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**TWO CLASSES OF COOPERATIVE GAMES RELATED TO
ONE-OBJECT AUCTION SITUATIONS**

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Two classes of cooperative games related to one-object auction situations¹

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Abstract

We consider a market situation with two corners. One corner consists of a single seller with one object, and the other corner consists of potential buyers who all want the object. We suppose that the valuations of the object for the different buyers is known by all of them. Then two cooperative games, which we call the auction game and the ring game, corresponding to such a market situation are considered.

Auction games are related to special total big boss games, while ring games are related to special convex games, the peer group games. It turns out that there exists a duality relation between the auction game and the ring game arising from the same two-corner market situation. For both classes of games relevant solution concepts are studied.

Key words: market games, ring games, one-object auction situations, big boss games, peer group games.

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1 Introduction

Our starting point is a two-corner market situation. One corner consists of a single agent, that we denote by $n+1$, who possesses one object with value w_{n+1} for him and wants to sell it. The other corner consists of n potential buyers (bidders), whose set we denote by $N = \{1, 2, \dots, n\}$, where agent $i \in N$ values the object w_i . We denote such a market situation by $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$ and the set of agents involved by $N' = \{1, \dots, n, n+1\}$. To make the following analysis simple, we suppose that the valuations w_1, w_2, \dots, w_n of the object are known by the potential buyers, and that $w_1 \geq w_2 \geq \dots \geq w_n \geq w_{n+1} \geq 0$. With such a market situation we associate two types of cooperative games: auction games and ring games. For both classes of games we study solutions

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such as the core (Gillies, 1953), the Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969), the τ -value (Tijs, 1981) and the AL-value (Tijs, 2005).

The auction game $\langle N', v \rangle$ with player set N' corresponding to the market situation $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$ is the game whose characteristic function $v : 2^N \rightarrow \mathbb{R}$ is such that for each $S \in 2^N$, $v(S) = 0$ and $v(S \cup \{n+1\}) = \max\{w_i | i \in S \cup \{n+1\}\}$. In cooperation, an optimal action of $S \cup \{n+1\}$ is a shift of the object from $n+1$ to a most eager player i in S , so $i \in \arg \max_{k \in S} w_k$. This kind of market games were already studied in von Neumann and Morgenstern (1944), page 564. We use the terms auction games and auction situations for these market games and market situations because we study the interaction of the players in the market via auctions. In Section 2 we study in detail properties of auction games and the core, and relate them with total big boss games (Muto et al., 1988).

The core of a big boss game consists of all vectors between two points: the big boss point and the union point. In case the interaction between the market corners is regulated via one of the four standard auctions, the resulting payoff allocation coincides in all four cases with the union point, which is the best point in the core for all potential buyers.

The ring game $\langle N, r \rangle$ corresponding to $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$ has only the bidders as player set. A subset $S \subseteq N$ can suppress the price to be paid for the object by forming a ring (Graham and Marshall (1987), McAfee and McMillan (1992)). In such a ring S one of the most eager players $\hat{s} \in \arg \max_{i \in S} w_i$ will be active in the auction and the other players in S will behave as if they have the value w_{n+1} . This leads to the cooperative game $\langle N, r \rangle$, which we call the ring game, in which only potential buyers are involved. For each $S \subseteq N$ the ring value is given by

$$r(S) := \begin{cases} w_1 - \max\{w_i | i \in N' \setminus S\}, & \text{if } \arg \max_{i \in N} w_i \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

The reader may notice that we obtain this ring game value via the following consideration. In each one of the four standard types of auctions (see Klemperer (2004), Krishna (2002), Milgrom (2004), Vickrey (1962)) the revenue equivalence theorem (RET) shows that the object goes to a bidder with the highest value for the price equal to the second highest value. If the ring contains all players with highest value then only one of them l^S , the leader of S , will act in the auction as such and the others act as if they have value w_{n+1} . So, the active second highest value appears outside of S . The object goes to l^S for the price $\max\{w_i | i \in N \setminus S\}$. It is remarkable that all the four ring games are the same, which is a consequence of RET. We denote the fact that all the four ring games are the same by GET (game equivalence theorem). In Section 3 we study in detail ring games and relate them with special convex games, namely peer group games (Branzei et al., 2002). Also a duality relation between the auction game and the ring game arising from the same market situation will be derived. Section 4 concludes and proposes some further research.

2 Total big boss games and auction games

Let $N = \{1, \dots, n\}$, $N' = N \cup \{n+1\}$ and $\mathbb{B} = \{S \in 2^{N'} \mid n+1 \in S\}$ the set of coalitions containing player $n+1$. A game $\langle N', v \rangle$ is a total big boss game (TBBG) with $n+1$ as big boss if the following properties hold:

- (i) *Monotonicity*: For all $S, T \in 2^{N'}$, $S \subseteq T$ implies $v(S) \leq v(T)$.
- (ii) *Big boss property*: For each $S \in 2^{N'}$ with $n+1 \notin S$, $v(S) = 0$ holds.
- (iii) *\mathbb{B} -concavity property*: For all $S, T \in \mathbb{B}$ with $S \subseteq T$ and $i \in N' \setminus T$, $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$ holds.

For an extensive description of (T)BBG we refer to Muto et al. (1988), Branzei et al. (2001), Branzei et al. (2005a) and Tijs et al. (2005). Note that each subgame $\langle S, v \rangle$ with $S \in \mathbb{B}$ is also a TBBG. The following well-known theorem holds.

Theorem 2.1 (Muto et al., 1988). *Let $\langle N', v \rangle$ be a TBBG. Then*

- (i) *The core $C(v)$ is given by*

$$C(v) = \left\{ x \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq M_i(v) \text{ for each } i \in N, \sum_{i=1}^{n+1} x_i = v(N') \right\},$$

where $M_i(v) = v(N') - v(N' \setminus \{i\})$.

- (ii) *The τ -value, $\tau(v)$, and the nucleolus, $Nu(v)$, coincide and are equal to $\tau(v) = Nu(v) = \left(\frac{1}{2}M_1(v), \frac{1}{2}M_2(v), \dots, \frac{1}{2}M_n(v), v(N') - \frac{1}{2}\sum_{i=1}^n M_i(v)\right)$.*

The reader may notice that $\tau(v)$ is the average between the big boss point $\mathbb{B}(v) = (0, 0, \dots, 0, v(N'))$ and the union point

$$\mathbb{U}(v) = \left(M_1(v), M_2(v), \dots, M_n(v), v(N') - \sum_{i=1}^n M_i(v) \right).$$

The following theorem shows that auction games are TBBG of a special type.

Theorem 2.2 *Let $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$ be an auction situation and let $\langle N', v \rangle$ be the corresponding auction game. Then*

- (i) *$\langle N', v \rangle$ is a TBBG,*
- (ii) *$C(v) = \text{conv}\{(0, 0, \dots, 0, w_1), (w_1 - w_2, 0, \dots, 0, w_2)\}$ is zero or one dimensional,*
- (iii) *$\tau(v) = Nu(v) = \left(\frac{1}{2}(w_1 - w_2), 0, \dots, 0, \frac{1}{2}(w_1 + w_2)\right)$,*

(iv) the union point $\mathbb{U}(v) = (w_1 - w_2, 0, \dots, 0, w_2)$ is the core-allocation arising in each of the four classical auction types.

Proof.

(i) From $v(S) = 0$ for each $S \in 2^N$ it follows the big boss property. The monotonicity property follows because $v(S \cup \{n+1\}) = \max\{w_i | i \in S \cup \{n+1\}\}$ is monotonic increasing in S . The \mathbb{B} -concavity property follows because for all $S, T \in \mathbb{B}$ with $S \subseteq T$ and $i \in N' \setminus T$ we have

$$\begin{aligned} v(S \cup \{i\}) - v(S) &= \max\{w_k | k \in S \cup \{i\}\} - \max\{w_k | k \in S\} \\ &= \max\{0, w_i - \max\{w_k | k \in S\}\} \\ &\geq \max\{0, w_i - \max\{w_k | k \in T\}\} = v(T \cup \{i\}) - v(T). \end{aligned}$$

(ii) From (i) it follows that

$$\begin{aligned} C(v) &= \left\{ x \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq M_i(v) \text{ for each } i \in N, \sum_{i=1}^{n+1} x_i = v(N') \right\} \\ &= \{x \in \mathbb{R}^{n+1} \mid 0 \leq x_1 \leq w_1 - w_2, x_2 = \dots = x_n = 0, x_1 + x_{n+1} = w_1\} \\ &= \text{conv}\{(0, 0, \dots, 0, w_1), (w_1 - w_2, 0, \dots, 0, w_2)\}. \end{aligned}$$

The reader may notice that if $w_1 = w_2$, then $C(v) = \{(0, 0, \dots, 0, w_1)\}$ is zero-dimensional and if $w_1 > w_2$, then $\dim(C(v)) = 1$.

(iii) Follows from the τ -value and the nucleolus formulas for TBBG in Theorem 2.1.

(iv) It follows from RET, which tells that the object goes for the second highest price to one of the highest bidders leading to the allocation $(w_1 - w_2, 0, \dots, 0, w_2)$. ■

Remark 2.3 *The average lexicographic value $AL(v)$ of $\langle N', v \rangle$ is equal to the nucleolus $Nu(v)$ and the τ -value $\tau(v)$ (see Tijjs, 2005). The Shapley value $\Phi(v)$ is a core allocation of $\langle N', v \rangle$ if and only if $w_2 = w_3 = \dots = w_n = w_{n+1}$.*

Next theorem together with Theorem 2.2 generates a characterization of auction games arising from market situations discussed in this paper: the auction games correspond to TBBG whose core has dimension zero or one.

Theorem 2.4 *Let $\langle N', v \rangle$ be a TBBG with zero or one-dimensional core. Suppose $x_2 = x_3 = \dots = x_n = 0$ for each $x \in C(v)$. Then there is a market situation $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$ such that the corresponding auction game equals $\langle N', v \rangle$.*

Proof. Note that $C(v) = \{(x_1, 0, \dots, 0, v(N') - x_1) \mid 0 \leq x_1 \leq M_1(v)\}$. We can distinguish two cases.

1. $\dim(C(v)) = 0$. Then $M_1(v) = 0$. Take a market situation $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$ with $w_1 = v(N') = w_2 \geq w_3 \geq \dots \geq w_n \geq w_{n+1}$.
2. $\dim(C(v)) = 1$. Then $M_1(v) > 0$. Take $w_1 = v(N')$ and $w_2 = v(N') - M_1(v) \geq w_3 \geq \dots \geq w_n \geq w_{n+1}$.

In both cases, it is easy to check that the corresponding auction game is $\langle N', v \rangle$. ■

The following example illustrates the previous results.

Example 2.5 Consider the market situation $\langle 210, 90, 60; 0 \rangle$. The corresponding auction game $\langle \{1, 2, 3, 4\}, v \rangle$ is given by

$$\begin{aligned} v(\{1, 4\}) &= v(\{1, 2, 4\}) = v(\{1, 3, 4\}) = v(\{1, 2, 3, 4\}) = 210 \\ v(\{2, 4\}) &= v(\{2, 3, 4\}) = 90 \\ v(\{3, 4\}) &= 60 \\ v(S) &= 0 \text{ otherwise.} \end{aligned}$$

By Theorem 2.2 we know that the above game is a TBBG with player 4 as big boss, and $C(v) = \text{conv} \{(0, 0, 0, 210), (120, 0, 0, 90)\}$ is a one dimensional set. In addition, $AL(v) = \tau(v) = Nu(v) = (60, 0, 0, 150)$ and the union point $\mathbb{U}(v) = (120, 0, 0, 90)$ is the core-allocation arising in the four classical auction types. However, $\Phi(v) = (70, 10, 5, 125) \notin C(v)$ since $\Phi_i(v) > 0$ for $i = 2, 3$.

3 Peer group games and ring games

A game $\langle N, v \rangle$ is called a peer group game (Branzei et al., 2002) if v is of the form $v = \sum_{k=1}^n c_k u_{[1,k]}$ where $c_k \geq 0$ for each $k \in N$, $[1, k] := \{1, 2, \dots, k-1, k\}$ and $u_{[1,k]}$ is the unanimity game related to the coalition $[1, k]$.

Each one-object auction situation with valuation vector $(w_1, w_2, \dots, w_n, w_{n+1})$, where $w_1 \geq w_2 \geq \dots \geq w_n \geq w_{n+1} \geq 0$, gives rise to a ring game $\langle N, r \rangle$ where $r = \sum_{k=1}^n (w_k - w_{k+1}) u_{[1,k]}$. Obviously, the ring game corresponding to $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$ is equal to the peer group game $\sum_{k=1}^n c_k u_{[1,k]}$ where $c_k = w_k - w_{k+1}$ for each $k \in N$. Conversely, the peer group game $\sum_{k=1}^n c_k u_{[1,k]}$ is equal to a ring game corresponding to the one-object auction situation with valuation vector of the form $(\alpha + \sum_{k=1}^n c_k, \alpha + \sum_{k=2}^n c_k, \dots, \alpha + c_n, \alpha)$, where $\alpha \geq 0$. Hence, we have the following theorem.

Theorem 3.1 Each ring game is a peer group game and conversely.

Because peer group games are convex games (Shapley, 1971), the Shapley value coincides with the average lexicographic value. So, for a peer group game $\langle N, v \rangle$ with $v = \sum_{k=1}^n c_k u_{[1,k]}$ we obtain

$$\Phi(v) = AL(v) = \left(\sum_{k=1}^n k^{-1} c_k, \sum_{k=2}^n k^{-1} c_k, \dots, n^{-1} c_n \right).$$

Hence, for a ring game $\langle N, r \rangle$ corresponding to the valuation vector $(w_1, w_2, \dots, w_n, w_{n+1})$ the Shapley value is given by

$$\Phi(r) = \left(\sum_{k=1}^n k^{-1}(w_k - w_{k+1}), \sum_{k=2}^n k^{-1}(w_k - w_{k+1}), \dots, n^{-1}(w_k - w_{k+1}) \right).$$

The τ -value for the ring game $r = \sum_{k=1}^n (w_k - w_{k+1})u_{[1,k]}$ is the feasible compromise between the minimum right vector

$$m(r) = (r(\{1\}), r(\{2\}), \dots, r(\{n\})) = (w_1 - w_2, 0, \dots, 0)$$

and the marginal vector $M(r)$, whose component $i \in N$ is given by:

$$M_i(r) = r(N) - r(N \setminus \{i\}) = (w_1 - w_{n+1}) - (w_1 - w_i) = w_i - w_{n+1}.$$

So, $M(r) = (w_1 - w_{n+1}, w_2 - w_{n+1}, \dots, w_n - w_{n+1})$. Hence

$$\tau(r) = (1 - \alpha)(w_1 - w_2, 0, \dots, 0) + \alpha(w_1 - w_{n+1}, \dots, w_n - w_{n+1}), \quad (1)$$

where α is chosen such that $\sum_{i=1}^n \tau_i(r) = r(N) = w_1 - w_{n+1}$.

The nucleolus of peer group games is studied in Branzei et al. (2000) and Branzei et al. (2005b).

The core $C(r)$ of a ring game $r = \sum_{k=1}^n (w_k - w_{k+1})u_{[1,k]}$ is given by the Minkowsky sum $\sum_{k=1}^n (w_k - w_{k+1})\Delta_{[1,k]}$ where

$$\Delta_{[1,k]} = \left\{ x \in \mathbb{R}_+^n \mid \sum_{i=1}^k x_i = 1, x_j = 0 \text{ for } j \in [k+1, n] \right\}.$$

Let us now study maps $F : W^{n+1} \rightarrow \mathbb{R}^n$ with special properties, where $W^{n+1} := \{w \in \mathbb{R}^{n+1} \mid w_1 \geq w_2 \geq \dots \geq w_n \geq w_{n+1} \geq 0\}$. Solutions for ring games generate such maps. Let for $w \in W^{n+1}$, the corresponding ring game be r^w . Then $S : W^{n+1} \rightarrow \mathbb{R}^n$ defined by $S(w) = \Phi(r^w)$ for each $w \in W^{n+1}$ is the map corresponding to the Shapley value. Further $T : W^{n+1} \rightarrow \mathbb{R}^n$ defined by $T(w) = \tau(r^w)$ for each $w \in W^{n+1}$ is the map corresponding to the τ -value. We want to give axiomatic characterizations of these maps. For this reason we introduce the following properties. We say that $F : W^{n+1} \rightarrow \mathbb{R}^n$ has the property

- (i) *efficiency* if $\sum_{i=1}^n F_i(w) = w_1 - w_{n+1}$ for each $w \in W^{n+1}$
- (ii) *symmetry* if $F_i(w) = F_j(w)$ for all $i, j \in \{1, 2, \dots, n\}$ and $w \in W^{n+1}$ with $w_i = w_j$
- (iii) *independence of larger values* if for each $i \in \{1, 2, \dots, n\}$ and each $w, w' \in W^{n+1}$ with $w_k = w'_k$ for $k \geq i$ we have $F_i(w) = F_i(w')$
- (iv) *highest bidder right* if for all $w \in W^{n+1} : F_1(w) = (w_1 - w_2) + F_1(w_2, w_2, w_3, \dots, w_{n+1})$

(v) *weak proportionality* if there is an $\alpha \in \mathbb{R}$ such $F_i(w) = \alpha(w_i - w_{n+1})$ for each $i \in \{1, 2, \dots, n\}$ and each $w \in W_{12}^{n+1} := \{w \in \mathbb{R}^{n+1} | w_1 = w_2\}$.

The reader may notice that the Shapley map $S : W^{n+1} \rightarrow \mathbb{R}^n$ satisfies efficiency, symmetry and independence of larger values. Van den Brink proved the following theorem (van den Brink, 2004).

Theorem 3.2 (*Characterization of the Shapley map S*). *There is a unique map $F : W^{n+1} \rightarrow \mathbb{R}^n$ with the properties efficiency, symmetry and independence of larger values and it is the Shapley map S .*

It follows from formula (1) that the T-map satisfies efficiency, weak proportionality and highest bidder right property. Moreover, we have

Theorem 3.3 (*Characterization of the T-map*). *There is a unique map $F : W^{n+1} \rightarrow \mathbb{R}^n$ with the properties efficiency, weak proportionality and highest bidder right and it is the T-map.*

Proof. We need only to prove that if F satisfies the three properties then $F = T$. Take $w \in W^{n+1}$. Then

$$\begin{aligned} F(w_1, w_2, \dots, w_n, w_{n+1}) &= (w_1 - w_2, 0, \dots, 0) + F(w_2, w_2, w_3, \dots, w_n, w_{n+1}) \\ &= (w_1 - w_2, 0, \dots, 0) + \alpha(w_2 - w_{n+1}, w_2 - w_{n+1}, \dots, w_n - w_{n+1}) \end{aligned}$$

with α such that $\sum_{k=1}^n F_i(w_1, w_2, \dots, w_n, w_{n+1}) = w_1 - w_{n+1}$, by efficiency. The first equality above follows from the highest bidder right property, and the second equality from the weak proportionality property. In view of formula (1) we obtain $F = T$. ■

Now, we show that a duality relation between the auction games and the ring games exists.

For a game $\langle N', v \rangle$ the dual game $\langle N', v^* \rangle$ is defined by $v^*(S) = v(N') - v(N' \setminus S)$ for each $S \in 2^N$.

In the following theorem we describe a duality relation between the auction game and the ring game corresponding to a market situation $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$.

Theorem 3.4 *Let $\langle N', v \rangle$ and $\langle N, r \rangle$ be the auction game and the ring game arising from the same auction situation $\langle w_1, w_2, \dots, w_n; w_{n+1} \rangle$. Then the restriction of $\langle N', v^* \rangle$ to N is equal to $\langle N, r \rangle$; i.e., $v^*(S) = r(S)$ for each $S \in 2^N$.*

Proof. Take $S \in 2^N \setminus \{\emptyset\}$. Let $m(S) = \max\{i \in S | [1, i] \subseteq S\}$. Then

$$r(S) = \sum_{k=1}^n (w_k - w_{k+1}) u_{[1, k]}(S) = \sum_{k=1}^{m(S)} (w_k - w_{k+1}) = w_1 - w_{m(S)+1}.$$

Note that $m(S) + 1 \notin S$ and this is the player with lowest index with this property. So, $m(S) + 1$ is in $N' \setminus S$ the player with highest value. Hence, $v(N' \setminus S) = w_{m(S)+1}$. This implies that $v^*(S) = v(N') - v(N' \setminus S) = w_1 - w_{m(S)+1} = r(S)$. ■

The following example illustrates our results on ring games.

Example 3.5 Consider the market situation given in Example 2.5. The corresponding ring game $\langle \{1, 2, 3\}, r \rangle$ is the following

$$\begin{aligned} r(\{1, 2, 3\}) &= 210 \\ r(\{1, 2\}) &= 150 \\ r(\{1\}) &= r(\{1, 3\}) = 120 \\ r(S) &= 0 \text{ otherwise.} \end{aligned}$$

Then $r = 120 \cdot u_{\{1\}} + 30 \cdot u_{\{1,2\}} + 60 \cdot u_{\{1,2,3\}}$. We obtain

$$\Phi(r) = AL(r) = (155, 35, 20),$$

$$C(r) = \{x \in \mathbb{R}_+^n \mid x_1 \geq 120, x_3 \leq 60, x_1 + x_2 + x_3 = 210\},$$

and

$$\tau(r) = \left(\frac{615}{4}, \frac{135}{4}, \frac{90}{4} \right) \in C(r).$$

Finally, the dual game $\langle \{1, 2, 3, 4\}, v^* \rangle$ of the auction game of Example 2.5 is given by

$$\begin{aligned} v^*(\{4\}) &= v^*(\{1, 4\}) = v^*(\{2, 4\}) = v^*(\{3, 4\}) = 210 \\ v^*(\{1, 2, 3\}) &= v^*(\{1, 2, 4\}) = v^*(\{1, 3, 4\}) = v^*(\{2, 3, 4\}) = 210 \\ v^*(\{1, 2, 3, 4\}) &= 210 \\ v^*(\{1, 2\}) &= 150 \\ v^*(\{1, 3\}) &= v^*(\{1\}) = 120 \\ v^*(S) &= 0 \text{ otherwise.} \end{aligned}$$

Notice that $v^*(S) = r(S)$ for each $S \in 2^N$.

4 Concluding remarks

We have seen that each one-object auction situation with complete information leads for the four classical auction regimes to the same auction game and the same ring game (GET), which are in a duality relation. The auction games coincide with total big boss games with a core of dimension zero or one and the ring games turn out to be peer group games (and conversely). Interesting characterizations of the Shapley value and the τ -value of ring games are discussed in terms of the valuations.

Our further research will concentrate on cooperative games related to multi-object auction situations. Also cooperative game theory has to be developed for two-corner market situations with incomplete knowledge of the valuations.

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