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THE COMPROMISE VALUE FOR COOPERATIVE GAMES WITH RANDOM PAYOFFS

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The Compromise Value for Cooperative Games with Random Payoffs

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Abstract

The compromise value is introduced for cooperative games with random payoffs, that is, for cooperative games where the payoff to a coalition of players is a random variable. It is a compromise between utopia payoffs and minimal rights. This solution concept is based on the compromise value for NTU games and the τ-value for TU games. It is shown that the nonempty core of a game is bounded by the utopia payoffs and the minimal rights. Further, we show that the compromise value of a cooperative game with random payoffs is determined by the τ-value of a related TU game if the players have special types of preferences. Finally, the compromise value and the marginal value, which is defined as the average of the marginal vectors, coincide on the class of one- and two-person games.

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1 Introduction

In this paper we introduce and study the compromise value for cooperative games with random payoffs which are introduced in Timmer, Borm and Tijs (2000a). In these games, the payoffs that the players can obtain by cooperation are not known with certainty and are modeled as random variables. The players cannot await the realizations of the payoffs before deciding upon an allocation of these payoffs. Hence, the preferences of the players over the uncertain payoffs play an important role in the analysis of such games. Further, the possible allocations of the payoffs are of a specific type.

Another model to analyze this kind of situations is that of stochastic cooperative games introduced by Suijs, Borm, De Waegenaere and Tijs (1999) and further developed by Suijs (2000). The main differences with cooperative games with random payoffs lie in the assumptions on the preferences and the structure of the set of allocations of the payoffs (see Timmer et al. (2000a) for more details).

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The compromise value for cooperative games with random payoffs is based on the compromise value for NTU games. In its turn, this latter value is an extension of the τ-value for TU games. The τ-value is a solution concept for quasi-balanced TU games introduced by Tijs (1981). It is a compromise between the utopia payoffs and the minimal rights of the players. The compromise value value for compromise admissible NTU games, introduced by Borm, Keiding, McLean, Oortwijn and Tijs (1992), is defined in a similar way. These utopia payoffs and minimal rights have several nice properties. First, if the core of the game is nonempty then the utopia payoffs and the minimal rights provide upper and lower bounds of the core. This implies that a TU or NTU game with a nonempty core is quasi-balanced or compromise admissible, respectively. Finally, if all marginal vectors of a TU game belong to the core then the minimal right of any player equals his individual payoff. For a survey on compromise values in cooperative game theory the reader is referred to Tijs and Otten (1993).

The compromise value for cooperative games with random payoffs is a compromise between the utopia payoffs and the minimal rights, whose definitions are based on their counterparts for NTU games. The properties of the utopia payoffs and minimal rights are similar to those for TU and NTU games. For all cooperative games with random payoffs with a nonempty core and with a nonzero payoff for the coalition of all players, the core is bounded by the utopia payoffs and the minimal rights. Consequently, such games are compromise admissible. If all marginal vectors belong to the core of the game then any player is indifferent between receiving his minimal right and receiving his individual payoff. Furthermore, if all players have a special type of preference relation then the compromise value is determined by the τ-value of a related TU game. Finally, we relate the compromise value to the marginal value, which is defined as the average of the marginal vectors. These values coincide on the class of one- and two-person games. Timmer, Borm and Tijs (2000b) characterized the marginal value on this class of games. Consequently, there exists a characterization of the compromise value on this class of games.

The organization of this paper is as follows. In section 2 a brief introduction to cooperative games with random payoffs is given. The compromise value for these games is introduced in section 3. We show that the core is bounded by the utopia payoffs and the minimal rights. In section 4 several properties of the compromise value are presented. After this, it is shown that the compromise value is determined by the τ-value of a related TU game if all the players have a specific type of preference relation. Section 4 is concluded by showing the coincidence of the compromise value and the marginal value on the class of one- and two-person games.

2 Cooperative games with random payoffs

In this section we recall some basic concepts of cooperative games with random payoffs as introduced in Timmer, Borm and Tijs (2000a). A cooperative game with random payoffs G is a tuple $(N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$. N is the finite player set. A nonempty subset of N is called a coalition. The nonnegative random payoff to coalition S is denoted by $R(S)$. $\mathcal{S}$ is the set of all coalitions with a nonzero payoff. The set $\mathcal{A}$ contains all the possible individual payoffs that a player may receive. The function α_i describes how player $i \in N$ compares any two individual payoffs.

In more detail, let $N = \{1, \ldots, n\}$. The set $\mathcal{L}_+$ is the set of all nonnegative random variables with finite expectation. 0 stands for the payoff zero for sure. Note that $0 \in \mathcal{L}_+$. The random payoff to coalition S, $R(S)$, is assumed to be an element of $\mathcal{L}_+$. $\mathcal{S} = \{S \subset N | S \neq \emptyset,\ R(S) \neq 0\}$ is the set of all coalitions with a payoff unequal to zero. Hence, for coalition S it holds that $S \notin \mathcal{S}$ if and only if $R(S) = 0$.

An allocation of the random payoff $R(S)$ is a vector $pR(S), p \in \mathbb{R}^S$, of multiples of $R(S)$. Such an allocation is efficient if $p \in \Delta^*(S) = \{p \in \mathbb{R}^S | \sum_{i \in S} p_i = 1\}$. $\mathcal{A}$ is the set of all the possible individual payoffs with regard to the random payoffs $R(S)$ to the coalitions, $\mathcal{A} = \{tR(S) | t \in \mathbb{R}, S \in \mathcal{S}\}$. $\mathcal{A}_{-0} = \{tR(S) \in \mathcal{A} | t \neq 0\}$ is the restriction of $\mathcal{A}$ to all the nonzero individual payoffs.

The preference relation $\succsim_i$ of player $i \in N$ has the following interpretation. Let $X, Y \in \mathcal{A}$ be two individual payoffs. Player i weakly prefers X to Y if $X \succsim_i Y$. She is indifferent between them, $X \sim_i Y$, if $X \succsim_i Y$ and $Y \succsim_i X$. Finally, she strictly prefers X to Y, $X \succ_i Y$, if $X \succsim_i Y$ and not $Y \succsim_i X$. We assume the following about this preference relation.

Assumption 2.1 *For all players $i \in N$ there exist surjective, strictly increasing and continuous functions $f_S^i : \mathbb{R} \to \mathbb{R}$, $S \in \mathcal{S}$, such that*

$$f_S^i(t)R(S) \succsim_i f_T^i(t')R(T) \text{ if and only if } t \geq t'$$

and $f_S^i(0) = 0$ for all $S, T \in \mathcal{S}$ and $t, t' \in \mathbb{R}$.

So, if player i compares the payoffs $pR(S)$ and $qR(T)$, $S, T \in \mathcal{S}$, then $pR(S) \succsim_i qR(T)$ if and only if $t = (f_S^i)^{-1}(p) \geq t' = (f_T^i)^{-1}(q)$. Hence, the function $(f_S^i)^{-1}$, $S \in \mathcal{S}$, may be interpreted as a kind of utility function with respect to multiples of $R(S)$ only. Two examples of preference relations $\succsim_i$ that satisfy this assumption are the following. Let $X, Y \in \mathcal{A}$. The first example is $X \succsim_i Y$ if and only if $\mathrm{E}(X) \geq \mathrm{E}(Y)$ with $\mathrm{E}(X)$ the expectation of the random variable X. We refer to this type of preferences as 'expectation preferences'. Define $f_S^i(t) = t/\mathrm{E}(R(S))$ for all $S \in \mathcal{S}$. This function satisfies assumption 2.1.

For the second example let $u_{\beta_i}^X = \sup\{t \in \mathbb{R} | \Pr\{X \leq t\} \leq \beta_i\}$ be the β_i-quantile of the random variable X where $0 < \beta_i < 1$ is such that $u_{\beta_i}^{R(S)} > 0$ for all $S \in \mathcal{S}$. Define the (utility) function $U_i : \mathcal{A} \to \mathbb{R}$ by $U_i(X) = u_{\beta_i}^X$ if $X \geq 0$ and $U_i(X) = u_{1-\beta_i}^X$ otherwise. The preference relation $X \succsim_i Y$ if and only if $U_i(X) \geq U_i(Y)$ is called a 'quantile preference relation'. The functions $f_S^i(t) = t/u_{\beta_i}^{R(S)}$ describe these preferences. Notice that both expectation preferences and quantile preferences have linear functions f^i. That is, $f_S^i(t) = tf_S^i(1)$ for all $t \in \mathbb{R}$.

In this paper we often like to know for which real number α_i it holds that $X \sim_i \alpha_i Y$ where $X \in \mathcal{A}$ and $Y \in \mathcal{A}_{-0}$. For this reason we define $\alpha_i(X, Y)$ to denote this number α_i. It follows from assumption 2.1 that the number $\alpha_i(X, Y)$ is unique and if $X = pR(S)$ and $Y = qR(T)$ then $\alpha_i(X, Y) = f_T^i((f_S^i)^{-1}(p))/q$. If we interpret $(f_S^i)^{-1}$, $S \in \mathcal{S}$, as some kind of utility function then $\alpha_i(X, Y)$ is that multiple of Y that gives player i the same utility as X, namely $(f_S^i)^{-1}(p)$.

Further, define $\alpha_i(0,0) = 1$. Again from assumption 2.1 it can be deduced that if $S \in \mathcal{S}$ then $p_i R(S) \succ_i 0$ if $p_i > 0$ and $0 \succ_i p_i R(S)$ if $p_i < 0$. Hence, there exists no real number α_i such that $p_i R(S) \sim_i \alpha_i \cdot 0 = 0$, $p_i \neq 0$. This is why $\alpha_i(X, 0)$ is not defined for any $X \in \mathcal{A}_{-0}$.

The lemma below, lemma 2.3 in Timmer, Borm and Tijs (2000b), presents some properties of the preference relations $\succsim_i$ and the functions α_i that we use in this paper.

Lemma 2.2 *For any $i \in N$, $h \in \mathbb{R}$ and $X \in \mathcal{A}_{-0}$ it holds that $\alpha_i(hX, X) = h$.*
If the functions f_S^i are linear for all $S \in \mathcal{S}$ then

- $\alpha_i(pR(S), qR(T)) = pf_T^i(1)/(qf_S^i(1))$ *for any $pR(S) \in \mathcal{A}$ and $qR(T) \in \mathcal{A}_{-0}$,*
- $pR(S) \succsim_i qR(T)$ *if and only if $p/f_S^i(1) \geq q/f_T^i(1)$ for any $pR(S), qR(T) \in \mathcal{A}$.*

Let $\mathcal{G}^N$ be the set of all cooperative games with random payoffs and player set N that satisfy assumption 2.1. Let $G \in \mathcal{G}^N$. The imputation set $I(G)$ contains all the allocations of $R(N)$ that are efficient and individual rational:

$$I(G) = \{pR(N) \,|p \in \Delta^*(N),\ p_i R(N) \succsim_i R(\{i\}) \text{ for all } i \in N\}\ .$$

The set of allocations of $R(N)$ that are dominated by coalition S is

$$\text{dom}(S) = \left\{pR(N) \,\middle|\, p \in \mathbb{R}^S,\ \exists q \in \Delta^*(S):\ q_i R(S) \succ_i p_i R(N) \text{ for all } i \in S\right\}\ .$$

The core $C(G)$ of the game contains those efficient allocations of $R(N)$ that are not dominated by any coalition:

$$C(G) = \{pR(N) \,|p \in \Delta^*(N),\ p_S R(N) \notin \text{dom}(S) \text{ for all coalitions } S\}$$

with $p_S = (p_i)_{i \in S}$ the restriction of p to coalition S. Notice that $C(G) \subset I(G)$.

Let $\Pi(N)$ be the set of all bijections $\sigma : \{1, \ldots, n\} \to N$. Let $\sigma \in \Pi(N)$. Coalition $S_i^\sigma = \{\sigma(k) | k \leq i\}$ consists of the first i players according to $\sigma \in \Pi(N)$. The marginal contribution $Y_{\sigma(1)}^\sigma$ of the first player $\sigma(1)$ is equal to his individual payoff, $Y_{\sigma(1)}^\sigma = R(\{\sigma(1)\}) = R(S_1^\sigma)$. The marginal contribution of player $\sigma(i)$, $i = 2, \ldots, n$, to coalition S_{i-1}^σ equals

$$Y_{\sigma(i)}^\sigma = \left[1 - \sum_{k=1}^{i-1} \alpha_{\sigma(k)}(Y_{\sigma(k)}^\sigma, R(S_i^\sigma))\right] R(S_i^\sigma).$$

Each player $j \in S_{i-1}^\sigma$ receives from player $\sigma(i)$ the random payoff $\alpha_j(Y_j^\sigma, R(S_i^\sigma))R(S_i^\sigma)$. Player j is indifferent between receiving this payoff and receiving her marginal contribution Y_j^σ. The marginal contribution of player $\sigma(i)$ is all that remains of the payoff $R(S_i^\sigma)$. The marginal vector $M^\sigma(G)$ corresponding to permutation $\sigma \in \Pi(N)$ is defined by $M_i^\sigma(G) = m_i^\sigma(G)R(N)$ for all $i \in N$, with $m_i^\sigma(G) = \alpha_i(Y_i^\sigma, R(N))$. To ensure that the marginal vectors are well defined, that is, to avoid $\alpha_i(X, 0)$ for some $X \in \mathcal{A}_{-0}$, we assume the following about the payoff structure of the game G only if we talk about marginal vectors.

Assumption 2.3 *If $R(T) = 0$ for some coalition T then $R(S) = 0$ for all coalitions S such that $S \subset T$.*

Let $\mathcal{H}^N \subset \mathcal{G}^N$ be a set of cooperative games with random payoffs and player set N. A solution Ψ for cooperative games with random payoffs is a function on $\mathcal{H}^N$ such that $\Psi(G)$ is an allocation $pR(N)$ for the game $G \in \mathcal{H}^N$. The marginal value Φ^m for a game $G \in \mathcal{G}^N$ that satisfies assumption 2.3 is the average of its marginal vectors:

$$\Phi_i^m(G) = \left[(n!)^{-1} \sum_{\sigma \in \Pi(N)} m_i^\sigma(G) \right] R(N).$$

This solution concept for cooperative games with random payoffs is introduced in Timmer et al. (2000a) and studied in Timmer et al. (2000b).

3 The compromise value

The compromise value for cooperative games with random payoffs is an extension of the compromise value for NTU games, which in its turn is an extension of the τ-value for TU games. The latter two values are introduced and studied in Borm, Keiding, McLean, Oortwijn and Tijs (1992) and Tijs (1981), respectively. These values are a compromise between utopia payoffs and minimal rights.

Let $G = (N, (R(S))_{S \in \mathcal{S}}, \mathcal{A}, (\alpha_i)_{i \in N})$ be a cooperative game with random payoffs and let $i \in N$. Suppose player i proposes the efficient allocation $pR(N)$. It seems reasonable to assume that the players $j \neq i$ will agree with this proposal if $p_{N \setminus \{i\}} R(N) \notin \text{dom}(N \setminus \{i\})$ and if $p_j R(N) \succsim_j R(\{j\})$. Therefore, the largest individual payoff that player i can expect when cooperating with coalition $N \setminus \{i\}$ is $K_i(G)R(N)$ where

$$K_i(G) = \sup \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists a \in \mathbb{R}^{N \setminus \{i\}} : \sum_{j \in N \setminus \{i\}} a_j + t = 1; \\ aR(N) \notin \text{dom}(N \setminus \{i\});\; a_j R(N) \succsim_j R(\{j\}),\; j \neq i \end{array} \right\}.$$

The payoff $K_i(G)R(N)$ is called the *utopia payoff* to player i. Attention will be paid only to games G with $K_i(G) \in \mathbb{R}$.

If player i is a member of coalition S then any player $j \in S \setminus \{i\}$ will not object against an efficient allocation of $R(S)$ in which she is better off than receiving her utopia payoff. Hence, the players $j \in S \setminus \{i\}$ will not disagree if player i claims his part $r(S,i)R(S)$ of such an allocation with

$$r(S,i) = \sup \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists a \in \mathbb{R}^{S \setminus \{i\}} : \sum_{j \in S \setminus \{i\}} a_j + t = 1; \\ a_j R(S) \succ_j K_j(G)R(N),\; j \in S \setminus \{i\} \end{array} \right\}.$$

The *remainder of i in S*, $\rho(S,i)$, is defined as the remainder $r(S,i)R(S)$ expressed as a multiple of $R(N)$:

$$\rho(S,i) = \alpha_i(r(S,i)R(S), R(N)).$$

The largest of these remainders of player i determines his *minimal right* $k_i(G)R(N)$ where

$$k_i(G) = \max_{S: i \in S} \rho(S,i).$$

We restrict ourselves to games G with $k_i(G) \in \mathbb{R}$ for all players i. The lemma below shows that this condition is satisfied for all games with $R(N) \neq 0$.

Lemma 3.1 *For any $G \in \mathcal{G}^N$ with $R(N) \neq 0$, $K_i(G) < \infty$ and $k_i(G) \in \mathbb{R}$ for all $i \in N$.*

Proof. Let $G \in \mathcal{G}^N$ be a cooperative game with random payoffs and $R(N) \neq 0$. First,

$$\begin{aligned} K_i(G) &= \sup\left\{ t \in \mathbb{R} \,\middle|\, \begin{array}{l} \exists a \in \mathbb{R}^{N\setminus\{i\}} : \sum_{j \in N\setminus\{i\}} a_j + t = 1; \\ aR(N) \notin \text{dom}(N \setminus \{i\});\ a_j R(N) \succsim_j R(\{j\}),\ j \neq i \end{array} \right\} \\ &\leq \sup\left\{ t \in \mathbb{R} \,\middle|\, \begin{array}{l} \exists a \in \mathbb{R}^{N\setminus\{i\}} : \sum_{j \in N\setminus\{i\}} a_j + t = 1; \\ a_j R(N) \succsim_j R(\{j\}),\ j \neq i \end{array} \right\} \\ &< \infty, \end{aligned}$$

where the last inequality follows from $R(N) \neq 0$.

Second, let S be a coalition of players. If $R(S) \neq 0$ then $r(S,i) \in \mathbb{R}$ and consequently $\rho(S,i) \in \mathbb{R}$. If $R(S) = 0$ then $r(S,i) = -\infty$ and $\rho(S,i) = \alpha_i(-\infty \cdot 0, R(N)) = 0$ since $R(N) \neq 0$. We conclude that $k_i(G) = \max_{S: i \in S} \rho(S,i) \in \mathbb{R}$. □

A game G is called *compromise admissible* if $K_i(G), k_i(G) \in \mathbb{R}$ for all $i \in N$,

$$k_i(G) \leq K_i(G) \text{ for all } i \in N, \text{ and } \sum_{i \in N} k_i(G) \leq 1 \leq \sum_{i \in N} K_i(G).$$

Hence, in a compromise admissible game the utopia payoff of a player is larger than his minimal right and there exists an efficient allocation of $R(N)$ between the allocation of utopia payoffs and the allocation of minimal rights. Denote by $\mathcal{C}^N$ the set of all compromise admissible games G with player set N.

The *compromise value* Φ^c on $\mathcal{C}^N$ is the unique efficient allocation between the minimal rights and the utopia payoffs,

$$\Phi^c(G) = (k(G) + \gamma(K(G) - k(G)))\, R(N),$$

where $0 \leq \gamma \leq 1$ is the unique real number such that $k(G) + \gamma(K(G) - k(G)) \in \Delta^*(N)$.

Example 3.2 Consider the game $G = (N, (R(S))_{S \in \mathcal{S}}, \mathcal{A}, (\alpha_i)_{i \in N})$ where $N = \{1, 2, 3\}$. The payoffs to the various coalitions are $R(\{1\}) = 0$, $R(\{2\}) = 1$, $R(\{3\}) = 0$, $R(\{1,2\}) = 4$, $R(\{1,3\}) = 1$, $R(\{2,3\}) = 3$, and $R(N) \sim U(4,8)$, that is, $R(N)$ is uniformly distributed over the interval [4,8]. Hence $\mathcal{S} = \{\{2\}, \{1,2\}, \{1,3\}, \{2,3\}, N\}$ and $\mathcal{A} = \{pR(S) | p \in \mathbb{R}, S \in \mathcal{S}\}$.

Let $\beta_1 = 1/2$ and $\beta_2 = \beta_3 = 1/4$. Recall from section 2 that $u^X_{\beta_i} = \sup\{t \in \mathbb{R} | \Pr\{X \leq t\} \leq \beta_i\}$ is the β_i-quantile of the random variable X. All the players $i \in N$ have quantile preferences, thus $f^i_S(t) = t / u^{R(S)}_{\beta_i}$ for all $S \in \mathcal{S}$, $t \in \mathbb{R}$. From this we obtain the maps α_i. Now

$$\begin{aligned} K_1(G) &= \sup\left\{ t \in \mathbb{R} \,\middle|\, \begin{array}{l} \exists a \in \mathbb{R}^{\{2,3\}} : a_2 + a_3 + t = 1; \\ aR(N) \notin \text{dom}(\{2,3\});\ a_j R(N) \succsim_j R(\{j\}),\ j = 2,3 \end{array} \right\} \\ &= \sup\{t \in \mathbb{R} | \exists a \in \mathbb{R}^{\{2,3\}} : t = 1 - a_2 - a_3;\ a_2 + a_3 \geq 3/5;\ a_2 \geq 1/5;\ a_3 \geq 0\} \\ &= 2/5 \end{aligned}$$

and similarly, $K_2(G) = 5/6$ and $K_3(G) = 3/10$. For player 2 we have $r(\{2\}, 2) = 1$,

$$\begin{aligned} r(\{1,2\},2) &= \sup\{t \in \mathbb{R} | \exists a \in \mathbb{R}: \; a + t = 1; \; aR(\{1,2\}) \succ_1 2R(N)/5\} \\ &= \sup\{t \in \mathbb{R} | \exists a \in \mathbb{R}: \; t = 1 - a; \; a > 3/5\} \\ &= 2/5, \end{aligned}$$

$r(\{2,3\}, 2) = 1/2$, and $r(N, 2) = 3/10$. The remainders for player 2 are

$$\rho(\{2\}, 2) = \alpha_2(r(\{2\}, 2)R(\{2\}), R(N)) = 1/5,$$

$\rho(\{1, 2\}, 2) = 8/25$, $\rho(\{2, 3\}, 2) = 3/10$ and $\rho(N, 2) = 3/10$. Thus, the minimal right for player 2 is $k_2(G)R(N)$ where

$$k_2(G) = \max\{1/5, 8/25, 3/10, 3/10\} = 8/25.$$

In a similar way we obtain $k_1(G) = 0$ and $k_3(G) = 0$. The game G is compromise admissible and

$$\Phi^c(G) = (102/455, 79/130, 153/910)R(N)$$

is the compromise value. $\diamond$

It is known for TU and NTU games that the minimal rights and utopia payoffs are lower and upper bounds, respectively, for the nonempty core of the game (cf. Tijs and Lipperts (1982) and Borm et al. (1992)). This result extends to cooperative games with random payoffs.

Theorem 3.3 *For all $G \in \mathcal{G}^N$ with $R(N) \neq 0$ and $C(G) \neq \emptyset$ and for any $pR(N) \in C(G)$, $k_i(G) \leq p_i \leq K_i(G)$ for all players i.*

Proof. Let $G \in \mathcal{G}^N$ be a game with $R(N) \neq 0$ and $C(G) \neq \emptyset$. Let $pR(N) \in C(G)$ and let $i \in N$. By definition of the utopia payoff and by $a_jR(N) \succsim_j R(\{j\})$ if and only if $a_jR(N) \notin \text{dom}(\{j\})$, it follows that

$$\begin{aligned} K_i(G) &\geq \sup\left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists a \in \mathbb{R}^{N\setminus\{i\}}: \; (a,t) \in \Delta^*(N); \\ (a,t)_S R(N) \notin \text{dom}(S) \text{ for all } S \subset N, \; S \neq \emptyset \end{array} \right\} \\ &= \sup\{t \in \mathbb{R} | \exists a \in \mathbb{R}^{N\setminus\{i\}}: \; (a,t)R(N) \in C(G)\} \\ &\geq p_i, \end{aligned}$$

where (a, t) is shorthand for the vector $(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n)$ in $\mathbb{R}^N$.

Next, let coalition T, $i \in T$, be such that $k_i(G) = \rho(T, i)$. If $T = \{i\}$ then $k_i(G) = \rho(\{i\}, i) = \alpha_i(R(\{i\}), R(N))$ where the second equality follows from the definition of $\rho(\{i\}, i)$. Because $pR(N) \in C(G) \subset I(G)$ we have $p_iR(N) \succsim_i R(\{i\}) \sim_i k_i(G)R(N)$. According to assumption 2.1 this implies $p_i \geq k_i(G)$.

Otherwise, $T \setminus \{i\} \neq \emptyset$. Suppose that $k_i(G) > p_i$. Then there exists an $\varepsilon_i > 0$ such that $k_i(G) > p_i + \varepsilon_i$. According to assumption 2.1 $(p_i + \varepsilon_i)R(N) \prec_i k_i(G)R(N) \sim_i r(T, i)R(T)$ and there exists an $\varepsilon_i' > 0$ such that

$$(p_i + \varepsilon_i)R(N) \sim_i (r(T, i) - \varepsilon_i')R(T).$$

By definition of $r(T,i)$ there exists a vector $\hat{a} \in \mathbb{R}^{T\setminus\{i\}}$ satisfying $\sum_{j\in T\setminus\{i\}} \hat{a}_j + r(T,i) - \varepsilon_i' = 1$ and $\hat{a}_j R(T) \succ_j K_j(G)R(N)$ for all $j \in T \setminus \{i\}$. Now we have

$$(r(T,i) - \varepsilon_i')R(T) \sim_i (p_i + \varepsilon_i)R(N) \succ_i p_i R(N)$$

and from $K_j(G) \geq p_j$

$$\hat{a}_j R(T) \succ_j K_j(G)R(N) \succsim_j p_j R(N)$$

for all $j \in T\setminus\{i\}$. This implies that $p_T R(N) \in \mathrm{dom}(T)$, which is in contradiction to $pR(N) \in C(G)$. We conclude that $k_i(G) \leq p_i$. □

Example 3.4 Consider the game in example 3.2. Recall that $K(G) = (2/5, 5/6, 3/10)$ and $k(G) = (0, 8/25, 0)$. The nonempty core of this game is

$$C(G) = \left\{ pR(N) \,\middle|\, \begin{array}{l} p \in \Delta^*(N),\ p_1 \geq 0,\ p_2 \geq 1/5,\ p_3 \geq 0,\ 6p_1 + 5p_2 \geq 4, \\ 6p_1 + 5p_3 \geq 1,\ 5p_2 + 5p_3 \geq 3 \end{array} \right\}.$$

It is easy to check that $k(G) \leq p \leq K(G)$ for all $pR(N) \in C(G)$. ◇

An immediate consequence of theorem 3.3 is that any game G with a nonempty core and with $R(N) \neq 0$ is compromise admissible.

Lemma 3.5 *Any game $G \in \mathcal{G}^N$ with $R(N) \neq 0$ and $C(G) \neq \emptyset$ is compromise admissible.*

Proof. Let $G \in \mathcal{G}^N$ be a game with $R(N) \neq 0$ and $C(G) \neq \emptyset$. Let $pR(N) \in C(G)$. According to theorem 3.3, $k_i(G) \leq p_i \leq K_i(G)$ for all $i \in N$. Hence, $k_i(G) \leq K_i(G)$ for all players i. Because $pR(N)$ is an element of the core $C(G)$ we know that $p \in \Delta^*(N)$ and so,

$$\sum_{i\in N} k_i(G) \leq \sum_{i\in N} p_i = 1 \leq \sum_{i\in N} K_i(G).$$

Further, by lemma 3.1, $K_i(G) < \infty$ and $k_i(G) \in \mathbb{R}$ for all $i \in N$. Together with $p_i \leq K_i(G)$ this results in $K_i(G) \in \mathbb{R}$ for all $i \in N$. The game G is compromise admissible. □

Another result for TU games is that if all the marginal vectors belong to the core of the game then the minimal right of any player is equal to her individual payoff. This result is based on the fact that for TU games where for some player i specific marginal vectors belong to the core, the minimal right of this player is equal to her individual payoff. A similar result holds for cooperative games with random payoffs.

Theorem 3.6 *If $G \in \mathcal{C}^N$ and if for some player i it holds that $M^\sigma(G) \in C(G)$ for all $\sigma \in \Pi(N)$ with $\sigma(1) = i$ then $k_i(G) = \alpha_i(R(\{i\}), R(N))$.*

Proof. Let $G \in \mathcal{C}^N$ and $i \in N$ be such that $M^\sigma(G) \in C(G)$ for all $\sigma \in \Pi(N)$ with $\sigma(1) = i$. Let $\sigma \in \Pi(N)$ be a permutation with $\sigma(1) = i$. By definition of a marginal vector $M^\sigma_{\sigma(1)} \sim_{\sigma(1)} R(\{\sigma(1)\})$. Since $\sigma(1) = i$ this reduces to $M^\sigma_i(G) \sim_i R(\{i\})$ and so,

$$M^\sigma_i(G) = \alpha_i(R(\{i\}), R(N))R(N) \tag{3.1}$$

From $G \in \mathcal{C}^N$ and from assumption 2.3 we obtain $R(N) \neq 0$ since the game with $R(S) = 0$ for all coalitions S is not an element of $\mathcal{C}^N$. Then by theorem 3.3, $k_i(G) \leq p_i$ for all $pR(N)$ in the core $C(G)$. In particular, $M^\sigma(G) \in C(G)$ and this implies with (3.1) $k_i(G) \leq \alpha_i(R(\{i\}), R(N))$.

On the other hand,

$$k_i(G) = \max_{S: i \in S} \rho(S, i) \geq \rho(\{i\}, i) = \alpha_i(R(\{i\}), R(N)).$$

We conclude that $k_i(G) = \alpha_i(R(\{i\}), R(N))$. □

Consequently, if for a game $G \in \mathcal{C}^N$ all marginal vectors belong to the core then $k_i(G) = \alpha_i(R(\{i\}), R(N))$ for all $i \in N$. Player i is indifferent between his individual payoff $R(\{i\})$ and the minimal right $k_i(G)R(N)$.

4 Properties

In this section we present several properties of the compromise value. We show that for a special class of games the compromise value is determined by the τ-value of a corresponding TU game. Further, the compromise value coincides with the marginal value on the class of one- and two-person games.

A solution concept Ψ on $\mathcal{C}^N$ is called

(i) *efficient* if for all $G \in \mathcal{C}^N$, $\Psi(G) = pR(N)$ for some $p \in \Delta^*(N)$.

(ii) *individual rational* if for all $G \in \mathcal{C}^N$ and for all $i \in N$, $\Psi_i(G) \succsim_i R(\{i\})$.

(iii) *anonymous* if for all $G \in \mathcal{C}^N$ and for all $\sigma \in \Pi(N)$ we have $\Psi(G^\sigma) = \sigma^*(\Psi(G))$ where $G^\sigma = (N, (R^\sigma(S))_{S \in \mathcal{S}^\sigma}, \mathcal{A}^\sigma, (\alpha^\sigma_i)_{i \in N})$, $R^\sigma(\sigma(U)) = R(U)$, $\mathcal{S}^\sigma = \{\sigma(S) | S \in \mathcal{S}\}$, $\mathcal{A}^\sigma = \{pR^\sigma(S) | p \in \mathbb{R},\ S \in \mathcal{S}^\sigma\}$, $\alpha^\sigma_{\sigma(i)} = \alpha_i$ and $(\sigma^*(pR(N)))_{\sigma(i)} = p_i R(N)$ for $i \in N$ and $p \in \mathbb{R}^N$.

(iv) *weakly proportional* if for all $G \in \mathcal{C}^N$ with $k(G) = 0$, $\Psi(G)$ is proportional to $K(G)R(N)$.

(v) *symmetric* if for all $G \in \mathcal{C}^N$, and for all $i, j \in N$ such that $\alpha_i = \alpha_j$ and $R(S \cup \{i\}) = R(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$, $\Psi_i(G) = \Psi_j(G)$.

(vi) *strongly symmetric* if for all $G \in \mathcal{C}^N$ and for all $i, j \in N$ with $k_i(G) = k_j(G)$ and $K_i(G) = K_j(G)$, $\Psi_i(G) = \Psi_j(G)$.

The compromise value satisfies these properties.

Lemma 4.1 *The compromise value Φ^c on $\mathcal{C}^N$ is efficient, individual rational, anonymous, weakly proportional, symmetric and strongly symmetric.*

Proof. We only show that Φ^c is individual rational and symmetric on $\mathcal{C}^N$. The remainder of the proof is left to the reader.

Let $G \in \mathcal{C}^N$ be a compromise admissible game and let $i \in N$. By definition of the compromise value

$$\begin{aligned}\alpha_i(\Phi_i^c(G), R(N)) &= k_i(G) + \gamma\,(K_i(G) - k_i(G)) \\ &\geq k_i(G) \geq \rho(\{i\}, i) = \alpha_i(R(\{i\}), R(N))\end{aligned}$$

and so,

$$\Phi_i^c(G) \succsim_i \alpha_i(R(\{i\}), R(N))R(N) \sim_i R(\{i\}).$$

The compromise value is individual rational.

Next, let $i, j \in N$ be such that $\alpha_i = \alpha_j$ and $R(S \cup \{i\}) = R(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$. From $R(N \setminus \{i\}) = R(N \setminus \{j\})$ and $\alpha_i = \alpha_j$ it follows that $K_i(G) = K_j(G)$. This implies that $r(S \cup \{i\}, i) = r(S \cup \{j\}, j)$ and $\rho(S \cup \{i\}, i) = \rho(S \cup \{j\}, j)$ for all $S \subset N \setminus \{i, j\}$. Consequently, $k_i(G) = k_j(G)$. It follows from the definition of Φ^c that $\Phi_i^c(G) = \Phi_j^c(G)$. The compromise value is symmetric. □

For games in $GLI^N \cap \mathcal{C}^N$, where GLI^N is the class of games in which all the players have linear functions f^i and identical preferences, the compromise value Φ^c and the τ-value are closely related. Before we can present this result we need to know what is a transferable utility (TU) game and what is the τ-value of such a game. A TU game is a pair (N, v) where N is the player set and v is a function that assigns to each coalition S a real number $v(S)$ with the convention $v(\emptyset) = 0$. The utopia payoff for player $i \in N$ is

$$M_i(v) = v(N) - v(N \setminus \{i\}) \tag{4.1}$$

and the minimal right for this player is defined by

$$m_i(v) = \max_{S: i \in S}\left(v(S) - \sum_{j \in S \setminus \{i\}} M_j(v)\right). \tag{4.2}$$

The game (N, v) is called *quasi-balanced* if

$$m(v) \leq M(v) \text{ and } \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v). \tag{4.3}$$

For quasi-balanced TU games (N, v) the τ-value is defined by

$$\tau(v) = m(v) + \delta(M(v) - m(v)) \tag{4.4}$$

with $0 \leq \delta \leq 1$ such that $\sum_{i \in N} \tau_i(v) = v(N)$.

Let $G \in GLI^N \cap \mathcal{C}^N$ and define $f_S = f_S^i$ for all $S \in \mathcal{S}$, $i \in N$. Define the TU game (N, v) corresponding to G by

$$v(S) = \begin{cases} 0, & S \notin \mathcal{S}, \\ 1/f_S(1), & S \in \mathcal{S}, \end{cases}$$

and $v(\emptyset) = 0$. The τ-value of this game determines the compromise value of G.

Theorem 4.2 *For all games $G \in GLI^N \cap \mathcal{C}^N$ with $R(N) \neq 0$ and with the corresponding TU game (N, v) satisfying $v(N \setminus \{i\}) \geq \sum_{j \in N \setminus \{i\}} v(\{j\})$ for all $i \in N$, $\Phi^c(G) = \tau(v)/v(N) \cdot R(N)$.*

Proof. Let $G \in GLI^N \cap \mathcal{C}^N$ be a game with $R(N) \neq 0$. Let (N, v) be the TU game corresponding to G with

$$v(N \setminus \{i\}) \geq \sum_{j \in N \setminus \{i\}} v(\{j\}) \text{ for all } i \in N. \tag{4.5}$$

The utopia payoff for player $i \in N$ is $K_i(G)R(N)$ with

$$\begin{aligned} K_i(G) &= \sup \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists a \in \mathbb{R}^{N \setminus \{i\}} : \sum_{j \in N \setminus \{i\}} a_j + t = 1; \\ aR(N) \notin \text{dom}(N \setminus \{i\});\; a_j R(N) \succsim_j R(\{j\}), j \in N \setminus \{i\} \end{array} \right\} \\ &= \sup \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists a \in \mathbb{R}^{N \setminus \{i\}} : t = 1 - \sum_{j \in N \setminus \{i\}} a_j; \\ \sum_{j \in N \setminus \{i\}} a_j v(N) \geq v(N \setminus \{i\});\; a_j v(N) \geq v(\{j\}),\; j \neq i \end{array} \right\} \\ &= 1 - v(N \setminus \{i\})/v(N) \\ &= M_i(v)/v(N), \end{aligned}$$

where the third and fourth equality follow from (4.5) and (4.1), respectively. Because G is compromise admissible, $R(\{i\}) \geq 0$ and $R(N) \neq 0$ we have

$$K_i(G) \geq k_i(G) \geq \rho(\{i\}, i) = \alpha_i(R(\{i\}), R(N)) \geq 0. \tag{4.6}$$

Further, if $R(S) \neq 0$ then

$$\begin{aligned} r(S, i) &= \sup \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists a \in \mathbb{R}^{S \setminus \{i\}} : \sum_{j \in S \setminus \{i\}} a_j + t = 1; \\ a_j R(S) \succ_j K_j(G) R(N),\; j \in S \setminus \{i\} \end{array} \right\} \\ &= \sup \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists a \in \mathbb{R}^{S \setminus \{i\}} : t = 1 - \sum_{j \in S \setminus \{i\}} a_j; \\ a_j v(S) > M_j(v),\; j \in S \setminus \{i\} \end{array} \right\} \\ &= 1 - \sum_{j \in S \setminus \{i\}} M_j(v)/v(S) \\ &= \left(v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right) / v(S). \end{aligned}$$

Now the remainder of i in S equals

$$\rho(S, i) = \alpha_i(r(S, i)R(S), R(N)) = \left(v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right) / v(N)$$

and the minimal right $k_i(G)R(N)$ is determined by

$$k_i(G) = \max_{S:i\in S} \rho(S,i) = \max_{S:i\in S} \left(v(S) - \sum_{j\in S\setminus\{i\}} M_j(v) \right) / v(N) = m_i(v)/v(N),$$

where the last equality follows from (4.2). If $R(S) = 0$ then

$$\begin{aligned} r(S,i) &= \sup \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists a \in \mathbb{R}^{S\setminus\{i\}} : \sum_{j\in S\setminus\{i\}} a_j + t = 1; \\ a_j \cdot 0 = 0 \succ_j K_j(G)R(N),\; j \in S \setminus \{i\} \end{array} \right\} \\ &= \sup \emptyset = -\infty. \end{aligned}$$

The set over which the supremum is taken, is empty because, by (4.6), $K_j(G) \geq 0$ which implies $K_j(G)R(N) \succsim_j 0$ for all $j \in N$. Thus,

$$\rho(S,i) = \alpha_i(-\infty \cdot 0, R(N)) = 0 \leq \rho(\{i\}, i) \leq k_i(G).$$

$R(S) = 0$ implies $v(S) = 0$ and so,

$$v(S) - \sum_{j\in S\setminus\{i\}} M_j(v) = -\sum_{j\in S\setminus\{i\}} M_j(v) \leq 0 \leq v(\{i\}) \leq m_i(v).$$

Hence, the remainder $\rho(S,i)$ will not determine the minimal right if $R(S) = 0$.

The game G is compromise admissible and therefore $k(G) \leq K(G)$, which is equivalent to $m(v) \leq M(v)$. Second, $\sum_{i\in N} k_i(G) \leq 1 \leq \sum_{i\in N} K_i(G)$ is equal to $\sum_{i\in N} m_i(v) \leq v(N) \leq \sum_{i\in N} M_i(v)$. According to (4.3), the game (N,v) is quasi-balanced. Finally, the compromise value is defined by

$$\Phi^c(G) = (k(G) + \gamma(K(G) - k(G)))\, R(N)$$

where $0 \leq \gamma \leq 1$ such that $k(G) + \gamma(K(G) - k(G)) \in \Delta^*(N)$. But then

$$\sum_{i\in N} (m_i(v) + \gamma(M_i(v) - m_i(v))) = v(N),$$

by (4.4) $\tau(v) = m(v) + \gamma(M(v) - m(v))$ and $\Phi^c(G) = \tau(v)/v(N) \cdot R(N)$. □

After this specific attention for games in $GLI^N \cap \mathcal{C}^N$ we will turn our attention to games in $\mathcal{C}^N$. Define the subgame G_T of $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ restricted to coalition T by $G_T = (T, (R(S))_{S\in\mathcal{S}_T}, \mathcal{A}_T, (\alpha_i)_{i\in T})$ where $\mathcal{S}_T = \{S \in \mathcal{S} | S \subset T\}$ and $\mathcal{A}_T = \{pR(S) \in \mathcal{A} | S \in \mathcal{S}_T\}$. Denote by $\bar{\mathcal{G}}^N = \cup_{M\subset N, M\neq\emptyset} \mathcal{G}^M$ the class of games in $\mathcal{G}^N$ and all of their subgames. Similarly we define $\bar{\mathcal{C}}^N$.

The remainder of this section deals with the marginal value. Therefore, assumption 2.3 is valid. If G is a one-person or a two-person game then the compromise value and the marginal value coincide.

Theorem 4.3 $\Phi^c(G) = \Phi^m(G)$ *for all* $G \in \bar{\mathcal{C}}^N$, $|N| = 2$.

Proof. It is obvious that $\Phi^c(G) = \Phi^m(G)$ for one-person games $G \in \bar{\mathcal{C}}^N$. Next, let $G \in \bar{\mathcal{C}}^N$ be a two-person game. Assume that $N = \{1, 2\}$. Then

$$\begin{aligned} K_1(G) &= \sup\{t \in \mathbb{R} | \exists a_2 \in \mathbb{R} : a_2 + t = 1;\ a_2 R(N) \succsim_2 R(\{2\})\} \\ &= \sup\{1 - a_2 | a_2 \geq \alpha_2(R(\{2\}), R(N))\} \\ &= 1 - \alpha_2(R(\{2\}), R(N)). \end{aligned}$$

Similarly, $K_2(G) = 1 - \alpha_1(R(\{1\}), R(N))$. Calculating the remainders results for player 1 in coalition $\{1\}$ in

$$r(\{1\}, 1) = 1, \rho(\{1\}, 1) = \alpha_1(R(\{1\}), R(N))$$

and for player 1 in coalition N

$$\begin{aligned} \rho(N, 1) &= r(N, 1) = \sup\{t \in \mathbb{R} | \exists a_2 \in \mathbb{R} : a_2 + t = 1;\ a_2 R(N) \succ_2 K_2(G) R(N)\} \\ &= \sup\{1 - a_2 | a_2 > K_2(G)\} \\ &= 1 - K_2(G) = \alpha_1(R(\{1\}), R(N)). \end{aligned}$$

The minimal right for player 1 is $k_1(G) = \max\{\rho(\{1\}, 1), \rho(N, 1)\} = \alpha_1(R(\{1\}), R(N))$. In a similar way, $k_2(G) = \alpha_2(R(\{2\}), R(N))$. Easy calculations show that

$$\begin{aligned} \Phi^c(G) = \tfrac{1}{2}(&1 + \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)), \\ &1 - \alpha_1(R(\{1\}), R(N)) + \alpha_2(R(\{2\}), R(N)))R(N), \end{aligned}$$

The marginal vectors are

$$M^{(1,2)}(G) = (\alpha_1(R(\{1\}), R(N)), 1 - \alpha_1(R(\{1\}), R(N)))R(N)$$

and

$$M^{(2,1)}(G) = (1 - \alpha_2(R(\{2\}), R(N)), \alpha_2(R(\{2\}), R(N)))R(N),$$

where (i, j) is shorthand for the permutation σ with $\sigma(1) = i$ and $\sigma(2) = j$. Their average, the marginal value, is

$$\begin{aligned} \Phi^m(G) = \tfrac{1}{2}(&1 + \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)), \\ &1 - \alpha_1(R(\{1\}), R(N)) + \alpha_2(R(\{2\}), R(N)))R(N), \end{aligned}$$

which is equal to the compromise value. □

Timmer et al. (2000b) characterized the marginal value on $\bar{\mathcal{G}}^N$ with $|N| = 2$. They use the following property, which is based on the balanced contributions property for cooperative TU games by Myerson (1980). A solution concept Ψ on $\bar{\mathcal{G}}^N$ is said to have

(vii) *balanced contributions* if for all games $G \in \mathcal{G}^N$, for all coalitions $T \subset N$ and for all $i, j \in T$, $i \neq j$,

$$\begin{aligned} &\alpha_i(\Psi_i(G_T), R(T)) - \alpha_i(\Psi_i(G_{T \setminus \{j\}}), R(T)) \\ &= \alpha_j(\Psi_j(G_T), R(T)) - \alpha_j(\Psi_j(G_{T \setminus \{i\}}), R(T)). \end{aligned}$$

Theorem 4.4 *The compromise value Φ^c is the unique solution concept on $\bar{\mathcal{C}}^N$ with $|N| = 2$ that is efficient and has balanced contributions.*

Proof. This result follows from theorem 4.3 and from noting that theorem 4.8 in Timmer et al. (2000b), which characterizes the marginal value on $\bar{\mathcal{G}}^N$ with $|N| = 2$, also holds for Φ^m on $\bar{\mathcal{C}}^N$. □

If we consider games with more than two players, then the compromise value and the marginal value may be different, as is shown in the example below.

Example 4.5 Consider the game in example 3.2. Let (i, j, k) denote the permutation $\sigma \in \Pi(N)$ with $\sigma(1) = i$, $\sigma(2) = j$, and $\sigma(3) = k$. Then

$$M^{(1,2,3)}(G) = M^{(1,3,2)}(G) = (0, 4/5, 1/5)R(N),$$
$$M^{(2,1,3)}(G) = (1/2, 1/5, 3/10)R(N),$$

and so on. The average of the six marginal vectors is

$$\Phi^m(G) = (11/45, 103/180, 11/60)R(N),$$

which is unequal to the compromise value. ◇

This example shows that we cannot use the characterization of the marginal value on GLI^N (see Timmer et al. (2000b)) to obtain a characterization of the compromise value on $GLI^N \cap \mathcal{C}^N$. A second possibility would be to search for a characterization of the compromise value with the help of the τ-value of the related TU games (N, v). One of the assumptions on the payoffs is that $R(S) \geq 0$ for all coalitions S. This implies $v(S) \geq 0$, the game (N, v) is nonnegative. As far as we know there does not exist a characterization of the τ-value on the class of nonnegative TU games or on any subclass thereof. Hence, it remains an open question whether there exists a characterization of the compromise value on $GLI^N \cap \mathcal{C}^N$ or on some other subclass of $\mathcal{C}^N$.

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