Protective behaviour in games

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Abstract

This paper introduces the notion of protective strategy combination in the context of finite games in strategic form. It shows that for matrix games the set of protective strategy combinations equals the set of proper equilibria. Moreover, in the context of bimatrix games, the notion of protective behaviour is used as a selection tool.

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1 Introduction

The notion of protective behaviour was first introduced in [1] in the context of implementation of social choice functions for social choice situations with a finite number of alternatives. Here, an agent behaves in a protective way if he reveals his preferences so as to protect himself from the worst eventuality as far as possible. This concept is closely related to the notion of prudent behaviour as [11] formulated. The main difference is that the latter assumes that each agent considers all possible preference profiles of other agents equally likely, whereas the former does not.

Protective behaviour as a binary decision criterium on the set of all finite-dimensional vectors of real numbers is axiomatically characterized [2]. Here, also an axiomatic comparison with the maximin decision criterium is offered. An application of protective behaviour towards matching models is presented in [3].

In this paper we want to proceed on this line of research by considering protective behaviour in mixed extensions of finite games in strategic form. First we define both protective and prudent strategies in a strategic form game based on the ideas of these two concepts in social choice situations. It is shown that the sets of prudent and protective strategies for each player in a strategic form game coincide. A protective strategy combination is defined to be a strategy combination that consists of a protective strategy for each player. Existence of protective strategy combinations is shown and it is seen that each protective strategy is also a maximin strategy. So, in particular, for matrix games, protective strategy combinations are equilibria and offer a selection of the saddle points. Moreover, it is proved that each protective strategy is a Dresher optimal strategy ([8]) and hence, for matrix games, protective strategy combinations coincide with proper equilibria à la Myerson ([13], cf. [7]), and the nucleolus introduced by [14] for matrix games.

For general strategic games, protective strategy combinations need not to be Nash equilibria. For two-person games, however, the notion of protective behaviour can be extended to provide selections of Nash equilibria, perfect equilibria à la Selten [15] and proper equilibria. In the definition of this extended concept of protective behaviour results on the structure of the set of Nash equilibria [16], [10], perfect equilibria [6] and proper equilibria [11] are used.

The paper is organized as follows. Section 2 provides the formal definition of protective strategy combinations for finite strategic games. Existence is shown and it is seen that protective strategies are minimax strategies. Section 3 concentrates on matrix games and it is shown that each protective strategy combination is proper and conversely. Section 4 discusses a possible way to use the concept of protective behaviour to obtain selections of Nash, perfect and proper equilibria for bimatrix games. An example is presented to clarify the main ideas.

2 Protective strategy combinations

Let $N = \{1, \ldots, n\}$ denote the set of players. A finite game Γ in strategic form with player set N is represented by $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$ where for each $i \in N$, the finite set S_i denotes the set of pure strategies for player i and $K_i : \prod_{j=1}^{n} S_j \longrightarrow \mathbb{R}$ denotes the payoff function for player i.

Considering mixed strategies we let $\Delta_j = \Delta(S_j)$ represent the set of all probability measures on S_j for all $j \in N$. The payoff functions $\{K_i\}_{i \in N}$ are extended to the set $\prod_{j \in N} \Delta_j$ of all mixed strategies combinations in the obvious way.

For notational convenience we set, for $i \in N$,

$$S = \prod_{j \in N} S_j, \; \Delta = \prod_{j \in N} \Delta_j, \; S_{-i} = \prod_{j \in N \setminus \{i\}} S_j \text{ and } \Delta_{-i} = \prod_{j \in N \setminus \{i\}} \Delta_j.$$

A pure strategy combination is denoted by $s \in S$, a mixed strategy combination by $\sigma \in \Delta$. Sometimes, given $i \in N$, we will write $s = (s_{-i}, s_i)$ and $\sigma = (\sigma_{-i}, \sigma_i)$.

A mixed strategy combination $\tilde{\sigma}$ is called a Nash equilibrium of Γ if for each player $i \in N$, $K_i(\tilde{\sigma}) \geq K_i(\tilde{\sigma}_{-i}, \sigma_i)$ for all $\sigma_i \in \Delta_i$. It is well known that every finite game in strategic form has at least one combination of mixed strategies which is a Nash equilibrium.

A mixed strategy $\tilde{\sigma}_i \in \Delta_i$ is called a maximin strategy for player i if

$$\min_{\sigma_{-i} \in \Delta_{-i}} K_i(\sigma_{-i}, \tilde{\sigma}_i) = \max_{\sigma_i \in \Delta_i} \min_{\sigma_{-i} \in \Delta_{-i}} K_i(\sigma_{-i}, \sigma_i).$$

For each player $i \in N$, the set Δ_i is compact and closed, and the payoff-function K_i is continuous. These properties guarantee the existence of maximin strategy combinations for every finite person game in strategic form.

As maximin strategies assure some payoff to a player, we will see that the notion of protectiveness and prudentness offers a possibility to select interesting strategies out of this set. The notion of protective and prudent domination is described below.

Definition 2.1 Let $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$. Let $i \in N$ and $\sigma_i \in \Delta_i$. Recursively, we define $a_i^r(\sigma_i) \in \mathbb{R}$ and $S_{-i}^r(\sigma_i) \subset S_{-i}$ by

(i) for $r = 1$,

$$a_i^1(\sigma_i) = \min\{K_i(s_{-i}, \sigma_i) |\ s_{-i} \in S_{-i}\}$$

$$S_{-i}^1(\sigma_i) = \{s_{-i} \in S_{-i} |\ K_i(s_{-i}, \sigma_i) = a_i^1(\sigma_i)\}$$

(ii) for $r > 1$,

$$a_i^r(\sigma_i) = \min\{K_i(s_{-i}, \sigma_i) |\ s_{-i} \in S_{-i} \setminus \bigcup_{k=1}^{r-1} S_{-i}^k(\sigma_i)\}$$

$$S_{-i}^r(\sigma_i) = \{s_{-i} \in S_{-i} |\ K_i(s_{-i}, \sigma_i) = a_i^r(\sigma_i)\}.$$

Definition 2.2 Let Γ be a finite game in strategic form. Let $i \in N$ and $\sigma_i, \tilde{\sigma}_i \in \Delta_i$. We say that $\tilde{\sigma}_i$ dominates σ_i in a protective way, in notation $\tilde{\sigma}_i \succ_{pro} \sigma_i$, if there exists an $l \in \mathbb{N}$, such that

(i) $a_i^r(\sigma_i) = a_i^r(\tilde{\sigma}_i)$ and $S_{-i}^r(\sigma_i) = S_{-i}^r(\tilde{\sigma}_i)$ for all $r \in \mathbb{N}$, $r < l$, and

(ii) $a_i^l(\sigma_i) < a_i^l(\tilde{\sigma}_i)$ or
$a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i)$ and $S_{-i}^l(\tilde{\sigma}_i) \subsetneqq S_{-i}^l(\sigma_i)$.

A mixed strategy $\hat{\sigma}_i \in \Delta_i$ is called protective for player i in Γ if it is undominated w.r.t. the protective dominance relation, i.e., if there does not exist a mixed strategy $\tilde{\sigma}_i \in \Delta_i$ such that $\tilde{\sigma}_i \succ_{pro} \hat{\sigma}_i$.

A combination of mixed strategies σ is called a protective strategy combination of Γ if σ_i is a protective strategy for player i for all $i \in N$.

So, a strategy is protective if it consecutively maximizes the worst possible payoff, thereby taking into account the sets of pure strategy combinations of the opponents which yield that minimal amount. It turns out that each protective strategy is maximin.

Lemma 2.1 Let Γ be an $n-$person game in strategic form. If σ_i is a protective strategy for player i in Γ, then σ_i is also a maximin strategy for player i in Γ.

Proof

Let $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$. Let $i \in N$ and let $\hat{\sigma}_i$ be a protective strategy for player i. As a consequence of the definition 2.2, we have that

$$a_i^1(\hat{\sigma}_i) = \min_{s_{-i} \in S_{-i}} K_i(s_{-i}, \hat{\sigma}_i) \geq \min_{s_{-i} \in S_{-i}} K_i(s_{-i}, \sigma_i)$$

for all $\sigma_i \in \Delta_i$. Then

$$\min_{s_{-i} \in S_{-i}} K_i(s_{-i}, \hat{\sigma}_i) = \max_{\sigma_i \in \Delta_i} \min_{s_{-i} \in S_{-i}} K_i(s_{-i}, \sigma_i)$$

and hence $\hat{\sigma}_i$ is a maximin strategy of player i. ■

However, the concepts of protective strategies and maximin strategies are not equivalent as we can see in the following example.

Example 2.1. Consider the 2×3 matrix game A given by

$$A = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{pmatrix} -1 & -2 & -1 \\ -5 & -2 & 0 \end{pmatrix} \end{array}$$

The set of maximin strategies of the row player is given by

$$\{(\sigma_1(e_1), \sigma_1(e_2)) \in \Delta_1 | \ \frac{3}{4} \leq \sigma_1(e_1) \leq 1\}$$

but only $\hat{\sigma}_1$ with $\hat{\sigma}_1(e_1) = 1$ is protective.

Even though the protective dominance relation need not be complete, the next lemma reveals that a protective strategy is dominant, up to payoff equivalence, with respect to the $\succ_{pro}$ relation.

Lemma 2.2 Let $\Gamma = \langle \{S_i\}_{i\in N}, \{K_i\}_{i\in N}\rangle$. Let $\tilde{\sigma}_i \in \Delta_i$ be a protective strategy of player $i \in N$ and let $\sigma_i \in \Delta_i$ be an arbitrary mixed strategy for player i. Then, either $\tilde{\sigma}_i$ and σ_i are payoff equivalent for player i or $\tilde{\sigma}_i \succ_{pro} \sigma_i$.

Proof

Assume that $\tilde{\sigma}_i$ and σ_i are not payoff equivalent and suppose that $\tilde{\sigma}_i$ does not dominate σ_i in a protective way. Taking into account that $\tilde{\sigma}_i$ is a protective strategy of player i, σ_i does not dominate $\tilde{\sigma}_i$ in a protective way. Then, according to definition 2.2, there exists $l \in \mathbb{N}$ such that

(i) $a_i^r(\sigma_i) = a_i^r(\tilde{\sigma}_i)$ and $S_{-i}^r(\sigma_i) = S_{-i}^r(\tilde{\sigma}_i)$ for all $r \in \mathbb{N}$, $r < l$, and

(ii) $a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i)$, $S_{-i}^l(\tilde{\sigma}_i) \backslash S_{-i}^l(\sigma_i) \neq \emptyset$ and $S_{-i}^l(\sigma_i) \backslash S_{-i}^l(\tilde{\sigma}_i) \neq \emptyset$.

Let $0 < \alpha < 1$ and let $\hat{\sigma}_i = \alpha\tilde{\sigma}_i + (1-\alpha)\sigma_i \in \Delta_i$ with the obvious interpretation. We will prove that $\hat{\sigma}_i \succ_{pro} \tilde{\sigma}_i$.

Clearly, from (i), $a_i^r(\sigma_i) = a_i^r(\tilde{\sigma}_i) = a_i^r(\hat{\sigma}_i)$ and $S_{-i}^r(\tilde{\sigma}_i) = S_{-i}^r(\sigma_i) \subset S_{-i}^r(\hat{\sigma}_i)$, for all $r < l$. One can even show that $S_{-i}^r(\tilde{\sigma}_i) = S_{-i}^r(\sigma_i) = S_{-i}^r(\hat{\sigma}_i)$ for all $r < l$.

Next, we will show that either $a_i^l(\hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$ or $S_{-i}^l(\hat{\sigma}_i) \subsetneqq S_{-i}^l(\tilde{\sigma}_i)$ and $a_i^l(\hat{\sigma}_i) = a_i^l(\tilde{\sigma}_i)$. Consider the following two cases:

(a) Let $S_{-i}^l(\tilde{\sigma}_i) \bigcap S_{-i}^l(\sigma_i) = \emptyset$. We will show that $a_i^l(\hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$. Take $\tilde{s}_{-i} \in S_{-i}^l(\tilde{\sigma}_i)$. Then, there is an $r > l$ such that $\tilde{s}_{-i} \in S_{-i}^r(\sigma_i)$ and so,

$$K_i(\tilde{s}_{-i}, \sigma_i) > K_i(\tilde{s}_{-i}, \tilde{\sigma}_i) = a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i).$$

Hence,

$$K_i(\tilde{s}_{-i}, \hat{\sigma}_i) > K_i(\tilde{s}_{-i}, \tilde{\sigma}_i) = a_i^l(\tilde{\sigma}_i).$$

For all $\bar{s}_{-i} \in S_{-i}^t(\tilde{\sigma}_i)$ for some $t > l$, it holds that

$$K_i(\bar{s}_{-i}, \tilde{\sigma}_i) > a_i^l(\tilde{\sigma}_i) \text{ and } K_i(\bar{s}_{-i}, \sigma_i) \geq a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i)$$

and, hence, $K_i(\bar{s}_{-i}, \hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$. We may conclude that $a_i^l(\hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$.

(b) Let $S_{-i}^l(\tilde{\sigma}_i) \bigcap S_{-i}^l(\sigma_i) \neq \emptyset$. Clearly, for every $\bar{s}_{-i} \in S_{-i}^l(\tilde{\sigma}_i) \bigcap S_{-i}^l(\sigma_i)$

$$a_i^l(\tilde{\sigma}_i) = K_i(\bar{s}_{-i}, \tilde{\sigma}_i) = K_i(\bar{s}_{-i}, \hat{\sigma}_i) = K_i(\bar{s}_{-i}, \sigma_i) = a_i^l(\sigma_i),$$

which implies that $a_i^l(\hat{\sigma}_i) = a_i^l(\tilde{\sigma}_i)$.
Furthermore, for every $s_{-i} \in \Delta_{-i} \backslash (\bigcup_{r=1}^{l-1} S_{-i}^r(\sigma_i) \bigcup (S_{-i}^l(\tilde{\sigma}_i) \bigcap S_{-i}^l(\sigma_i)))$, either

$K_i(s_{-i}, \tilde{\sigma}_i) > a_i^l(\tilde{\sigma}_i)$ or $K_i(s_{-i}, \sigma_i) > a_i^l(\sigma_i)$ and, consequently, $K_i(s_{-i}, \hat{\sigma}_i) > a_i^l(\tilde{\sigma}_i) = a_i^l(\hat{\sigma}_i)$ and $S_{-i}^l(\hat{\sigma}_i) = S_{-i}^l(\tilde{\sigma}_i) \bigcap S_{-i}^l(\sigma_i) \subsetneqq S_{-i}^l(\tilde{\sigma}_i)$.

In both cases $\hat{\sigma}_i \succ_{pro} \tilde{\sigma}_i$. A contradiction results and the assertation of the theorem holds. ■

Next we define the prudent domination criterium. There is a slight difference between prudent and protective domination. Even though both criteria compare payoff levels, the former only compares , for each player, the cardinality of the sets of pure strategy combinations of the opponents where those payoff levels are achieved instead of the inclusion relation used in the latter.

Definition 2.3 Let $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$. Let $i \in N$ and $\sigma_i, \tilde{\sigma}_i \in \Delta_i$. We say that $\tilde{\sigma}_i$ dominates σ_i in a prudent way, in notation $\tilde{\sigma}_i \succ_{pru} \sigma_i$, if there exists $l \in \mathbb{N}$ such that

(i) $a_i^r(\sigma_i) = a_i^r(\tilde{\sigma}_i)$ and $|S_{-i}^r(\sigma_i)| = |S_{-i}^r(\tilde{\sigma}_i)|$ for all $r \in \mathbb{N}$, $r < l$, and

(ii) $a_i^l(\sigma_i) < a_i^l(\tilde{\sigma}_i)$ or
$a_i^l(\sigma_i) = a_i^l(\tilde{\sigma}_i)$ and $|S_{-i}^l(\tilde{\sigma}_i)| < |S_{-i}^l(\sigma_i)|$.

A mixed strategy $\hat{\sigma}_i \in \Delta_i$ is called prudent for player i in Γ if it is undominated w.r.t. the prudent dominance relation, i.e., if there does not exist any mixed strategy $\tilde{\sigma}_i \in \Delta_i$ such that $\tilde{\sigma}_i \succ_{pru} \hat{\sigma}_i$.

If a strategy of player i is prudent, then it is also protective because from the definitions 2.2 and 2.3 it follows that $\hat{\sigma}_i \succ_{pro} \sigma_i$ implies $\hat{\sigma}_i \succ_{pru} \sigma_i$. The next example shows that the converse need not hold.

Example 2.2. Consider the 2×3 matrix game given by

$$\begin{array}{cc} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ A = \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \end{array}$$

Take the mixed strategies $\sigma_1 = (\sigma_1(e_1), \sigma_1(e_2)) = (\frac{1}{4}, \frac{3}{4})$ and $\tilde{\sigma}_1 = (\frac{3}{4}, \frac{1}{4})$ of the row player. Then, σ_1 does not dominate $\tilde{\sigma}_1$ in a protective way because

$$a_1^1(\sigma_1) = a_1^1(\tilde{\sigma}_1) = \frac{5}{4}, \text{ and}$$

$$S_{-1}^1(\sigma_1) = \{e_2\} \text{ and } S_{-1}^1(\tilde{\sigma}_1) = \{e_1, e_3\}.$$

We may, however, conclude that $\sigma_1 \succ_{pru} \tilde{\sigma}_1$. In this game there is only one protective and prudent strategy for player 1 given by $\hat{\sigma}_1 = (\frac{1}{2}, \frac{1}{2})$, which is also the unique maximin strategy of the row player.

However, as a consequence of the lemma 2.2, the next theorem holds.

Theorem 2.1 In a finite strategic form game Γ a mixed strategy is protective if and only if it is prudent.

We use this equivalence between protective and prudent strategies to prove the existence of protective strategy combinations in a finite game in strategic form.

Theorem 2.2 Every finite game in strategic form has at least one protective strategy combination.

Proof

We prove this result in a constructive way finding the set of prudent strategies of a finite game in strategic form of an arbitrary player. Using theorem 2.1 we obtain also the set of all the protective strategies. Let $\Gamma = \langle \{S_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle$. We define for each $i \in N$,

$$M_i^1 := \{\hat{\sigma}_i \in \Delta_i | \; a_i^1(\hat{\sigma}_i) = \max\{a_i^1(\sigma_i) | \; \sigma_i \in \Delta_i\}\}$$

$$P_i^1 := \{\hat{\sigma}_i \in M_i^1 | \; |S_{-i}^1(\hat{\sigma}_i)| \leq |S_{-i}^1(\sigma_i)| \text{ for all } \sigma_i \in M_i^1\}$$

and for $r > 1$, we define

$$M_i^r := \{\hat{\sigma}_i \in P_i^{r-1} | \; a_i^r(\hat{\sigma}_i) = \max\{a_i^r(\sigma_i) | \; \sigma_i \in M_i^{r-1}\}\}$$

$$P_i^r := \{\hat{\sigma}_i \in M_i^r | \; |S_{-i}^r(\hat{\sigma}_i)| \leq |S_{-i}^r(\sigma_i)| \text{ for all } \sigma_i \in M_i^r\}.$$

Note that M_i^1 is the set of maximin strategies of player i and that $a_i^r(\sigma_i) = \infty$ and $|S_{-i}^r(\sigma_i)| = 0$ if $\bigcup_{k=1}^{r-1} S_{-i}^k(\sigma_i) = S_{-i}$.

Clearly, $M_i^r \neq \emptyset$ and $P_i^r \neq \emptyset$ for all $r \in \mathbb{N}$. Since $|S_{-i}^r(\bar{\sigma}_i)| = |S_{-i}^r(\hat{\sigma}_i)|$ for all $\bar{\sigma}_i, \hat{\sigma}_i \in P_i^r$ and $r \geq 1$, and S_{-i} is a finite set, we can take the smallest $t \in \mathbb{N}$ such that $P_i^t = P_i^r$ for all $r \geq t$.

By definition, P_i^t precisely contains all prudent strategies of player i in the game Γ. ■

Since the protective strategy combinations (and therefore the prudent ones) are maximin strategy combinations, the previous theorem and example 2.1 show that protective strategy combinations constitute a selection of the maximin strategies for every finite game in strategic form.

3 Protective behaviour in matrix games

In this section we consider finite zero-sum games $A = \langle S_1, S_2, K \rangle$ in strategic form with the payoff function K for player 1 determined by an $m \times n$ matrix A in the following way

$$K(p, q) := pAq$$

for all $p \in \Delta_1$ and $q \in \Delta_2$. A matrix game will be denoted by A.

Dresher [8] proposed a criterion to select Nash equilibria of a matrix game based on the assumption that each player follows a conservative plan of action and tries to maximize the minimum gain resulting from the opponent's deviations. For describing the Dresher procedure, we need some basic facts of matrix games.

Let $A = \langle S_1, S_2, K \rangle$ be an $m \times n$ matrix game where $S_1 = \{e_1, \ldots, e_m\}$ and $S_2 = \{f_1, \ldots, f_n\}$. Its value, $v(A)$, is given by

$$v(A) := \max_{p \in \Delta_1} \min_{1 \leq j \leq n} pAf_j = \min_{q \in \Delta_2} \max_{1 \leq i \leq m} e_i Aq.$$

The sets of optimal strategies for player 1 and 2 are given by the polytopes

$$O_1(A) := \{p \in \Delta_1 |\ pAf_j \geq v(A) \text{ for all } j \in \{1, \ldots, n\}\}$$

and

$$O_2(A) := \{q \in \Delta_2 |\ e_i Aq \leq v(A) \text{ for all } i \in \{1, \ldots, m\}\}.$$

Furthermore we define the carrier of a strategy $p \in \Delta_1$ by

$$C_1(p) := \{e_i \in \{1, \ldots, m\} |\ p(e_i) > 0\},$$

the carrier of the set of optimal strategies by

$$C_1(A) := \bigcup_{p \in O_1(A)} C_1(p)$$

and the equalizer set by

$$E_1(A) := \{e_i \in \{1, \ldots, m\} |\ e_i A q = v(A) \text{ for all } q \in O_2(A)\}.$$

The sets $C_2(q)$, $C_2(A)$ and $E_2(A)$ are defined in an analogous way. It is well known that $C_1(A) = E_1(A)$ and $C_2(A) = E_2(A)$ (Bohnenblust, Karlin and Shapley [4], Gale and Sherman [9]).

Dresher [8] constructs a sequence of matrix games A^k in the following way. Let $A^1 = A$. In the game A^2 player 1 has as pure strategy set $S_1(2)$ the extreme points of $O_1(A)$ and the set of pure strategies for player 2 is given by $S_2(2) = S_2 \setminus C_2(A)$. The payoff functions in A^2 are just the restrictions of the original ones.

If A^{k-1} is defined and $C_2(A^{k-1}) \neq S_2(k-1)$, then the set $S_1(k)$ of pure strategies of player 1 in A^k is constituted by the extreme points of $O_1(A^{k-1})$ and the set $S_2(k)$ of pure strategies of player 2 is $S_2(k-1) \setminus C_2(A^{k-1})$. Clearly, after a finite number of steps t, A^t has been defined and $C_2(A^t) = S_2(t)$. Then $O_1(A^t)$ is called the set of D-optimal strategies of player 1 denoted by $D_1(A)$.

In a similar way, one defines the set $D_2(A)$ of D-optimal strategies of player 2.

The set of proper equilibria of a matrix game was characterized in [7] as the set of combinations of D-optimal strategies of the game. In [14] has been proved that the nucleolus of a matrix game equals its set of proper equilibria. We offer another characterization using the concept of protective strategy combinations.

Theorem 3.1 For every finite matrix game the set of protective strategy combinations coincides with the set of proper equilibria.

Proof

Let $A = \langle S_1, S_2, K\rangle$ be an $m \times n$ matrix game $S_1 = \{e_1, \ldots, e_m\}$ and $S_2 = \{f_1, \ldots, f_n\}$.

Let $\bar{p}$ be a protective strategy of player 1 in A. We will prove that $\bar{p}$ is a D-optimal strategy. Using lemma 2.1, we know that $\bar{p} \in O_1(A)$. Assume that $\bar{p} \in O_1(A^k)$ for all $k \in \{1, \ldots, t-1\}$ for some $t \geq 2$ where A^k is as described in the Dresher procedure. It suffices to prove that $\bar{p} \in O_1(A^t)$ if $S_2(t) = S_2(t-1) \setminus C_2(A^{t-1}) \neq \emptyset$.

Let $\tilde{p} \in O_1(A^t)$. If $\bar{p}$ and $\tilde{p}$ are payoff equivalent, the proof is finished. Otherwise, according to lemma 2.2 we have that $\bar{p} \succ_{pro} \tilde{p}$. We need to show that

$$\min_{f_j \in S_2(t)} \bar{p}Af_j = \min_{f_j \in S_2(t)} \tilde{p}Af_j.$$

We know that

$$\min_{f_j \in S_2(k)} \bar{p}Af_j = \min_{f_j \in S_2(k)} \tilde{p}Af_j$$

for all $k \in \{1, \ldots, t-1\}$ and

$$\min_{f_j \in S_2(t)} \bar{p}Af_j \leq \min_{f_j \in S_2(t)} \tilde{p}Af_j$$

Suppose we have a strict inequality. Then, since $\bar{p} \succ_{pro} \tilde{p}$ there is a game A^r with $r \in \{1, \ldots, t-1\}$ and an integer l such that $v(A^r) = a_1^l(\bar{p}) = a_1^l(\tilde{p})$, $S_{-1}^l(\bar{p}) \subsetneqq S_{-1}^l(\tilde{p})$ and for all $k < l$, $a_1^k(\bar{p}) = a_1^k(\tilde{p})$ and $S_{-1}^k(\bar{p}) = S_{-1}^k(\tilde{p})$.

Choose $f_s \in S_{-1}^l(\tilde{p}) \setminus S_{-1}^l(\bar{p})$. By definition, $\bar{p}Af_s > \tilde{p}Af_s$ and, hence, $f_s \in S_2(t)$. For, suppose $f_s \notin S_2(t)$. Then $f_s \in C_2(A^k) = E_2(A^k)$ for some $k \in \{1, \ldots, t-1\}$ and, consequently, since $\bar{p} \in O_1(A^k)$ and $\tilde{p} \in O_1(A^k)$, it would hold that $\bar{p}Af_s = \tilde{p}Af_s$, which is a contradiction. We may conclude that

$$\min_{f_j \in S_2(t)} \tilde{p}Af_j = \tilde{p}Af_s = a_1^l(\tilde{p}) = a_1^l(\bar{p}) \leq \min_{f_j \in S_2(t)} \bar{p}Af_j$$

and, there is equality. Hence, every protective strategy of player 1 is D-optimal. We can proceed in an analogous way to prove that every protective strategy of player 2 is also a D-optimal strategy.

Since each combination of D-optimal strategies is a proper equilibrium, it follows that each protective strategy combination is also proper. Moreover, since A has at least one protective strategy combination, this game has a protective D-optimal combination of strategies. Taking into account that all D-optimal strategies are payoff equivalent [7], it follows that all proper equilibria of this game are protective. ■

4 Protective behaviour as a refinement tool for bimatrix games

A bimatrix game (A, B) is a two-person game $\Gamma = \langle\{S_1, S_2\}, \{K_1, K_2\}\rangle$ in strategic form where $S_1 = \{e_1, \ldots, e_m\}$ and $S_2 = \{f_1, \ldots, f_n\}$ are the pure strategy spaces for player 1 and 2, respectively and the payoff functions are given by

$$K_1(p, q) := pAq \quad \text{and} \quad K_2(p, q) := pBq \tag{1}$$

for all $p \in \Delta_1$ and $q \in \Delta_2$, respectively.

Definition 4.1 Let (A, B) be an $m \times n$ bimatrix game. Let $P \subset \Delta_1$ be a non-empty, closed and convex set. Let $\bar{p} \in P$. We say that $\bar{p}$ is a protective strategy for player i w.r.t. P if and only if there does not exist any $p \in P$ such that $p \succ_{pro} \bar{p}$.
In a similar way one defines a protective strategy of player 2 w.r.t a closed and convex set $Q \subset \Delta_2$.

The following game in extensive form illustrates the notion of protective behavior with respect to a closed and convex subset of strategies.

Example 3.1.

In each endpoint of the tree, the first coordinate is the payoff to player 1 and the second coordinate is the payoff to player 2, if they reach this node after a play. Its representation in reduced strategic form is given by the 2×4 bimatrix game (A, B), given by

$$(A,B) = \begin{array}{c} \\ L \\ R \end{array} \begin{array}{c} \begin{array}{cccc} out & (in,l) & (in,m) & (in,r) \end{array} \\ \left(\begin{array}{cccc} (0,6) & (2,0) & (1/2,-4) & (0,8) \\ (0,6) & (0,8) & (1/2,-4) & (1,0) \end{array} \right) \end{array}$$

We can obtain the set of Nash equilibria, perfect equilibria and proper equilibria of this game using the GC-approach described in [5]. The set of Nash equilibria and perfect equilibria of this game are the same and, using the obvious notation, they are given by

$$T^1 \times T^2 = \{(\alpha, 1-\alpha) | \ \frac{1}{4} \leq \alpha \leq \frac{3}{4}\} \times \{(1,0,0,0)\}.$$

This game has a unique proper equilibrium: $((\frac{1}{2}, \frac{1}{2}), (1,0,0,0))$. If we use the concept of protective strategy with respect to T^1 for player 1 and with respect to T^2 for player 2, we select the perfect equilibrium: $((\frac{1}{3}, \frac{2}{3}), (1,0,0,0))$.

Let us examine the differences between them. A comparison between the two should take into account the actions player 2 will take if the information set **b** is reached. Clearly, player 2 will never choose m if he is called upon to act at that stage. In expectation, the protective strategy $(\frac{1}{3}, \frac{2}{3})$ leads to a payoff of $\frac{2}{3}$ while the proper equilibrium strategy leads to either 1 (in case of l) or $\frac{1}{2}$ (in case of r). So the choice of the proper equilibrium seems riskier for player 1 than the choice of the protective strategy.

For the protective dominance relation defined in 4.1, one can derive the same type of results as we obtain in section 2 from the conditions that the set P satisfies. So, for a protective strategy $p \in P$ and an arbitrary $\bar{p} \in P$ either p is payoff equivalent with $\bar{p}$ or $p \succ_{pro} \bar{p}$ and consequently a strategy $p \in P$ is protective w.r.t. P if and only if it is prudent w.r.t. P. Furthermore, each finite game in strategic form has for every P at least one protective strategy combination with respect to it. Given the fact that the set of Nash equilibria of a bimatrix game [16], [10], the set of perfect equilibria of a bimatrix game [6] and the set of proper equilibria of a bimatrix game [11] are the union of a finite number of polytopes, we can select out Nash equilibria, perfect equilibria and proper equilibria considering protective strategies with respect to Nash, Selten and Myerson sets, respectively, for each player.

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