and 3 is our main result in this section.

Theorem 4 *For each charge system $\mathcal{C} = \{C^1, \ldots, C^{|N|}\}$ on N the following statements are equivalent:*

i) $\mathcal{C}$ is a conservative charge system;

ii) $\mathcal{C}$ has the patch property;

iii) the CC-rule w.r.t. to $\mathcal{C}$ is an Obligation rule.

From Theorems 2 and 4 we conclude that conservative CC-rules are stable and cost monotonic solutions for mcst situations.

6 Conservative CC-rules and sharing values for cost games

In this section the set of Obligation rules, and consequently the set of conservative CC-rules, will be considered from a value-theoretic point of view.

A *sharing system* is a map $q : 2^N \setminus \{\emptyset\} \to I\!R^N_+$ such that $q(S) \in \Delta(S)$, for every nonempty coalition S.
With every sharing system q one can associate a *sharing value* m^q, defined by

$$m^q_i(c) = \sum_{S:i\in S} q_i(S)\alpha_S(c) \tag{21}$$

for every $c \in \mathcal{G}^N$, every $i \in N$ and where $\alpha_S(c)$ for each $S \in 2^N \setminus \{\emptyset\}$ is the dual unanimity coefficient. In particular, with every obligation function o one can associate a special sharing value m^o.

The following lemmas are helpful in relating sharing values with Obligation rules.

Lemma 3 *Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^\sigma$. Then*

i) $\overline{w} = \overline{w}(\sigma(1))\overline{e^{\sigma,1}} + \sum_{k=2}^{|E_{N'}|}\big(\overline{w}(\sigma(k)) - \overline{w}(\sigma(k-1))\big)\overline{e^{\sigma,k}}$;

ii) $c_{\overline{w}} = \overline{w}(\sigma(1))c_{\overline{e^{\sigma,1}}} + \sum_{k=2}^{|E_{N'}|}\big(\overline{w}(\sigma(k)) - \overline{w}(\sigma(k-1))\big)c_{\overline{e^{\sigma,k}}}$.

Proof The proof directly follows from relation (2) and by Proposition 6 in Tijs et al. (2005). ■

The core of the game $c_{\overline{w}}$ is a refinement of the core of the game c_w and has been characterized in Tijs et al. (2005) via monotonicity and additivity properties.

Let o be an obligation function and let $\hat{o}$ be the corresponding obligation map on $\Theta(N')$. From relation (2) and the definition of Obligation rule in Tijs et al. (2004) it follows that an alternative way of calculating $\phi^{\hat{o}}(w)$ for each $w \in \mathcal{W}^{N'}$ as linear combination of $\phi^{\hat{o}}(e^{\sigma,k})$, $k \in \{1,\dots,|E_{N'}|\}$, where $\sigma \in \Sigma_{E_{N'}}$ is such that $w \in K^\sigma$, will be useful in the following. In formula,

$$\phi^{\hat{o}}(w) = w(\sigma(1))\phi^{\hat{o}}(e^{\sigma,1}) + \sum_{k=2}^{|E_{N'}|}\big(w(\sigma(k)) - w(\sigma(k-1))\big)\phi^{\hat{o}}(e^{\sigma,k}). \tag{22}$$

Let $k \in \{1,\dots,|E_{N'}|\}$. Consider $e^{\sigma,k} \in K^\sigma$. By relation (15) it follows that

$$\phi^{\hat{o}}(e^{\sigma,k}) = \sum_{r=k+1}^{|E_{N'}|}\big(\hat{o}(\pi^{\sigma,r-1}) - \hat{o}(\pi^{\sigma,r})\big) = \hat{o}(\pi^{\sigma,k}) = \sum_{V\in\pi^{\sigma,k}:0\notin V} o(V). \tag{23}$$

Recall also that in Tijs et al. (2005) (see their Proposition 5 and Remark 5) it has been proved that

$$\phi^{\hat{o}}(w) = \phi^{\hat{o}}(\bar{w}) \tag{24}$$

for every $w \in \mathcal{W}^{N'}$.

Now, we can introduce the following lemma.

Lemma 4 *Let $\sigma \in \Sigma_{E_{N'}}$ and let $e^{\sigma,k} \in K^{\sigma}$, $k \in \{1, \ldots, |E_{N'}|\}$. Let $\hat{o}$ be an obligation map on $\Theta(N')$. Then*

i) $c_{\overline{e^{\sigma,k}}} = \sum_{V \in \pi^{\sigma,k} : 0 \notin V} u^*_V$;

ii) $m^o(c_{\overline{e^{\sigma,k}}}) = \phi^{\hat{o}}(\overline{e^{\sigma,k}})$.

Proof First note that $\overline{e^{\sigma,k}} \in K^{\sigma}$ (see Lemma 2 in Tijs et al. (2005)).

i) follows from the fact that for each $S \in 2^N \setminus \{\emptyset\}$,

$$c_{\overline{e^{\sigma,k}}}(S) = |\{V \ : \ V \text{ is a } (\overline{e^{\sigma,k}}, N') - \text{component}, V \cap S \neq \emptyset, 0 \notin V\}|,$$

where the $(\overline{e^{\sigma,k}}, N')$-components are precisely the elements of the partition $\pi^{\sigma,k}$;

ii) From i) and the linearity of m^o it follows that

$$\begin{aligned} &m^o(c_{\overline{e^{\sigma,k}}}) = m^o(\textstyle\sum_{V \in \pi^{\sigma,k} : 0 \notin V} u^*_V) \\ &= \textstyle\sum_{V \in \pi^{\sigma,k} : 0 \notin V} m^o(u^*_V) \\ &= \textstyle\sum_{V \in \pi^{\sigma,k} : 0 \notin V} o(V) = \phi^{\hat{o}}(\overline{e^{\sigma,k}}), \end{aligned} \tag{25}$$

where the first equality follows by point i), the second equality follows from linearity of m^o, the third equality follows from relation (21) and the last equality follows from relations (23) and (24).

■

Now, we are able to prove the main result of this section.

Theorem 5 *Let $w \in \mathcal{W}^{N'}$ and let $\sigma \in \Sigma_{E_{N'}}$ be such that $w \in K^{\sigma}$. Let $\hat{o}$ be an obligation map on $\Theta(N')$. Then*

$$m^o(c_{\overline{w}}) = \phi^{\hat{o}}(\overline{w}).$$

Proof Note that

$$\begin{aligned}
&m^o(c_{\overline{w}})\\
&= \overline{w}(\sigma(1))m^o(c_{\overline{e^{\sigma,1}}}) + \textstyle\sum_{k=2}^{|E_{N'}|}\big(\overline{w}(\sigma(k)) - \overline{w}(\sigma(k-1))\big)m^o(c_{\overline{e^{\sigma,k}}})\\
&= \overline{w}(\sigma(1))\phi^{\hat{o}}(\overline{e^{\sigma,1}}) + \textstyle\sum_{k=2}^{|E_{N'}|}\big(\overline{w}(\sigma(k)) - \overline{w}(\sigma(k-1))\big)\phi^{\hat{o}}(\overline{e^{\sigma,k}})\\
&= \phi^{\hat{o}}(\bar{w}),
\end{aligned}$$

where the first equality follows from Lemma 3.ii and the linearity of m^o, the second equality from Lemma 4.ii and the third equality follows from relation (22). ■

Corollary 1 [1] *The P-value on $\overline{w}$ equals the Shapley value on $c_{\overline{w}}$.*

Proof Consider the charge system of Example 1. As we already said in Remark 5, such a charge system leads to a conservative CC-rule which corresponds to the P-value (Branzei et al.(2004)). The obligation function o^* obtained from the charge system of Example 1 via relation (18) is such that $o^*(S) = \frac{e^S}{|S|}$ for each $S \in 2^N \setminus \{\emptyset\}$ (see also Example 1 in Tijs et al. (2004)), where e^S is the $|N|$-vector such that $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \in N \setminus S$. Then, directly from relation (4) it follows that $m^{o^*}(c_{\overline{w}})$ is the Shapley value of the game $c_{\overline{w}}$.

■

7 Final Remarks

This paper deals with Construct and Charge rules for mcst situations based on the Kruskal algorithm.

The Prim algorithm (Prim (1957)) also generates a sequence of edges which form an mcst. In the first step an edge of minimal cost between a node in N and the source 0 is formed. In every subsequent step an edge

[1] In Bergañtinos and Vidal-Puga (2004) this result has also been proved in a different way.

of minimal cost between a node in N which is not connected yet with the source (directly or indirectly) and a node in N which is already connected with the source is formed. In every step of the algorithm there is precisely one node in N which gets a connection with the source, so the algorithm also stops after precisely n steps. C.G.Bird (1976) proposes a way to share the costs of each edge constructed via the Prim algorithm where each agent pays the first edge in which he is involved. This situation can also be seen as a construct and charge protocol, where an mcst tree is constructed edge by edge and where one of the players (the player who is just connected with the source) pays the edge just constructed. For more information on this rule see Feltkamp (1995).

Since we were interested in linearity properties of the construct and charge protocols with respect to mcst situations with the same orderings of the cost of the edges, we have focused our analysis only on construct and charge protocols based on the Kruskal algorithm.

Finally, note that in view of Theorem 4, Obligation rules on mcst situations can be seen as *weighted Shapley values* (Kalai and Samet (1987); see also Derks et al. (2000)) of the corresponding mcst games.

References

Bergañtinos, G., Vidal-Puga, J.J., 2004b. Defining rules in cost spanning tree problems through the canonical form, EconPapers, RePEc:wpa:wuwpga: 0402004.

Bird, C.G., 1976. On cost allocation for a spanning tree: a game theoretic approach, Networks, 6, 335-350.

Branzei, R., Moretti, S., Norde, H., Tijs, S., 2004. The *P*-value for cost sharing in minimum cost spanning tree situations, Theory and Decision 56, 47-61 (and also CentER DP 2003 nr.129, Tilburg University, The Netherlands).

Derks, J., Haller, H., Peters, H., 2000. The selectope for cooperative games, International Journal of Game Theory, 29, 23-38.

Feltkamp, V., 1995. Cooperation in controlled network structures, PhD Dissertation, Tilburg University, The Netherlands.

Muto, S., 1994a. Minimum cost spanning extension problems: the proportional rule and the decentralized rule, CentER DP 1994 nr.96, Tilburg University, The Netherlands.

Feltkamp, V., Tijs, S., Muto, S., 1994b. On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems, CentER DP 1994 nr.106, Tilburg University, The Netherlands.

Kalai, E., Samet, D., 1987. On weighted Shapley values, International Journal of Game Theory, 16, 205-222.

Kruskal, J.B., 1956. On the shortest spanning subtree of a graph and the traveling salesman problem, Proceedings of the American Mathematical Society, 7, 48-50.

Norde, H., Moretti, S., Tijs, S., 2004. Minimum cost spanning tree games and population monotonic allocation schemes, European Journal of Operational Research, 154, 84-97 (and also CentER DP 2001 nr.18, Tilburg University, The Netherlands).

Prim, R.C., 1957. Shortest connection networks and some generalizations, Bell Systems Technical Journal 36, 1389-1401.

Shapley, L.S., 1953. A Value for n-Person Games, in Contributions to the Theory of Games II (Annals of Mathematics Studies 28), H. W. Kuhn and A. W. Tucker (eds.), Princeton University Press, 307-317.

Sprumont, Y., 1990. Population monotonic allocation schemes for cooperative games with transferable utility, Games and Economic Behavior, 2, 378-394.

Tijs, S., Branzei, R., Moretti, S., Norde, H., 2004. Obligation rules for minimum cost spanning tree situations and their monotonicity properties, CentER DP 2004-53, Tilburg University, The Netherlands (to appear in European Journal of Operational Research).

Tijs, S., Moretti, S., Branzei, R., Norde, H., 2005. The Bird core for minimum cost spanning tree problems revisited: monotonicity and additivity aspects, CentER DP 2005 nr.3, Tilburg University, The Netherlands (to appear in Recent Advances in Optimization, Lectures Notes in Economics and Mathematical Systems, Springer-Verlag ed.).