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Combinatorial Integer Labeling Theorems on Finite Sets with an Application to Discrete Systems of Nonlinear Equations*

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Abstract

Tucker's well-known combinatorial lemma states that for any given symmetric triangulation of the n-dimensional unit cube and for any integer labeling that assigns to each vertex of the triangulation a label from the set $\{\pm 1, \pm 2, \cdots, \pm n\}$ with the property that antipodal vertices on the boundary of the cube are assigned opposite labels, the triangulation admits a 1-dimensional simplex whose two vertices have opposite labels. In this paper we are concerned with an arbitrary finite set D of integral vectors in the n-dimensional Euclidean space and an integer labeling that assigns to each element of D a label from the set $\{\pm 1, \pm 2, \cdots, \pm n\}$. Using a constructive approach we prove two combinatorial theorems of Tucker type, stating that under some mild conditions there exists two integral vectors in D having opposite labels and being cell-connected in the sense that both belong to the set $\{0, 1\}^n + q$ for some integral vector q. These theorems will be used to show in a constructive way the existence of an integral solution to a system of nonlinear equations under certain natural conditions.

Keywords: Sperner lemma, Tucker lemma, integer labeling, simplicial algorithm, discrete nonlinear equations.

JEL classification: C61, C62, C68, C72, C58.

1 Introduction

Fixed point theorems are fundamental tools for establishing the existence of a solution to various problems in many fields of study such as economics, mathematics and engineering. It is well-known that Brouwer's fixed point theorem is implied by Sperner's combinatorial lemma on the unit simplex and that the Borsuk-Ulam antipodal point theorem is implied by Tucker's combinatorial lemma on the unit cube.

For any given triangulation of the $(n-1)$-dimensional unit simplex $S^n = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$, where $\mathbb{R}^n_+$ is the nonnegative orthant of the n-dimensional Euclidean space $\mathbb{R}^n$, and any given labeling function assigning to every vertex of the triangulation an integer label from the set $\{1, 2, \cdots, n\}$, such that a vertex x is not assigned label k whenever $x_k = 0$, Sperner's lemma states that the triangulation contains a completely labeled $(n-1)$-dimensional simplex, being a simplex whose n vertices carry all the labels from 1 up to n. In his pioneering work, Scarf (1967, 1973) gave the first constructive method for finding a completely labeled simplex in Sperner's lemma and thus led to a constructive proof of Brouwer's theorem as well. Later on, more efficient and more flexible simplicial algorithms were developed by Kuhn (1969), Eaves (1972), Merrill (1972), van der Laan and Talman (1979) among many others.

Tucker's lemma (see Tucker (1945) and Lefschetz (1949)) asserts that for any given symmetric triangulation of the n-dimensional unit cube $C^n = \{x \in \mathbb{R}^n \mid |x_i| \leq 1, i = 1, 2, \cdots, n\}$ and any given labeling function assigning to every vertex of the triangulation an integer label from the set $\{\pm 1, \pm 2, \cdots, \pm n\}$, such that antipodal vertices of the triangulation on the boundary of C^n are assigned opposite labels (labels that sum to zero), there exists a complementary one-dimensional simplex, i.e., its two vertices have opposite labels. Tucker's lemma implies the famous Borsuk-Ulam antipodal point theorem, a more powerful result than Brouwer's theorem. Based on the $2n$-ray integer labeling algorithm of van der Laan and Talman (1981) and Reiser (1981), Freund and Todd (1981) have given an elegant constructive proof for Tucker's lemma, and thus a constructive proof of Borsuk-Ulam theorem as well, see also van der Laan (1984) and Yang (1999).

The combinatorial lemmas of Sperner and Tucker have been extended in various ways. For instance, Fan (1967) studied Sperner's lemma in a broader and more abstract setting. Shapley (1973) generalized Sperner's lemma by using a set labeling rule instead of an integer labeling rule. Bapat (1989) established a permutation-based generalization of Sperner's lemma. Freund (1986, 1989), van der Laan and Talman (1982), Yamamoto (1988), and van der Laan, Talman and Yang (2001) proved the existence of a completely labeled simplex for an arbitrarily given triangulation of the hypercube, the simplotope and any bounded polyhedron, respectively. Su (1999) presented an interesting application of Sperner's lemma to a fair division problem. Prescott and Su (2005) gave a constructive proof of Ky Fan's generalization of Tucker's lemma.

In this paper we present for the class of finite integrally convex sets several new results on the existence of two vectors in the set having opposite labels and being cell-connected, i.e., both vectors belong to the finite set $\{0,1\}^n + q$ for some integral vector q. The concept of integrally convex set was introduced by Favati and Tardella (1990) and has been well studied in the literature, see e.g. Murota (2003) and Fujishige (2005). Many well-known and important integral polyhedra generate integrally convex sets, for instance, the set $P \cap \mathbb{Z}^n$ is integrally convex for any base polyhedron or generalized polymatriod P, where $\mathbb{Z}^n$ is the integer lattice in $\mathbb{R}^n$. The basic property of an integrally convex set is that its convex hull admits a cubical triangulation for which the set of vertices is equal to the set itself and every simplex is contained in a cube of size one, i.e., a cell equal to $[0,1]^n + \{q\}$ for some $q \in \mathbb{Z}^n$.

For a finite integrally convex set D in $\mathbb{R}^n$ and an integer labeling function $\ell\colon D \to \{\pm 1, \pm 2, \cdots, \pm n\}$, our combinatorial theorems establish the existence of two cell-connected points in D with opposite labels under certain mild conditions. To prove the theorems, we apply the $2n$-ray integer labeling algorithm of van der Laan and Talman (1981) and Reiser (1981). This algorithm, originally developed to approximate a zero point of a continuous function from the n-dimensional unit cube to $\mathbb{R}^n$, is adapted in such way that it can be used on a cubical triangulation of the convex hull of D. Starting with an arbitrary point in D, the algorithm generates a sequence of adjacent simplices of variable dimension in the triangulation and finds in a finite number of steps a complementary one-dimensional simplex and therefore two cell-connected points with opposite labels. Until now, no such algorithm has been proposed on an arbitrary finite integrally convex set.

The current research was motivated by the recently renewed interest on discrete fixed (or zero) point problems or discrete nonlinear equation problems and by the study on economies with indivisibility; see Iimura (2003), Iimura, Murota and Tamura (2005), Danilov and Koshevoy (2004), Yang (2004, 2007), van der Laan, Talman and Yang (2005, 2006, 2007), Talman and Yang (2005). The research on this subject can date back to Tarski (1955), who proved the existence of a fixed point for a monotone function mapping from a finite complete lattice into itself. Zhou (1994) extended Tarski's theorem to point-to-set mappings. The discrete fixed point theorems obtained by the recent authors, however, need not impose the restrictive monotonicity assumption and thus can be applied to a broader range of problems. The combinatorial results in this paper will be used to establish in a constructive way more general discrete zero (or fixed) point theorems by showing the existence of an integral solution to a system of nonlinear equations.

This paper is organized as follows. In Section 2 we present basic concepts and definitions. In Section 3 we establish and prove the two main combinatorial labeling theorems.

In Section 4 we apply these theorems to show the existence of an integral solution to a general system of nonlinear equations.

2 Basic concepts and definitions

For a given positive integer n, let N denote the set $\{1, 2, \ldots, n\}$, $\mathbb{R}^n$ the n-dimensional Euclidean space, and $\mathbb{Z}^n$ the integer lattice of $\mathbb{R}^n$. For $x,\ y \in \mathbb{R}^n$, $x \cdot y$ stands for the inner product of x and y. For $i \in N$, $e(i)$ denotes the ith unit vector of $\mathbb{R}^n$ and $e(-i)$ denotes the ith negative unit vector, i.e., $e(-i) = -e(i)$. The symbol 0^n stands for the n-vector of zeroes. A set $X \subset \mathbb{R}^n$ is *symmetric* if $x \in X$ implies $-x \in X$. For an arbitrary set $X \subset \mathbb{R}^n$, $\overline{X}$ and ∂X denote the convex hull of X and the relative boundary of X, respectively.

Two integral points x and y in $\mathbb{Z}^n$ are *cell-connected* if both x and y belong to the set $\{0, 1\}^n + \{q\}$ for some $q \in \mathbb{Z}^n$. For an integer t, $0 \leq t \leq n$, the t-dimensional convex hull of $t+1$ affinely independent points $x^1, \ldots, x^{t+1}$ in $\mathbb{R}^n$ is a *t-simplex* or *simplex* and will be denoted by $\sigma(x^1, \ldots, x^{t+1})$. The extreme points $x^1, \ldots, x^{t+1}$ of a t-simplex $\sigma(x^1, \ldots, x^{t+1})$ are the *vertices* of σ. The convex hull of any subset of $k+1$ vertices of a t-simplex σ, $1 \leq k \leq t$, is a k-dimensional *face* or *k-face* of σ. A k-face of a t-simplex σ is a *facet* of σ if $k = t-1$, i.e., if the number of vertices is just one less than the number of vertices of the simplex. A simplex is *integral* if all of its vertices are integral vectors, i.e., all vertices are points in $\mathbb{Z}^n$. An integral simplex is *cubical* if all its vertices lie in a cube $\{0, 1\}^n + \{q\}$ for some $q \in \mathbb{Z}^n$. Notice that any two vertices of a cubical simplex are cell-connected.

Given an m-dimensional convex set X in $\mathbb{R}^n$, a collection $\mathcal{T}$ of m-dimensional simplices is a *triangulation* of X, if (i) X is the union of all simplices in $\mathcal{T}$, (ii) the intersection of any two simplices of $\mathcal{T}$ is either empty or a common face of both, and (iii) any neighborhood of any point in X only meets a finite number of simplices of $\mathcal{T}$. We refer to Todd (1976) and Yang (1999) for a more detailed description of triangulations. A triangulation is *cubical* if all its simplices are cubical. A triangulation $\mathcal{T}$ of a symmetric convex set is *symmetric* if $\sigma \in \mathcal{T}$ implies $-\sigma \in \mathcal{T}$.

A set $D \subseteq \mathbb{Z}^n$ is *integrally convex* if (i) $D = \mathbb{Z}^n \cap \overline{D}$ and (ii) the set $\overline{D}$ is a union of cubical simplices. Straightforward examples of integrally convex sets are the sets $\mathbb{Z}^n$, $\mathbb{Z}^n_+$, $\{-1, 0, 1\}^n$ and $P \cap \mathbb{Z}^n$ for any base polyhedron or generalized polymatroid P. The concept of integrally convex set was originally introduced by Favati and Tardella (1990) and has been well studied in the literature, see Murota (2003) and Fujishige (2005). It is known from Iimura, Murota and Tamura (2005, Lemma 1) that for any integrally convex set D there exists a cubical triangulation of its convex hull. From their proof it also follows that if D is a symmetric integrally convex set, there exists a symmetric cubical triangulation of the convex hull of D. An integrally convex set D in $\mathbb{Z}^n$ is *regular* if $\overline{D}$ is n-dimensional and contains an integral point in its interior. Let

$$\mathcal{I} = \{T \subset N \cup (-N) \mid i \in T \text{ implies } -i \notin T\}$$

and for any $T \in \mathcal{I}$, let $e(T) = \sum_{i \in T} e(i)$. Then the convex hull of a regular integrally convex set D in $\mathbb{Z}^n$ can be expressed as

$$\overline{D} = \{x \in \mathbb{R}^n \mid e(T) \cdot x \leq b_T \text{ for any } T \in \mathcal{J}\} \tag{2.1}$$

for a family $\mathcal{J} \subset \mathcal{I}$ and integers b_T, $T \in \mathcal{J}$, so that no constraint is redundant. Further, for any subset $\mathcal{K}$ of $\mathcal{J}$, define

$$F(\mathcal{K}) = \{x \in \overline{D} \mid e(T) \cdot x = b_T \text{ for all } T \in \mathcal{K}\}.$$

Then either $F(\mathcal{K})$ is empty or it is a face of $\overline{D}$. A set $\mathcal{K} \subset \mathcal{J}$ is *admissible*, when $F(\mathcal{K})$ is a face of $\overline{D}$. For an admissible set $\mathcal{K}$, we define $I(\mathcal{K}) = \cup_{T \in \mathcal{K}} T$. A vector $w \in \mathbb{R}^n$ is *outgoing* at $x \in D$ if $x + \epsilon w \notin \overline{D}$ for any $\epsilon > 0$. Notice that for $h \in N \cup (-N)$ the vector $e(h)$ is outgoing at $x \in D$ if and only if $h \in I(\mathcal{K})$, where $\mathcal{K} \subset \mathcal{J}$ is such that x lies in the relative interior of the face $F(\mathcal{K})$ of $\overline{D}$. Recall that $F(\emptyset) = \overline{D}$. We conclude this section with an example.

Example 1. Let the set D be given by the set of integral points

$$D = \{x \in \mathbb{Z}^2 \mid -2 \leq x_i \leq 2,\ i = 1, 2, \text{ and } -2 \leq x_1 + x_2 \leq 2\}.$$

Then D is a finite and regular integrally convex set. The family $\mathcal{J}$ is given by

$$\mathcal{J} = \{\{1\}, \{-1\}, \{2\}, \{-2\}, \{1, 2\}, \{-1, -2\}\}$$

and $b_T = 2$ for all $T \in \mathcal{J}$. The point $(2, 0)^\top$ lies in $F(\{\{1\}, \{1, 2\}\})$, so that $e(1)$ and $e(2)$ are the unique outgoing unit vectors at $(2, 0)^\top$. The point $(2, -1)^\top$ lies in (the relative interior of) $F(\{\{1\}\})$, so that $e(1)$ is the only outgoing unit vector at $(2, -1)^\top$. Finally, the point $(2, -2)^\top$ lies in $F(\{\{1\}, \{-2\}\})$, so that $e(1)$ and $e(-2)$ are the unique outgoing unit vectors at $(2, -2)^\top$. □

3 Combinatorial Integer Labeling Theorems

Given a set $X \subset \mathbb{R}^n$ and a finite set of integers K, an integer labeling function $\ell \colon X \to K$ assigns to every point in X an element of K. $\ell(x)$ is called the label of x. When X is an integrally convex set D in $\mathbb{Z}^n$ and $\ell \colon D \to N \cup (-N)$ is an integer labeling function, two points $x, y \in D$ are said to be *opposite labeled* if $\ell(x) = -\ell(y)$. In this section we provide constructive proofs for two new combinatorial theorems on the existence of two opposite labeled cell-connected points in a finite and regular integrally convex set. In the first theorem we impose a condition of Sperner type on the labeling function to hold for points in D that lie on the boundary of the convex hull of D.

Theorem 3.1 *Let D be a finite and regular integrally convex set in $\mathbb{Z}^n$ and let $\ell\colon D \to N \cup (-N)$ be an integer labeling function such that for any admissible set $\mathcal{K} \subset \mathcal{J}$ and integral boundary point $x \in F(\mathcal{K}) \cap D$ it holds that $\ell(x) \notin I(\mathcal{K})$. Then D contains two opposite labeled cell-connected points.*

The boundary condition in this theorem can be formulated equivalently as follows. For any integral point of D in the boundary of $\overline{D}$ the integer labeling function satisfies that $\ell(x) \neq h$ whenever $e(h)$ is an outgoing vector at x. Recall that for $h \in -N$, $e(h) = -e(-h)$.

To prove the theorem we adapt the $2n$-ray algorithm of van der Laan and Talman (1981) and Reiser (1981). For an arbitrarily chosen integral point v in the interior of $\overline{D}$ (since D is regular such a point exists), we define for any $T \in \mathcal{I}$ the set $A(T)$ by

$$A(T) = \{x \in \overline{D} \mid \ x_i \geq v_i \text{ if } i \in T;\ x_i \leq v_i \text{ if } -i \in T; \text{ and } x_i = v_i \text{ otherwise}\}.$$

Observe that $A(\emptyset) = \{v\}$ and that $A(T)$ is a t-dimensional convex set for any $T \in \mathcal{I}$, where $t = |T|$. Since D is integrally convex, there exists a cubical triangulation of $\overline{D}$. Let $\mathcal{T}$ be such a triangulation of $\overline{D}$. Since $\mathcal{T}$ is cubical, $\mathcal{T}$ induces for every $T \in \mathcal{I}$ a subdivision of $A(T)$ into t-dimensional cubical simplices, with $t = |T|$. Any facet τ of a t-simplex σ in $A(T)$ is either a facet of exactly one other t-simplex σ' in $A(T)$ or lies on the boundary of $A(T)$. In the latter case τ is either a $(t-1)$-simplex in $A(T \setminus \{k\})$ for some unique $k \in T$ or a facet in $A(T)$ on the boundary of $\overline{D}$. Moreover, any t-dimensional simplex in $A(T)$ is cubical and therefore any two vertices of a t-dimensional simplex in $A(T)$ are cell-connected.

Given a t-simplex σ in $A(T)$, with $t = |T|$, a facet τ of σ is said to be *T-complete* if

$$\{\ell(x) \mid x \text{ is a vertex of } \tau\} = T.$$

In other words, a facet τ in $A(T)$ is T-complete if the t vertices of the $(t-1)$-dimensional τ are all labeled differently by the t integers in the set T. The next lemma states that a facet of a simplex in $A(T)$ on the boundary of D cannot be T-complete.

Lemma 3.2 *When the integer labeling function $\ell\colon D \to N \cup (-N)$ satisfies the boundary condition of Theorem* 3.1*, there does not exist a T-complete facet in $A(T) \cap \partial\overline{D}$ for any $T \in \mathcal{I}$.*

Proof. Suppose that τ is a T-complete facet in $A(T)$ for some $T \in \mathcal{I}$ and that τ lies on the boundary of $\overline{D}$. Then $\tau \subset A(T) \cap F(\mathcal{K})$ for some minimal admissible set $\mathcal{K} \subset \mathcal{J}$. We will first show that $T \cap I(\mathcal{K}) \neq \emptyset$. Suppose to the contrary that $T \cap I(\mathcal{K}) = \emptyset$. Take any point x in the relative interior of τ. Clearly, x is in the relative interior of $F(\mathcal{K})$. Since $T \cap I(\mathcal{K}) = \emptyset$, there exists $\bar{\epsilon} > 0$ such that for any ϵ, $0 \leq \epsilon \leq \bar{\epsilon}$,

$$x + \epsilon e(h) \in \overline{D}, \text{ for all } \quad h \in T.$$

Furthermore, since x is in the interior of τ and τ cannot lie in $A(T \setminus \{h\})$ for any $h \in T$, there exist unique $\lambda_i > 0$, $i \in T$, such that

$$x = v + \sum_{i \in T} \lambda_i e(i).$$

For $i \in T$, let μ_i be equal to

$$\mu_i = \lambda_i / \sum_{j \in T} \lambda_j,$$

and, for some ϵ, $0 \leq \epsilon \leq \bar{\epsilon}$, let

$$y = x + \epsilon \sum_{i \in T} \mu_i e(i).$$

Then the point

$$y = \sum_{h \in T} \mu_h (x + \epsilon e(h))$$

is a convex combination of elements of $\overline{D}$. Since $\overline{D}$ is a convex set, $y \in \overline{D}$. On the other hand, it holds that $x = \lambda v + (1 - \lambda) y$ for some $0 < \lambda < 1$. Since v is in the interior of $\overline{D}$ and y is in $\overline{D}$, the point x must lie in the interior of $\overline{D}$, contradicting that x lies on the boundary of $\overline{D}$. Consequently, $T \cap I(\mathcal{K}) \neq \emptyset$. Take any $k \in T \cap I(\mathcal{K})$. Since τ is T-complete, some vertex y of τ carries label k, but the boundary condition implies that no vertex of τ is labeled with k. This yields a contradiction. Therefore there can be no T-complete facet of a simplex in $A(T)$ lying in $\partial \overline{D}$. □

We are now ready to give the constructive proof of Theorem 3.1 by applying the $2n$-ray algorithm.

Proof of Theorem 3.1. Let $\ell(v)$ be the label of the point v. Then, with $T = \{\ell(v)\}$, $\{v\}$ in $A(\emptyset)$ is a T-complete 0-dimensional facet of the unique 1-dimensional simplex $\sigma^0 = \sigma(x^1, x^2)$ in $A(T)$ with vertices $x^1 = v$ and $x^2 = v + e(\ell(v))$. The algorithm starts from σ^0 with $T = \{\ell(v)\}$ and generates, for varying sets $T \in \mathcal{I}$, a sequence of adjacent t-simplices in $A(T)$ with T-complete common facets, where $t = |T|$. When the algorithm encounters a t-simplex σ in $A(T)$ having a vertex carrying a label k for some $k \notin T \cup (-T)$, then k is added to T and the algorithm continues in $A(T \cup \{k\})$ with the unique $(t+1)$-simplex in $A(T \cup \{k\})$ having σ as its facet. When the algorithm generates in $A(T)$ a T-complete facet τ lying in $A(T \setminus \{k\})$ for some $k \in T$, then k is deleted from T and the algorithm continues in $A(T \setminus \{k\})$ with the facet opposite the vertex of τ carrying label k. Since each step of the algorithm is uniquely determined, it follows by the well-known Lemke-Howson argument that a simplex will never be visited more than once, see van der Laan and Talman (1981) and Reiser (1981) for a more detailed description. Since the number of simplices in the triangulation of $\overline{D}$ is finite, the algorithm will terminate within a finite number of steps. The only two possibilities to stop is that either the algorithm generates a T-complete facet τ in the boundary $A(T) \cap \partial\overline{D}$ or the algorithm encounters a vertex carrying label $-k$ for some $k \in T$. By Lemma 3.2 the first case can not occur. Therefore the algorithm terminates with a cubical simplex having two vertices with labels k and $-k$ for some $k \in N$ and thus finds within a finite number of steps two opposite labeled cell-connected points in D. $\square$

Theorem 3.1 holds for any finite (regular) integrally convex set D, i.e., for any finite set D of integral points for which the convex hull $\overline{D}$ is a polyhedron as described in formula (2.1) and any integral point of $\overline{D}$ belongs to D itself. A special case is that $\overline{D}$ is an n-dimensional rectangular or cube given by

$$\overline{D} = C^n(a,b) = \{x \in \mathbb{R}^n \mid a_i \le x_i \le b_i,\ i \in N\},$$

where a, b are two integral n-vectors satisfying $b_i > a_i + 1$ for all $i \in N$. In this case the boundary condition of the labeling function reduces to $\ell(x) \neq -i$ whenever $x_i = a_i$ and $\ell(x) \neq i$ whenever $x_i = b_i$. We now have the following corollary, see also Freund (1986).

Corollary 3.3 *Let the set D be given by $D = \{x \in \mathbb{Z}^n \mid a_i \le x_i \le b_i,\ i \in N\}$, for some vectors $a, b \in \mathbb{Z}^n$ satisfying $b_i > a_i + 1$ for all $i \in N$, and let $\ell\colon D \to N \cup (-N)$ be an integer labeling function such that $\ell(x) \neq -i$ if $x_i = a_i$ and $\ell(x) \neq i$ if $x_i = b_i$. Then D contains two opposite labeled cell-connected points.*

The following corollary follows immediately from Theorem 3.1 by observing that when the boundary condition is not satisfied the algorithm either ends with two opposite labeled cell-connected points or with a T-complete facet in $A(T) \cap \partial\overline{D}$ for some T. According to the proof of Lemma 3.2, in the latter case we must have that for at least one element $k \in T$ it holds that $e(k)$ is an outgoing vector at τ.

Corollary 3.4 *Let $D \subset \mathbb{Z}^n$ be a finite and regular integrally convex set and let $\ell\colon D \to N \cup (-N)$ be an arbitrary integer labeling function. Then D contains two opposite labeled* *cell-connected points or a point x in the boundary of $\overline{D}$ such that for some $k \in N \cup (-N)$, $\ell(x) = k$ and $e(k)$ is an outgoing vector at x.*

The next corollary follows immediately by applying the previous corollary and Theorem 3.1 to the integer labeling function $\widetilde{\ell} : D \to N \cup (-N)$ given by $\widetilde{\ell}(x) = -\ell(x)$ for all $x \in D$.

Corollary 3.5 *Let $D \subset \mathbb{Z}^n$ be a finite and regular integrally convex set and let $\ell\colon D \to N \cup (-N)$ be an arbitrary integer labeling function. Then D contains two opposite labeled cell-connected points or a point x in the boundary of $\overline{D}$ such that for some $k \in N \cup (-N)$, $\ell(x) = k$ and $e(-k)$ is an outgoing vector at x. Moreover, when ℓ is such that for any admissible set $\mathcal{K} \subset \mathcal{J}$ and integral boundary point $x \in F(\mathcal{K}) \cap D$ it holds that $\ell(x) \neq -k$ for any $k \in I(\mathcal{K})$, then D contains two opposite labeled cell-connected points.*

We now state our second major existence theorem. This theorem generalizes the well-known Tucker's lemma to a finite and regular, symmetric integrally convex set. To prove the theorem we adapt the $2n$-algorithm in such a way that when a T-complete facet τ in $A(T)$ on the boundary of $\overline{D}$ is generated, the algorithm continues with the $(-T)$-complete facet $-\tau$ in $A(-T)$. This step is called *the reflection step* and is originally due to Freund and Todd (1981) who modified the original $2n$-ray algorithm of van der Laan and Talman (1981) and Reiser (1981) to provide a constructive proof for Tucker's lemma.

Theorem 3.6 *Let $D \subset \mathbb{Z}^n$ be a finite, regular and symmetric integrally convex set and let $\ell\colon D \to N \cup (-N)$ be an integer labeling function such that for any $x \in \partial\overline{D} \cap D$, $\ell(-x) = -\ell(x)$. Then the set D contains two opposite labeled cell-connected points.*

Proof. Since D is regular, integrally convex and symmetric, there exists a symmetric cubical triangulation $\mathcal{T}$ of $\overline{D}$. Moreover, the origin 0^n is an element of D and lies in the interior of $\overline{D}$. We take the origin as the starting point v of the $2n$-algorithm, i.e., for any $T \in \mathcal{I}$ the set $A(T)$ is defined by

$$A(T) = \{x \in \overline{D} \ \mid \ x_i \geq 0 \ \text{if} \ i \in T; \ x_i \leq 0 \ \text{if} \ -i \in T; \ \text{and} \ x_i = 0, \text{otherwise}\}.$$

Since $\mathcal{T}$ is cubical, $\mathcal{T}$ induces for every $T \in \mathcal{I}$ a subdivision of $A(T)$ into t-dimensional cubical simplices, where $t = |T|$. Moreover, by the symmetry of D it follows that $A(-T) = -A(T)$ and by the symmetry of $\mathcal{T}$ that if σ is a t-dimensional simplex in $A(T)$, then $-\sigma$ is a t-dimensional simplex in $A(-T)$. In particular it holds that when τ is a facet of a t-dimensional simplex in $A(T)$ lying on the boundary of $\overline{D}$, then $-\tau$ also lies on the boundary of $\overline{D}$ and is a facet of precisely one t-dimensional simplex in $A(-T)$.

The algorithm starts with the unique 1-dimensional simplex σ^0 in $A(T)$ having 0^n as vertex, where $T = \{\ell(0^n)\}$, and generates in the same way as in the proof of Theorem

3.1 a sequence of adjacent t-simplices in $A(T)$ with T-complete common facets for varying $T \in \mathcal{I}$, with the only difference that when a T-complete facet τ of a simplex in $A(T)$ is generated that lies on the boundary of $\overline{D}$, then a reflection step is made, i.e., the set T of labels becomes $-T$ and the algorithm jumps from τ to $-\tau$ and continues with the unique simplex in $A(-T)$ having $-\tau$ as its facet in order to generate adjacent t-simplices in $A(-T)$ with $(-T)$-complete common facets. Again each step of the algorithm is uniquely determined and it follows by the Lemke-Howson argument that no simplex can be visited more than once. Since D is finite and thus $\overline{D}$ is bounded, the number of simplices is finite. Therefore the algorithm terminates within a finite number of steps with a simplex in $\mathcal{T}$ having two opposite labeled vertices. Since any simplex in $\mathcal{T}$ is cubical, these two vertices are cell-connected. □

4 Application to Discrete Nonlinear Equations

Many problems arising from economics, game theory, engineering and other fields are formulated as a system of nonlinear equations. The most widely used tool for showing the existence of a solution to the system is Brouwer's and Kakutani's fixed point theorems. Brouwer's theorem states that a continuous function from a convex and compact subset of $\mathbb{R}^n$ to itself leaves one point fixed. The latter one generalizes the former to upper semi-continuous point-to-set mappings.

In many circumstances, for example, when the domain of interest is a discrete set or the presence of indivisibility in an economic model is significant, the continuity (upper semi-continuity) or convexity requirement of Brouwer's (Kakutani's) theorem is no longer fulfilled; see e.g., Kelso and Crawford (1982), Fudenberg and Tirole (1991), Scarf (1994), Zhou (1994), Gul and Stachetti (1999), Sun and Yang (2006). The purpose of this section is to investigate such cases in a general framework. Precisely, we wish to study two problems: (1) what condition guarantees the existence of a solution to the system of nonlinear equations

$$f_i(x_1, \cdots, x_n) = 0, \quad i = 1, \ldots, n, \tag{4.2}$$

where $f = (f_1, f_2, \cdots, f_n)^\top$ is a nonlinear function from $\mathbb{Z}^n$ to $\mathbb{R}^n$, and (2) how to find a solution, if a solution exists. A solution to the system (4.2) is also called a *discrete zero point* of f. In the last few years several authors have obtained results on this topic mainly for functions that are *direction preserving* or *locally gross direction preserving*. In this section we establish two existence theorems for the system (4.2) that are built upon the new class of coordinatewisely opposite extrema free functions. The relationship between (locally gross) direction preserving functions and coordinatewisely opposite extrema free functions will be discussed at the end of the section.

Definition 4.1 *Let $D \subset \mathbb{Z}^n$ be an integrally convex set. A function $f\colon D \to \mathbb{R}^n$ is* coordinatewisely opposite extrema free (COEF) *if for any two cell-connected points $x, y \in D$, there is no $k \in N$ such that $f_k(x)f_k(y) = -\max_{h\in N}|f_h(x)|\max_{h\in N}|f_h(y)| < 0$.*

The COEF condition concerns only those components of the function having nonzero absolute value extrema and states that no component of the function should have both the highest and the lowest value at two vertices within one cell. We can further relax the COEF condition so that it relies only on a particular cubical triangulation.

Definition 4.2 *Let $D \subset \mathbb{Z}^n$ be an integrally convex set. A function $f\colon D \to \mathbb{R}^n$ is* simplicially coordinatewise opposite extrema free (SCOEF) *if there exists a cubical triangulation $\mathcal{T}$ of $\overline{D}$ such that for any two vertices x and y of a simplex of $\mathcal{T}$, there exists no $k \in N$ which satisfies*

$$f_k(x)f_k(y) = -\max_{h\in N}|f_h(x)|\max_{h\in N}|f_h(y)| < 0.$$

When this condition is satisfied for some specific triangulation $\mathcal{T}$, the function is SCOEF with respect to $\mathcal{T}$.

We now present two theorems for the existence of a discrete zero point. The first theorem can be seen as the discrete analogue of Brouwer's fixed point theorem by observing that a zero point of f is a fixed point of the function g given by $g(x) = f(x) + x$. The second theorem can be seen as a discrete analogue of the famous Borsuk-Ulam theorem, see Freund and Todd (1981), van der Laan (1984) and Yang (1999). It should be observed that both theorems are stated for arbitrary (symmetric) integrally convex sets and are not restricted to the cube. Both theorems will be proved by applying the results of Section 3.

Theorem 4.3 *Let $D \subset \mathbb{Z}^n$ be a finite and regular integrally convex set and let $f\colon D \to \mathbb{R}^n$ be a SCOEF function. Suppose that for any $x \in \partial\overline{D} \cap D$ with $f(x) \neq 0^n$, $f_h(x) < \max_{i\in N}|f_i(x)|$ if $e(h)$ is outgoing at x for some $h \in N$, and $f_h(x) > -\max_{i\in N}|f_i(x)|$ if $e(-h)$ is outgoing at x for some $h \in N$. Then f has a discrete zero point in D.*

Theorem 4.4 *Let $D \subset \mathbb{Z}^n$ be a finite, regular and symmetric integrally convex set and let $f\colon D \to \mathbb{R}^n$ be a SCOEF function with respect to some symmetric cubical triangulation of $\overline{D}$. Suppose that for any $x \in \partial\overline{D} \cap D$ there exists some $k \in N$ such that $f_k(x)f_k(-x) = -\max_{i\in N}|f_i(x)|\max_{i\in N}|f_i(-x)|$. Then f has a discrete zero point in D.*

We prove both theorems by using the results of Section 3. To prove Theorem 4.3, we assign to each point x of the set D an integer from the set $N \cup (-N) \cup \{0\}$. The labeling rule is induced from the function f as follows. For given $x \in D$, let k be any element in N such that $|f_k(x)| = \max_{h \in N} |f_h(x)|$. Then, $\ell(x) = 0$ if $f_k(x) = 0$, $\ell(x) = k$ if $f_k(x) > 0$ and $\ell(x) = -k$ if $f_k(x) < 0$. Observe that $\ell(x) = 0$ if and only if $f(x) = 0^n$. The following lemma shows that this integer labeling rule excludes the possibility that any two cell-connected points are opposite labeled when the function satisfies the SCOEF property for some cubical triangulation of $\overline{D}$.

Lemma 4.5 *Let $f\colon D \to \mathbb{R}^n$ be a SCOEF function with respect to some cubical triangulation $\mathcal{T}$ of $\overline{D}$. Then there exists no simplex in $\mathcal{T}$ carrying both labels k and $-k$ for some $k \in N$.*

Proof. Suppose to the contrary that there is a simplex in $\mathcal{T}$ carrying labels k and $-k$ for some $k \in N$. Then for the vertex x with label k we have $f_k(x) = \max_h |f_h(x)| > 0$, while for the vertex y with label $-k$ we have $f_k(y) = -\max_h |f_h(x)| < 0$. Since x and y are in the same simplex, this contradicts the (simplicially) coordinatewise opposite extrema free property of the function f. $\square$

We now prove the first theorem.

Proof of Theorem 4.3. Let $\mathcal{T}$ be a cubical triangulation of $\overline{D}$ for which f satisfies the SCOEF property and let ℓ be the integer labeling function on D induced by f. From the boundary condition that for every $h \in N$ it holds that $f_h(x) < \max_{i \in N} |f_i(x)|$ if $e(h)$ is outgoing at x and that $f_h(x) > -\max_{i \in N} |f_i(x)|$ if $e(-h)$ is outgoing at x for any $x \in \partial\overline{D} \cap D$ with $f(x) \neq 0^n$, it follows that the function ℓ satisfies the boundary condition of Theorem 3.1 for any $x \in \partial\overline{D} \cap D$ satisfying $f(x) \neq 0^n$. Suppose that the function f has no discrete zero point and thus $\ell(x) \neq 0$ for any $x \in D$. Then it follows from Theorem 3.1 that there exists two cell-connected integral points with opposite labels. However, this contradicts Lemma 4.5. Hence, there must be a point in D with label 0, which proves the existence of a discrete zero point of f in D. $\square$

The following two corollaries follow similarly from Corollaries 3.3 and 3.4, respectively.

Corollary 4.6 *Let $f\colon \mathbb{Z}^n \to \mathbb{R}^n$ be a SCOEF function. Suppose that there exist a, $b \in \mathbb{Z}^n$ with $a_i + 1 < b_i$ for every $i \in N$ such that for any $x \in \mathbb{Z}^n$, $x_j = b_j$ and $\max_{h \in N} |f_h(x)| > 0$ imply $f_j(x) < \max_{h \in N} |f_h(x)|$, and $x_j = a_j$ and $\max_{h \in N} |f_h(x)| > 0$ imply $f_j(x) > -\max_{h \in N} |f_h(x)|$. Then f has a discrete zero point in the finite set $D = \{x \in \mathbb{Z}^n \mid a_i \leq x_i \leq b_i,\ i \in N\}$.*

In particular, observe that the boundary conditions of the corollary above are satisfied when, for any $x \in \mathbb{Z}^n$, $x_j = b_j$ implies $f_j(x) \leq 0$ and $x_j = a_j$ implies $f_j(x) \geq 0$.

Corollary 4.7 *Let $D \subset \mathbb{Z}^n$ be a finite and regular integrally convex set and let $f\colon D \to \mathbb{R}^n$ be a SCOEF function. Then f has a discrete zero point in D or there exist a point x in D on the boundary of $\overline{D}$ such that for some $k \in N$, either $f_k(x) = \max_{h \in N} |f_h(x)|$ and $e(k)$ is an outgoing vector at x, or $f_k(x) = -\max_{h \in N} |f_h(x)|$ and $e(-k)$ is outgoing at x.*

The property in the last corollary might be considered as the discrete equivalent of a stationary point. It is well known that every continuous function f from a compact convex n-dimensional set C in $\mathbb{R}^n$ to $\mathbb{R}^n$ has a stationary point x^*, being a zero point of f when x^* lies in the interior of C and having a function value $f(x^*)$ lying in the normal cone $N(x^*, C)$ at x^* when x^* lies on the boundary of C. In case of a discrete set D, continuity is replaced by the SCOEF property and the normal cone condition is replaced by the condition that the function value $f(x^*)$ must lie in some cone of outgoing vectors at x^*.

To prove Theorem 4.4, we adapt the labeling rule as follows. For any point interior in $\overline{D}$, let the label $\ell(x)$ be the label induced by f as above. For a point x on the boundary of $\overline{D}$ we adapt the labeling as follows. When $f(x) = 0^n$, then $\ell(x) = 0$. When $f(x) \neq 0^n$ and $f(-x) = 0^n$, then $\ell(x)$ is the label induced by f as before. When both $f(x) \neq 0^n$ and $f(-x) \neq 0^n$, take any $k \in N$ such that $f_k(x)f_k(-x) = -\max_{i \in N} |f_i(x)| \max_{i \in N} |f_i(-x)|$. Then $\ell(x) = k$ if $f_k(x) > 0$ and $\ell(x) = -k$ if $f_k(x) < 0$. Clearly, the labeling rule $\ell(\cdot)$ satisfies that two antipodal points on the boundary of $\overline{D}$ either have opposite labels or at least one of the two points has label 0. The next lemma shows that no two vertices of some simplex can have opposite labels.

Lemma 4.8 *Let $D \subset \mathbb{Z}^n$ be a finite, regular and symmetric integrally convex set and let $f\colon D \to \mathbb{R}^n$ be a function satisfying the conditions of Theorem 4.4 with respect to some symmetric cubical triangulation $\mathcal{T}$ of $\overline{D}$. Then for the modified integer labeling function induced by f, there exists no simplex in $\mathcal{T}$ carrying both labels k and $-k$ for some $k \in N$.*

Proof. Let $\ell(\cdot)$ be the modified integer labeling function induced by f on D and let x and y be two vertices of any simplex of $\mathcal{T}$. When both x and y are in the interior of $\overline{D}$, then it follows similarly as in the proof of Lemma 4.5, that x and y can not have opposite labels. Next, let x be a vertex on the boundary of $\overline{D}$ with $\ell(x) \neq 0$. Without loss of generality, suppose that $\ell(x) = k$ for some $k \in N$. From the labeling rule it follows that $f_k(x) > 0$ and $f_k(x)f_k(-x) = -\max_{i \in N} |f_i(x)| \max_{i \in N} |f_i(-x)|$. This implies that $f_k(x) = \max_{h \in N} |f_h(x)|$. We now consider two cases. First, when y is in the interior of $\overline{D}$, it follows by the SCOEF condition that y can not have label $-k$. The second case is that also y is on the boundary of $\overline{D}$. Since D is regular and thus has an interior point, two opposite points on the boundary can not belong to the same cell and therefore also not to the same simplex of $\mathcal{T}$. Hence, y is not opposite to x, i.e., $y \neq -x$. Suppose that y has label $-k$. Then, by the labeling rule we must have that $f_k(y) < 0$ and $f_k(y)f_k(-y) = -\max_{i\in N} |f_i(y)| \max_{i\in N} |f_i(-y)|$. However, this implies that $f_k(y) = -\max_{h\in N} |f_h(x)|$, which again contradicts the SCOEF condition. □

Proof of Theorem 4.4. Let $\mathcal{T}$ be a symmetric cubical triangulation of $\overline{D}$ with respect to which f satisfies the SCOEF property and let $\ell(\cdot)$ be the modified integer label function on D induced by f. Suppose that the function f has no discrete zero point at all, thus $\ell(x) \neq 0$ for any $x \in D$. Then $\ell(\cdot)$ satisfies the boundary condition of Theorem 3.6 and thus there exists two cell-connected integral points with opposite labels. However, this contradicts Lemma 4.8. Hence, f must have a discrete zero point in D. □

We now conclude this section with a comparison between the class of (S)COEF functions introduced in this paper and the various other classes of functions known in the literature. A function $f\colon \mathbb{Z}^n \to \mathbb{R}^n$ is *direction preserving* if for any two cell-connected points x and y in $\mathbb{Z}^n$ it holds that $f_j(x)f_j(y) \geq 0$ for all $j \in N$. A function $f\colon \mathbb{Z}^n \to \mathbb{R}^n$ is *locally gross direction preserving* if $f(x) \cdot f(y) \geq 0$ for any two cell-connected points x and y in $\mathbb{Z}^n$. A function $f\colon \mathbb{Z}^n \to \mathbb{R}^n$ is *positive maximum component sign preserving* if for any two cell connected points x and y in $\mathbb{Z}^n$, $f_j(x) = \max_{h\in N} f_h(x) > 0$ implies $f_j(y) \geq 0$.

Direction preserving functions were introduced by Iimura (2003), locally gross direction preserving functions by Yang (2004), and (simplicially) positive maximum component sign preserving functions by Talman and Yang (2005). Their simplicial extensions can be found in Iimura and Yang (2005), van der Laan, Talman and Yang (2007) and Yang (2007).

Clearly, (S)COEF functions are more general than positive maximum component sign preserving functions, which are more general than direction preserving functions. Also locally gross direction preserving functions are obviously more general than direction preserving functions. However, the following examples show that (S)COEF functions and locally gross direction preserving functions are incomparable in the sense that they do not imply each other.

Example 2. Let $f\colon \mathbb{Z}^6 \to \mathbb{R}^6$ be defined by $f(x) = (2, 1, 1, 1, 1, 1)$ for $x = (0, 0, 0, 0, 0, 0)$, $f(x) = (-2, 1, 1, 1, 1, 1)$ for $x = (1, 1, 1, 1, 1, 1)$, and $f(x) = (0, 0, 0, 0, 0, 0)$ otherwise. Then f is locally gross direction preserving but not SCOEF.

Example 3. Let $f\colon \mathbb{Z}^2 \to \mathbb{R}^2$ be defined by $f(x) = (2, 1)$ for $x = (0, 0)$, $f(x) = (-2, -3)$ for $x = (1, 1)$, and $f(x) = (0, 0)$ otherwise. Then f is COEF but not locally gross direction preserving.

In the literature the existence of a discrete zero (or discrete fixed point) is often shown by proving that the piecewise linear extension of the function has a zero point. It is interesting, however, to note that the piecewise linear extension of a (S)COEF function may have a zero point, but the function itself may have no discrete zero point at all. We illustrate this point in the following example. This means that in order to prove the existence of a discrete zero point of a SCOEF function one cannot apply simplicial algorithms based on vector labeling to this class of functions, and neither can one directly apply the usual, such as Brouwer's or Kakutani's, fixed point argument.

Example 4. Let $f\colon \mathbb{Z}^3 \rightarrow \mathbb{R}^3$ be defined by $f(x) = (2, -1, -1)$ for $x = (1, 0, 0)$, $f(y) = (-1, 2, -1)$ for $y = (0, 1, 0)$, $f(z) = (-1, -1, 2)$ for $z = (0, 0, 1)$, otherwise $f(w) = (1, 1, 1)$. Then f is SCOEF. However, f has no discrete zero point, while the barycenter of the triangle with vertices x, y and z is a zero point of the piecewise linear extension of f.

To prove the existence of a discrete zero (or fixed) point for the direction preserving functions (see Iimura, Murota and Tamura (2005), Danilov and Koshevoy (2004)) and for the locally gross direction preserving functions (see Yang (2004)), they first prove the existence of a zero point x^* for the piecewise linear extension of the original function f by using Brouwer fixed point theorem or its extensions and then show that at least one vertex of the integral simplex containing x^* must be a discrete (fixed) zero point of the original function f. The same argument applies to positive maximum component sign preserving functions. In sharp contrast, Example 4 illustrates that the piecewise linear extension of a (simplicially) coordinatewisely opposite extrema free function f may have a zero point in the interior of a cubical simplex, but no vertex of the simplex is a discrete zero point of f. This indeed implies that one cannot directly apply simplicial algorithms based on vector labeling to the class of SCOEF or COEF functions and neither can one directly apply the usual, such as Brouwer's, fixed point argument.

In van der Laan, Talman and Yang (2005, 2006, 2007), simplicial methods are proposed to show existence theorems for direction preserving functions and simplicially local gross direction preserving functions, while Talman and Yang (2005) establish theorems also via a constructive method for (simplicially) positive maximum component sign preserving functions.

References

[1] Bapat, R.B., 1989. A constructive proof of a permutation-based generalization of Sperner's lemma, *Mathematical Programming* **44**, 113-120.

[2] Danilov, V., Koshevoy, G., 2004. Existence theorem of zero point in a discrete case, Moscow, draft.

[3] Eaves, B.C., 1972. Homotopies for computation of fixed points, *Mathematical Programming* **3**, 1-22.

[4] Fan, K., 1967. Simplicial maps from an orientable n-pseudomanifold into S^m with the octahedral triangulation, *Journal of Combinatorial Theory* **2**, 588-602.

[5] Favati, P., Tardella, F., 1990. Convexity in nonlinear integer programming, *Ricerca Operativa* **53**, 3-44.

[6] Freund, R.M., 1986. Combinatorial theorems on the simplotope that generalize results on the simplex and cube, *Mathematics of Operations Research* **11**, 169-179.

[7] Freund, R.M., 1989. Combinatorial analogs of Brouwer's fixed point theorem on a bounded polyhedron, *Journal of Combinatorial Theory Ser. B* **2**, 192-219.

[8] Freund, R.M., Todd, M.J., 1981. A constructive proof of Tucker's combinatorial lemma, *Journal of Combinatorial Theory* **30**, 321-325.

[9] Fudenberg, D., Tirole, J., 1991. *Game Theory*, The MIT Press, Boston.

[10] Fujishige, S., 2005. *Submodular Functions and Optimization*, 2nd edition, Elsevier, Amsterdam.

[11] Gul, F., Stachetti, E., 1999. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory* **87**, 95–124.

[12] Iimura, T., 2003. A discrete fixed point theorem and its applications, *Journal of Mathematical Economics* **39**, 725-742.

[13] Iimura, T., Murota, K., Tamura, A., 2005. Discrete fixed point theorem reconsidered, *Journal of Mathematical Economics* **41**, 1030-1036.

[14] Iimura, T., Yang, Z. 2005. A study on the demand and response correspondences in the presence of indivisibilities, preprint, Yokohama National University, Yokohama.

[15] Kelso, A.S., Crawford, V.P., 1982. Job matching coalition formation and gross substitutes. *Econometrica* **50**, 1483–1504.

[16] Kuhn, H.W., 1969. Simplicial approximation of fixed points, *Proc. Nat. Acad. of Sci.* **61** 1238-1242.

[17] Laan, G. van der, 1984. On the existence and approximation of zeros, *Mathematical Programming* **28**, 1-14.

[18] Laan, G. van der, Talman, A.J.J., 1979. A restart algorithm for computing fixed points without an extra dimension, *Mathematical Programming* **17**, 74-84.

[19] Laan, G. van der, Talman, A.J.J., 1981. A class of simplicial restart fixed point algorithms without an extra dimension, *Mathematical Programming* **20**, 33-48.

[20] Laan, G. van der, Talman, A.J.J., 1982. On the computation of fixed points in the product space of unit simplices and an application to noncooperative N-person game, *Mathematics of Operations Research* **7**, 1-13.

[21] Laan, G. van der, Talman, A.J.J., Yang, Z., 2001. Existence of balanced simplices on polytopes, *Journal of Combinatorial Theory Ser. A* **96**, 288-302.

[22] Laan, G. van der, Talman, A.J.J., Yang, Z., 2005. Computing integral solutions of complementarity problems, Discussion Paper No. 2005-006, Tinbergen Institute, Amsterdam, forthcoming in *Discrete Optimization*.

[23] Laan, G. van der, Talman, A.J.J., Yang, Z., 2006. Solving discrete zero point problems, *Mathematical Programming* **108**, 127-134.

[24] Laan, G. van der, Talman, A.J.J., Yang, Z., 2007. A vector labeling method for solving discrete zero point and complementarity problems, *SIAM Journal on Optimization* **18**, 290-308.

[25] Lefschetz, S., 1949. *Introduction to Topology*, Princeton University Press, Princeton.

[26] Merrill, O.H., 1972. Applications and Extensions of an Algorithm that Computes Fixed Points of Certain Upper Semi-Continuous Point-to-Set Mappings, PhD Thesis, University of Michigan, Ann Arbor.

[27] Murota, K., 2003. *Discrete Convex Analysis*, SIAM Publisher, Philadephia.

[28] Prescott, T., Su, F.E., 2005. A constructive proof of Ky Fan's generalizations of Tucker's lemma, *Journal of Combinatorial Theory Ser. A* **111**, 257-265.

[29] Reiser, P.M., 1981. A modified integer labeling for complementarity algorithms, *Mathematics of Operations Research* **6**, 129-139.

[30] Scarf, H., 1967. The approximation of fixed points of a continuous mapping, *SIAM Journal on Applied Mathematics* **15**, 1328-1343.

[31] Scarf, H., 1973. *The Computation of Economic Equilibria*, Yale University Press, New Haven.

[32] Scarf, H., 1994. The allocation of resources in the presence of indivisibilities, *Journal of Economic Perspectives* **4**, 111–128.

[33] Shapley, L.S., 1973. On balanced games without side payments, in "Mathematical Programming", T.C. Hu and S.M. Robinson (eds.), Academic Press, New York, pp. 261-290.

[34] Su, F.E., 1999. Rental harmony: Sperner's lemma in fair division, *American Mathematical Monthly* **106**, 930-942.

[35] Sun, N., Yang, Z., 2006. Equilibria and indivisibilities: gross substitutes and complements, *Econometrica* **74 (5)**, 1385-1402.

[36] Talman, A.J.J., Yang, Z., 2005. A discrete zero point theorem and its applications, FBA Working Paper No. 223, Yokohama National University, Yokohama. forthcoming in *European Journal of Operational Research.*

[37] Tarski, A., 1955. A lattice-theoretical fixpoint theorem and its applications, *Pacific Journal of Mathematics* **5**, 285-309.

[38] Todd, M.J., 1976. *Computation of Fixed Points and Applications*, Springer-Verlag, Berlin.

[39] Tucker, A.W., 1945. Some topological properties of disk and sphere, *Proceedings of the First Canadian Mathematical Congress*, Montreal, pp. 285-309.

[40] Yamamoto, Y., 1988. Orientability of a pseudomanifold and generalization of Sperner's lemma, *Journal of Operations Research Society of Japan* **31**, 19-42.

[41] Yang, Z., 1999. *Computing Equilibria and Fixed Points*, Kluwer, Boston.

[42] Yang, Z., 2004. Discrete fixed point analysis and its applications, FBA Working Paper No. 210, Yokohama National University, Yokohama.

[43] Yang, Z., 2007. "On the solutions of discrete nonlinear complementarity and related problems", forthcoming in *Mathematics of Operations Research*.

[44] Zhou, L., 1994. The set of Nash equilibria of a supermodular game is a complete lattice, *Games and Economic Behavior* **7**, 295-300.