



No. 2002-22

AUCTIONS WITH FINANCIAL EXTERNALITIES

By Emiel Maasland and Sander Onderstal

March 2002

ISSN 0924-7815

Discussion paper

Auctions with Financial Externalities[♠]

Emiel Maasland[♣] and Sander Onderstal[♢]

March 18, 2002

Abstract

We study sealed-bid auctions with financial externalities, i.e., auctions in which losers' utilities depend on how much the winner pays. In the unique symmetric equilibrium of the first-price sealed-bid auction (FPSB), larger financial externalities result in lower bids and in a lower expected revenue. The unique symmetric equilibrium of the second-price sealed-bid auction (SPSB) reveals ambiguous effects. We further show that a resale market does not have an effect on the equilibrium bids and that FPSB yields a lower expected revenue than SPSB. With a reserve price, we find an equilibrium for FPSB that involves pooling at the reserve price. For SPSB we derive a necessary and sufficient condition for the existence of a weakly separating equilibrium, and give an expression for the equilibrium.

Keywords: Auctions, financial externalities, reserve price, resale market.

JEL classification: D44

1 Introduction

In this paper, we study sealed-bid auctions with financial externalities. Financial externalities arise when losers benefit directly or indirectly from a high price paid by the winner(s). In auction theory, it is generally assumed that losers are indifferent about how much the winner(s) pay(s) in an auction. However, in real life auctions, this assumption may be false. In reality, an auction is not an isolated game, as winners and losers also interact after the auction. Paying a high price in the auction may make a winner a weaker competitor later.

[♠]For valuable discussions and comments we like to thank Eric van Damme and Jacob Goeree, and seminar participants at Tilburg University and the 2001 Econometric Society European Meeting.

[♣]SEOR, Erasmus University, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands, +31-10-408-1513, fax +31-10-408-9220, emaasland@few.eur.nl. The main part of this work was conducted while Maasland was affiliated with CentER, Tilburg University.

[♢]CPB Netherlands Bureau for Economic Policy Analysis, P.O. Box 80510, 2508 GM Den Haag, The Netherlands, +31-70-338-6116, fax +31-70-338-3350, A.M.Onderstal@cpb.nl, <http://www.cpb.nl/eng/general/org/homepages/amo/>. Onderstal's work was supported by NWO grant 510 010 501. The main part of this work was conducted while Onderstal was affiliated with CentER, Tilburg University.

The series of UMTS auctions that took place in Europe offers a concrete example of auctions where losers benefit indirectly from a high price paid by the winners. In this context, there are at least three ways how firms that do not acquire a license may benefit from a winning firm paying a high price. First, the share values of winning firms may drop, which makes the winner vulnerable to a hostile take-over by competing firms. For instance, the drop of the share value of the Dutch telecom company KPN with about 95% is partly explained by the huge amount of money the company spent to acquire British, Dutch and German UMTS licences.^{1,2} Second, if firms are budget constrained, a high payment in the first auction may give competing firms an advantage in the later auctions. Third, high payments may force the winning firms to cut their budget for investment, which may be favorable for the losers' position in the telecommunications market, as the losing firms are not only competitors of the winning firms in the auction, but in the telecommunications market as well. Börger and Dustmann (2001) argue that financial externalities may have led to seemingly irrational bidding in the British UMTS auction.

Financial externalities occur directly when losing bidders get money from the winner(s). For instance, this may happen in the case of bidding rings, in which a member of the ring receives money when she does not win the object (McAfee and McMillan, 1992). Also, partnerships are dissolved using an auction in which losing partners obtain part of the winner's bid (Cramton et al., 1987). Finally, the owner of a large estate may specify in his last will that after his death, the estate is sold to one of the heirs in an auction, where the auction revenue is divided among the losers (Engelbrecht-Wiggans, 1994).³

In Section 2, we present a model of bidding in sealed-bid auctions with financial externalities. The first-price sealed-bid auction (FPSB) or the second-price sealed-bid auction (SPSB) is used to sell an indivisible object. We assume an independent private signals model, with private values and common value models as special cases. Financial externalities are exogenously given and modelled by a parameter λ that is inserted in the bidders' utility functions. This is the simplest extension of the independent private signals model which incorporates financial externalities. Despite its admitted simplicity, this model appears to be sufficiently rich to generate interesting insights.

In Section 3, we derive results for FPSB and SPSB without reserve price. We find a unique symmetric and efficient bid equilibrium for each of the two auction types. Equilibrium bids in FPSB decrease as λ increases. An intuition for this result is that larger financial externalities make losing more attractive for the bidders so that they submit lower bids. The effect of financial externalities on the equilibrium bids in SPSB is ambiguous. A possible explanation is that in SPSB, a bidder is not only inclined to bid less the higher is λ (as she gets positive utility from losing), she also has an incentive to bid higher, because, given that she loses, she is able to influence directly the level of payments made

¹ In the UK, KPN bought part of the TIW license after the auction. In Germany, KPN has a majority share in E-plus.

² The other part of the drop is probably explained by the changed sentiment in the market.

³ More examples can be found in Goeree and Turner, 2001.

by the winner. We also study the effect of a resale market. Haile (1999) shows in an independent private values model that the efficient equilibria of FPSB and SPSB remain unaffected when the auction is followed by a resale market. We show that this result still holds in our model, and that it extends to any auction which leads to an efficient assignment of the object. Finally, we give a revenue comparison between FPSB and SPSB. We find that in the two-bidder case, SPSB results in a higher expected revenue than FPSB.

In Section 4, we characterize equilibrium bid strategies for the case that a reserve price is imposed in FPSB and SPSB. For simplicity, we assume independent private values and two bidders. In this section, we introduce the concept of a weakly separating Bayesian Nash equilibrium, which is an equilibrium in which all types below a threshold type abstain from bidding, and all types above this type submit a bid according to a strictly increasing bid function. We find that FPSB has no weakly separating Bayesian Nash equilibrium. However, we derive an equilibrium in which bidders with low signals abstain from bidding, bidders with intermediate signals pool at the reserve price, and bidders with high signals submit a bid according to a strictly increasing bid function. For SPSB, we derive a necessary and sufficient condition for the existence of a weakly separating Bayesian Nash equilibrium. For low values of the reserve price, such an equilibrium exists, for high values it does not. If a weakly separating Bayesian Nash equilibrium exists, then all types above the threshold type submit the same bid as in the case of no reserve price.

A closely related paper is Engelbrecht-Wiggans (1994), who considers an auction game in which each bidder receives an equal share θ of the revenue. He characterizes equilibrium bid functions for both FPSB and SPSB, and gives a revenue comparison between these two auction types.⁴ It is straightforwardly checked that his model is isomorphic to our model. Therefore, the equilibrium bids in our model can directly be derived from the equilibrium bids in Engelbrecht-Wiggans (1994). However, the comparative statics in our model and Engelbrecht-Wiggans' model (the effect of θ respectively θ on the equilibrium bids and the seller's revenue) turn out to be different. Engelbrecht-Wiggans shows that the equilibrium bid functions of FPSB and SPSB are increasing in θ . In our model, the effect of θ on the equilibrium bids can be both increasing and decreasing: We add to Engelbrecht-Wiggans' analysis that, if attention is restricted to symmetric Bayesian Nash equilibria, these equilibrium bid functions are unique. Also, in addition to Engelbrecht-Wiggans' study, we analyze the effect of a resale market, and of a reserve price on the equilibrium bids.

There are several other papers related to ours. Our companion paper (Maasland

⁴Several other papers make use of Engelbrecht-Wiggans' model. Ettinger (2000) extends the model by allowing the revenue shares to differ among the bidders and by introducing reserve prices. Engers and McManus (2000) study charity auctions, in which bidders receive a warm glow from the auction revenue, so that their utility depends on the auction revenue. Goeree and Turner (2002) compare standard auctions with k-th price all-pay auctions in Engelbrecht-Wiggans' environment. Simultaneously and independently of us, Engers and McManus, and Goeree and Turner derive similar results as Engelbrecht-Wiggans and we with respect to equilibrium bidding in FPSB and SPSB, and the revenue comparison among these two auction types.

and Onderstal, 2002) focuses on optimal auction design in the context of financial externalities. In that paper, we show that in a Double Coasean World, in which the seller cannot prevent a perfect resale market, nor withhold the object, the lowest-price all-pay auction is optimal.⁵ Moreover, in a Myersonean World, in which the seller can both prevent resale after the auction and fully commit to not selling the object, we find a two-stage mechanism that is revenue maximizing. In the first stage of this mechanism, bidders are asked whether they accept to pay an entry fee. If and only if all choose to accept, then in the second stage, bidders play the lowest-price all-pay auction with a reserve price.

Jehiel and Moldovanu (1996, 2000), and Jehiel et al. (1996, 1999) study auctions in which losing bidders receive positive or negative allocative externalities from the winner. Since the utility of the bidders is affected by the identity of the winner and not by how much she pays, these externalities are clearly different from financial externalities. Jehiel and Moldovanu (2000) derive equilibrium bid strategies that involve some pooling at the reserve price for SPSB with a reserve price and positive externalities. This equilibrium structure is similar to the one we found in FPSB.

Benoit and Krishna (2001) study a two-bidder model with complete information in which two objects are sold sequentially. As bidders are budget constrained, a particular bidder's payoff is affected by the price paid by a rival bidder, so that their model can be interpreted as a model with endogenously determined financial externalities.

2 The model

We consider a situation with $n \geq 2$ risk neutral bidders, numbered $1; 2; \dots; n$, who bid for one indivisible object. The auction being used is either FPSB or SPSB. Each of these auction types may or may not have a reserve price.

Essentially, we use Milgrom and Weber's (1982) model with independent signals instead of affiliated signals as a starting point. We assume that each bidder i receives a one-dimensional private signal t_i which is drawn, independently from all the other signals, from a cumulative distribution function F . (We also say that bidder i is of type t_i .) F has support on an interval $[t; \bar{t}]$, and continuous density f with $f(t_i) > 0$ for every $t_i \in [t; \bar{t}]$.

We will let $v_i(t)$ denote the value of the object for bidder i given the vector $t = (t_1; \dots; t_n)$ of all signals. Special cases are private value models ($v_i(t)$ only depends on t_i), and common value models ($v_i(t) = v_j(t)$ for all $i, j; t$).

We make the following assumptions on the functions v_i .

Value Differentiability: v_i is differentiable in all its arguments, for all $i; t$:

Value Monotonicity: $v_i(t) \geq 0$; $\frac{\partial v_i(t)}{\partial t_i} > 0$; and $\frac{\partial v_i(t)}{\partial t_j} \leq 0$, for all $i, j; t$:

⁵ In this auction, the bidder that submits the highest bid wins the object, and every bidder pays the lowest submitted bid.

Symmetry: $F_i = F_j$ for all i, j ; and $v_i(t_i, t_j) = v_j(t_j, t_i)$ for all t_i, t_j .

Value Differentiability is imposed to make the calculations on the equilibria tractable. Value Monotonicity indicates that all bidders are serious, and that bidders' values are strictly increasing in their own signal, and weakly in the signals of the others. Symmetry may be crucial for the existence of efficient equilibria in standard auctions.⁶ Value Differentiability, Value Monotonicity, and Symmetry together ensure that the bidder with the highest signal is also the bidder with the highest value, so that these assumptions imply that the seller assigns the object efficiently if and only if the bidder with the highest signal gets it.

We define $F^{[1]}$ and $f^{[1]}$ as the cumulative distribution function and density function respectively of $\max_{j \in i} t_j$. Also, let us define $v(x; y)$ as the expected value that bidder i assigns to the object, given that her signal is x , and that the highest signal of all the other bidders is equal to y :

$$v(x; y) = E[v_i(t_i) | t_i = x; \max_{j \in i} t_j = y].$$

By Symmetry, $F^{[1]}$, $f^{[1]}$, and v do not depend on i .

The bidders are expected utility maximizers. Each bidder is risk neutral, and cares about what other bidders pay in the auction. More specifically, the utility function of bidder i is defined as follows:

$$u_i(j; b) = \begin{cases} \frac{1}{2} v_i(j; b) & \text{if } j = i \\ \frac{1}{2} b & \text{if } j \neq i, \end{cases}$$

where v_i is the value that i attaches to the object, j is the winner of the object and b is the payment by j . It is a natural assumption to let a bidder's interest in her own payments be larger than her interest in the payments by the other bidders, so that we assume $\frac{1}{2} \leq \frac{1}{n+1}$.

A specific interpretation of the model is a situation of an auction in which all losing bidders receive an equal share of the auction revenue. In particular, when $\frac{1}{2} = \frac{1}{n+1}$, the entire auction revenue is divided among all losing bidders, which may be the case in situations of dissolving partnerships, or heirs bidding for a family estate. If $n = 2$ and $\frac{1}{2} = 1$, then FPSB and SPSB are special cases of the k -double auction with $k = 0$ and $k = 1$ respectively.^{7,8}

⁶Bulow et al. (1999) show that a slight asymmetry in value functions may have dramatic effects on bidding behavior in the English auction in a common value setting, as the bidder with the lower value function faces a strong winner's curse, and therefore bids zero in equilibrium.

⁷The k -double auction has the following rules. Both bidders submit a bid. The highest bidder wins the object, and pays the loser an amount equal to $kb_L + (1-k)b_W$, where b_L is the loser's bid, b_W the winner's bid, and $k \in [0, 1]$.

⁸Cramton et al. (1987) study k -double auctions in a private values environment with symmetric value distributions. It is shown that partners with equal shares may dissolve a partnership efficiently using these auctions. McAfee and McMillan (1992) show that the 0-double auction is a mechanism that allows a bidding ring to allocate the obtained object

3 Zero reserve price

Consider FPSB and SPSB with a zero reserve price.

3.1 First-price sealed-bid auction

The following proposition characterizes the equilibrium bid function for FPSB. To derive equilibrium bidding, we suppose that in equilibrium, all bidders use the same bid function. By a standard argument, this bid function must be strictly increasing and continuous. Let $U(t; s)$ be the utility for a bidder with signal t who behaves as if having signal s , whereas the other bidders play according to the equilibrium bid function. A necessary equilibrium condition is that

$$\frac{\partial U(t; s)}{\partial s} = 0$$

at $s = t$. From this condition, a differential equation can be derived, from which the equilibrium bid function is uniquely determined (at least if we restrict our attention to differentiable bid functions). The auction outcome is efficient. Observe that in the case of private values ($v(x; y)$ only depends on x), the bid function is strictly increasing in n .

Proposition 1 The unique symmetric differentiable Bayesian Nash equilibrium of FPSB is characterized by

$$B_1(\cdot; t) = v(t; t) - \frac{1}{1 + \frac{1}{n}} v(t; t) - \frac{1}{1 + \frac{1}{n}} \int_t^t \frac{dv(y; y)}{dy} \frac{F^{[1]}(y)}{F^{[1]}(t)}^{n-1} dy, \quad (1)$$

where $B_1(\cdot; t)$ is the bid of a bidder with signal t . The outcome of this auction is efficient.

Proof. A higher type of a bidder cannot submit a lower bid than a lower type of the same bidder. (If the low type gets the same expected surplus from strategies with two different probabilities of being the winner of the object, the high type strictly prefers the strategy with the highest probability of winning, so the high type will not submit a lower bid than the low type.) Also, $B_1(\cdot; t)$ cannot be constant on an interval $[t^0; t^0]$. (By bidding slightly higher, a type t^0 can largely improve its probability of winning, while only marginally influencing the payments by her and the other bidders.) Moreover, $B_1(\cdot; t)$ cannot be discontinuous at any t . (Suppose that $B_1(\cdot; t)$ makes a jump from \underline{b} to \bar{b} at t^a . A type just above t^a has an incentive to deviate from $\bar{b} + \epsilon$ to \underline{b} . Doing so, she is able to decrease the auction price, while just slightly affecting its probability of winning the object. As ϵ is small enough, this type is able to improve its

efficiently among the ring members. Van Damme (1992) shows that k -double auctions may lead to unfair equilibrium outcomes. Angeles de Frutos (2000) and Kittsteiner (2001) generalize the model of Cramton et al. (1987) allowing for asymmetric value distributions and interdependent valuations respectively.

utility.) Hence, a symmetric equilibrium bid function must be strictly increasing and continuous.

Define the utility $U(t; s)$ for a bidder with signal t who misrepresents herself as having signal s , whereas the other bidders report truthfully, if the bid function is indeed strictly increasing. Then,

$$U(t; s) = \int_t^{z^s} v(t; y) dF^{[1]}(y) - F^{[1]}(s) B_1(' ; s) + \int_s^{z^t} B_1(' ; y) dF^{[1]}(y).$$

The first two terms of the RHS of this expression refer to the case that this bidder wins the object. The third term refers to the case that she does not win. Assume that $B_1(' ; s)$ is differentiable in s . Maximizing $U(t; s)$ with respect to s and equating s to t gives the FOC of the equilibrium

$$f^{[1]}(t)v(t; t) - f^{[1]}(t)B_1(' ; t) - F^{[1]}(t)B_1^0(' ; t) - B_1(' ; t)f^{[1]}(t) = 0.$$

With some manipulation we get

$$F^{[1]}(t)' f^{[1]}(t)v(t; t) = (1 + ')B_1(' ; t)f^{[1]}(t)F^{[1]}(t)' + B_1^0(' ; t)F^{[1]}(t)^{1+'}, \quad (2)$$

or, equivalently

$$C_1 + \int_t^{z^t} F^{[1]}(y)' f^{[1]}(y)v(y; y)dy = F^{[1]}(t)^{1+'} B_1(' ; t),$$

where C_1 is a constant. Substituting $t = \underline{t}$ gives $C_1 = 0$, so that the bid function is given by

$$B_1(' ; t) = \frac{1}{F^{[1]}(t)} \int_t^{z^t} \frac{F^{[1]}(y)}{F^{[1]}(t)} f^{[1]}(y)v(y; y)dy. \quad (3)$$

It is readily checked that the second order condition $\text{sign} \frac{\partial^2 U(t; s)}{\partial s^2} = \text{sign}(t - s)$ is fulfilled. Using integration by parts, (3) can be rewritten as (1).

From (2), we infer that $\frac{\partial B_1(' ; t)}{\partial t} > 0$ if and only if $B_1(' ; t) < \frac{v(t; t)}{1+'}$, so that indeed $B_1(' ; t)$ is strictly increasing in t ; as Value Monotonicity implies that $\frac{dv(y; y)}{dy} > 0$ for all y . Finally, by Value Differentiability, Value Monotonicity, and Symmetry, the efficiency of the auction outcome is established. ■

Each of the terms of the RHS of (1) has an attractive interpretation. The first term is the equilibrium bid for a bidder with type t in SPSB without financial externalities, as in the absence of financial externalities, in SPSB, a bidder will submit a bid equal to her maximal willingness to pay given that her strongest opponent has the same signal as she (Milgrom and Weber, 1982). The second

term can be interpreted as the bid shading because of financial externalities. The reason for bid shading is that in the case of financial externalities, the willingness to pay of a bidder with type t bidding against an opponent who has the same signal is given by $\frac{1}{1+\lambda}v(t; t)$. This can be seen as follows. When a bidder wins at a bid of b , her utility is $v(t; t) - b$. When her opponent wins at the same bid, her utility is $-b$. Equating these utilities results in a bid of $\frac{1}{1+\lambda}v(t; t)$. The third term can be interpreted as the strategic bid shading because in FPSB, a bidder has to pay her own bid rather than the second highest bid which she has to pay in SPSB.

This interpretation of the equilibrium bid function suggests that this function is decreasing in λ , which in fact holds, as Proposition 2 shows. From Proposition 2, it immediately follows that the expected revenue is decreasing in λ .

Proposition 2 Increasing λ decreases $B_1(\lambda; t)$:

Proof. The proof immediately follows from Proposition 1, since $F^{[1]}(y) < F^{[1]}(t)$ for every $y \geq t$. ■

Corollary 3 Increasing λ decreases the seller's expected revenue.

3.2 Second-price sealed-bid auction

Equilibrium bids for SPSB are obtained using the same logic as for FPSB. The analysis reveals, just as in situations without financial externalities, uniqueness and efficiency of the equilibrium bid function. Observe that in the case of private values, the bid function does not depend on n .⁹

Proposition 4 The unique symmetric differentiable Bayesian Nash equilibrium of SPSB is characterized by

$$B_2(\lambda; t) = v(t; t) - \frac{\lambda}{1+\lambda}v(t; t) + \frac{\lambda}{(1+\lambda)(1+2\lambda)} \int_t^{\bar{t}} \frac{dv(y; y)}{dy} \frac{1 - F(y)}{1 - F(t)}^{1+\lambda} dy \quad (4)$$

where $B_2(\lambda; t)$ is the bid of a bidder with signal t . The outcome of this auction is efficient.

Proof. Following the lines of the proof of Proposition 1 it can be established that a symmetric equilibrium function must be strictly increasing and continuous. The utility for a bidder with signal t acting as if she had signal s is given

⁹This is actually a quite subtle observation, as n does not appear in the expression for the equilibrium bid. However, in general, $v(t; t)$ depends on n .

by

$$U(t; s) = \int_t^{z^s} [v(t; y) - B_2(' ; y)] dF^{[1]}(y) + ' \mathbb{1}(s) B_2(' ; s) + \int_{y=s}^{z^{\bar{t}}} B_2(' ; y) d\mathbb{1}(y),$$

where $\mathbb{1}(s)$ denotes the probability that there is exactly one opponent with a signal larger than s . The first term of the RHS refers to the case that this bidder wins, the second term to the case that she submits the second highest bid, and the third case to her bid being the third or higher. Assume that $B_2(' ; s)$ is differentiable in s . The FOC of the equilibrium is

$$[v(t; t) - B_2(' ; t)] f^{[1]}(t) + ' \frac{\partial \mathbb{1}(t) B_2(' ; t)}{\partial t} - B_2(' ; t) \mathbb{1}'(t) = 0$$

or, equivalently

$$v(t; t) f^{[1]}(t) = B_2^0(' ; t) \mathbb{1}(t) + B_2(' ; t) [(1 + ') f^{[1]}(t)]. \quad (5)$$

The general solution to the above differential equation is equal to

$$B_2(' ; t) (1 - F(t))^{1+'} = C_2 \int_t^{z^t} (1 - F(y))^{-1} v(y; y) f(y) dy;$$

where C_2 is a constant. Substituting $t = \bar{t}$ yields a unique solution for C_2 :

$$C_2 = \int_t^{z^{\bar{t}}} (1 - F(y))^{-1} v(y; y) f(y) dy.$$

The only possible differentiable bid function that may constitute a symmetric equilibrium is given by

$$B_2(' ; t) = \frac{1}{(1 - F(t))^{1+'}} \int_t^{z^{\bar{t}}} \frac{1 - F(y)}{1 - F(t)} f(y) v(y; y) dy. \quad (6)$$

It is readily checked that the second order condition $\text{sign} \frac{\partial^2 U(t; s)}{\partial s^2} = \text{sign}(t - s)$ holds. Using integration by parts on $B_2(' ; t)$, we see that (6) can also be written as (4).

To complete the proof, we must show that $B_2(' ; t)$ is indeed increasing in t . From (6), it follows that

$$B_2(' ; t) > \frac{v(t; t) \int_t^{z^{\bar{t}}} (1 - F(y))^{-1} f(y) dy}{(1 - F(t))^{1+'}} = \frac{v(t; t)}{1 + '};$$

As (5) implies that $B_2^0(\beta; t) > 0$ if and only if $B_2(\beta; t) > \frac{v(t;t)}{1+\beta}$, $B_2(\beta; t)$ is indeed strictly increasing in t . Then, by Value Differentiability, Value Monotonicity, and Symmetry, it follows that the outcome of the auction is efficient. ■

Each term of the RHS of (4) has its attractive interpretation. From the discussion of FPSB it follows that the first term is the bid in SPSB in the absence of financial externalities. The second term is the bid shading due to positive externalities from the payment of the winning bidder. The third term increases the bid due to the fact that each bidder is willing to drive up the final price, as it is the second highest bid that is paid by the winner.

In contrast to FPSB, the effect of an increase in β on the equilibrium bids in SPSB is dependent on a bidder's type. From (4), it is clear that the equilibrium bid of the highest type is decreasing in β . The reason is that as this bidder does not have a type above her, she does not have an incentive to drive up the price. However, the effect of β on the equilibrium bids of the other types is not clear. The effect of the second term of the RHS of (4) (without the minus sign) may be larger as well as smaller than the third term. The following example illustrates how equilibrium bidding is affected when β is varied.

Example 5 (Effect of β on equilibrium bidding) Let $F(t) = t$ (uniform distribution), $v(t; t) = t$ (independent private values) for all $t \in [0; 1]$. The equilibrium bid function is given by

$$B_2(\beta; t) = \frac{\beta}{(1+\beta)(1+2\beta)} + \frac{1}{1+2\beta}t, t \in [0; 1].$$

As B_2 is a continuous function in both β and t , the following can be derived. First, there is a strictly positive mass of types close to zero for which the effect of β is ambiguous in the sense that for β close to 0, an increase in β leads to higher bids and for β close to 1, an increase in β leads to lower bids. This follows from the following observations.

$$\frac{\partial B_2(0; 0)}{\partial \beta} = 1 > 0,$$

and

$$\frac{\partial B_2(1; 0)}{\partial \beta} = -\frac{1}{36} < 0.$$

Intuitively, if β is large enough, $B_2(\beta; t)$ decreases as for each bidder, losing becomes more interesting due to higher financial externalities. Second, the equilibrium bids of types close to 1 are decreasing in β . This follows from the fact that $B_2(\beta; 1) = \frac{1}{1+\beta}$.

Also, the effect of β on the expected revenue may be ambiguous. This follows from Example 6, in which the expected revenue is increasing if β is small, and decreasing if β is large.

Example 6 (Effect of α on the expected revenue) Let $F(t) = t$ (uniform distribution), $v(t; t) = t$ (independent private values) with $t \in [0; 1]$ and $n = 2$ (two bidders): The expected revenue is equal to the expectation of $B_2(\alpha; t^{(2)})$ with respect to the second highest signal $t^{(2)}$, which is given by

$$E_{t^{(2)}} B_2(\alpha; t^{(2)}) = \frac{1 + 4\alpha}{3(1 + \alpha)(1 + 2\alpha)}.$$

This continuous function is increasing for α close to 0 and decreasing for α close to 1, as

$$\frac{\partial E_{t^{(2)}} B_2(0; t^{(2)})}{\partial \alpha} = \frac{1}{3} > 0,$$

and

$$\frac{\partial E_{t^{(2)}} B_2(1; t^{(2)})}{\partial \alpha} = -\frac{11}{108} < 0.$$

3.3 Resale market

The presence of a resale market does not have any effect on equilibrium behavior. In order to obtain this result, the following assumptions are made. First, trade is voluntary. None of the bidders can be forced to be involved in an exchange if she is made worse off by it. Thus, trade only takes place if it is mutual profitable for the bidders. Second, the participants in the resale market are the same as in the auction. There are no third parties involved.

We assume the following conditions for trade to occur in the resale market. Let bidder i be the winner of the object in the auction, and bidder j be another bidder, who desires to buy the object from bidder i in the resale market. Let p be the price of the object in the resale market. As trade is voluntary, none of the bidders may be worse off by the trade. For bidder i , the following condition for trade must be fulfilled:

$$p + \alpha p \geq v_i \quad (7)$$

In words, bidder i prefers receiving a price of p , which also yields her a financial externality of αp , to keeping the object, which gives her a value of v_i . For bidder j a similar condition holds:

$$v_j \geq p + \alpha p \geq 0 \quad (8)$$

which is equivalent to

$$p + \alpha p \leq v_j \quad (9)$$

Note that, αp in (8) is a correction factor. This can be seen as follows. Without trade, the utility of bidder j is αp , where p is the price paid by bidder i in

the auction. With trade in the resale market, bidder i has paid $p_i - p$. This would give bidder j a utility increase of $v_j - (p_i - p)$ due to financial externalities. Therefore, bidder j loses an extra $v_j - p$, if she decides to buy the object in the resale market. Observe that, (7) and (9) exclude inefficient trade (trade from a bidder with a high value to one with a low value). Moreover, for both bidders the maximal gains from trade are $v_j - v_i$.

Proposition 7 shows that equilibrium bidding is not affected by the presence of a resale market if the equilibrium of the auction without resale market leads to an efficient outcome. We prove this proposition by assuming that all bidders, apart from bidder i , bid "as usual", i.e., they bid in the auction as if there were no resale market. Then we calculate bidder i 's utility both for the case when she submits a lower bid than "usual", and for the case that she submits a higher bid. In both cases, we separately calculate bidder i 's utility from the auction, and the maximal utility she can obtain in the resale market, which is the difference between her value, and the highest value among the other bidders. Adding these, we show that bidder i has no incentive to deviate from bidding "as usual".

Proposition 7 A Bayesian Nash equilibrium of an auction (without resale market) which leads to an efficient assignment of the object, is also a Bayesian Nash equilibrium when the same auction is followed by a resale market where the same bidders participate. In equilibrium, no trade will take place in the resale market.

Proof. To prove that "bidding as usual" is still an equilibrium, suppose that all bidders, apart from bidder i , bid as usual. Then it should be a best response for bidder i to bid as usual as well. Let $U(t; s)$ be the expected surplus for bidder i from the auction plus the resale market, when she has signal t , but behaves as if she has type s .

Suppose for the moment that all bidders play the efficient Bayesian Nash equilibrium of the auction game without resale market. Then, by Value Differentiability, Value Monotonicity, and Symmetry, the bidder with the highest signal wins the auction. Moreover, by the assumption of voluntary trade (which exclude inefficient trade), no trade will take place after the auction. From the Revenue Equivalence Theorem,¹⁰ $U(t; t)$ is given by

$$U(t; t) = U(\underline{t}; \underline{t}) + \int_{\underline{t}}^t \int_{\underline{t}}^x \frac{\partial v(x; y)}{\partial x} dF^{[1]}(y) dx.$$

When we change the order of integration and integrate the inner integral we get

$$U(t; t) = U(\underline{t}; \underline{t}) + \int_{\underline{t}}^t [v(t; y) - v(y; y)] dF^{[1]}(y): \quad (10)$$

¹⁰ See Maasland and Onderstal (2002) for this result in the context of financial externalities.

The utility $\Theta(t; s)$ from the auction alone for bidder i who has type t , but represents herself as if she has type s is given by

$$\Theta(t; s) = \Theta(s; s) + \int_t^s [v(t; y) - v(s; y)] dF^{[1]}(y). \quad (11)$$

Suppose that bidder i misrepresents herself as having a signal $s > t$. Trade will only take place when bidder i wins the auction, and there is another bidder j who has a higher valuation for the object. The gains from trade for bidder i from the resale market are at most the absolute difference between her value and the value of bidder j , which is, by Value Differentiability, Value Monotonicity, and Symmetry, the bidder with the highest signal. Let y be bidder j 's signal, then bidder j 's value is at most $v(y; y)$. Bidder i 's value is given by $v(t; y)$. Then, with (10) and (11),

$$\begin{aligned} U(t; s) - U(t; t) &= \Theta(s; s) + \int_t^s [v(t; y) - v(s; y)] dF^{[1]}(y) + \\ &\quad + \int_t^s [v(y; y) - v(t; y)] dF^{[1]}(y) - U(t; t) \\ &= \int_t^s [v(s; y) - v(y; y)] dF^{[1]}(y) - \int_t^s [v(t; y) - v(y; y)] dF^{[1]}(y) \\ &\quad + \int_t^s [v(t; y) - v(s; y)] dF^{[1]}(y) + \int_t^s [v(y; y) - v(t; y)] dF^{[1]}(y) \\ &= 0. \end{aligned}$$

So, bidder i cannot gain from deviating to a higher signal.

Suppose instead that bidder i deviates to a lower signal. Then, similarly,

$$\begin{aligned}
 U(t; s) - U(t; t) &= \int_t^{z^s} \theta(s; s) + [v(t; y) - v(s; y)]dF^{[1]}(y) + \\
 &\quad + \int_s^{z^t} [v(t; y) - v(y; t)]dF^{[1]}(y) - U(t; t) \\
 &= \int_t^{z^s} [v(s; y) - v(y; y)]dF^{[1]}(y) - \int_t^{z^t} [v(t; y) - v(y; y)]dF^{[1]}(y) \\
 &\quad + \int_t^{z^s} [v(t; y) - v(s; y)]dF^{[1]}(y) + \int_s^{z^t} [v(t; y) - v(y; y)]dF^{[1]}(y) \\
 &= 0.
 \end{aligned}$$

Hence a deviation to a lower type is not profitable. So, indeed it is a best response for bidder i to bid as usual.

As the equilibrium of the auction is efficient, it is always the bidder with the highest value who obtains the object after the auction. As inefficient trade is excluded in the resale market, no trade will take place there. ■

A corollary of the above result is that the equilibrium bids in FPSB and SPSB do not change when resale market opportunities are introduced. This immediately follows from the fact that both auctions have efficient equilibria, as was shown in Propositions 1 and 4. Moreover, no trade will take place in the resale market.

3.4 Revenue comparison for $n = 2$

For the tractable case of two bidders, if $0 < \lambda < 1$, SPSB generates a strictly higher expected revenue than FPSB.¹¹ This revenue ranking result is obtained by proving that the utility of the lowest type is strictly higher for FPSB than for SPSB. This short-cut immediately follows from the Revenue Equivalence Theorem (Myerson, 1981) which remains valid in case financial externalities are introduced (Maasland and Onderstal, 2002). According to the Revenue Equivalence Theorem, two auctions which are both efficient, and yield zero utility for the lowest type, yield the same expected revenue. For $\lambda = 1$, both auctions are revenue equivalent, which follows as the utility of the lowest type is the same for both auctions.

To obtain the proof of Proposition 9, the following lemma appears to be useful.

¹¹ Engelbrecht-Wiggans (1994) claims the same result for n bidders, but his proof is not correct, even not for the case of two bidders.

Lemma 8 For every $y \in (0; 1)$ and $\alpha \in (0; 1)$, the following inequality is satisfied:

$$y + \frac{\alpha}{1+\alpha}(1-y)^{\frac{1+\alpha}{\alpha}} - \frac{1}{1+\alpha}y^{1+\alpha} - \frac{\alpha}{1+\alpha} < 0;$$

If $\alpha = 1$, then for every $y \in (0; 1)$,

$$y + \frac{\alpha}{1+\alpha}(1-y)^{\frac{1+\alpha}{\alpha}} - \frac{1}{1+\alpha}y^{1+\alpha} - \frac{\alpha}{1+\alpha} = 0;$$

Proof. See the Appendix. ■

Proposition 9 For $\alpha < 1$ and $n = 2$, SPSB generates a strictly higher expected revenue than FPSB. For $\alpha = 1$ and $n = 2$, FPSB and SPSB are revenue equivalent.

Proof. Let $U_1(t)$ and $U_2(t)$ be the equilibrium utility of the lowest type in FPSB and SPSB respectively. As the outcome of both auctions is efficient, a bidder with type t loses the auction with probability 1, and gets financial externalities as the other bidder has to pay. So, $U_1(t)$ and $U_2(t)$ are respectively given by

$$U_1(t) = \int_t^{\bar{t}} B_1(\alpha; t) dF(t)$$

and

$$U_2(t) = \int_t^{\bar{t}} B_2(\alpha; t) dF(t).$$

Applying integration by parts twice on the expression for $\int_t^{\bar{t}} B_1(\alpha; t) dF(t)$, we obtain

$$\begin{aligned} \int_t^{\bar{t}} B_1(\alpha; t) dF(t) &= \int_t^{\bar{t}} \frac{1}{F(t)} \int_t^{\bar{t}} \frac{\mu}{F(y)} f(y) v(y; y) dy dF(t) \\ &= \int_t^{\bar{t}} (1 - F(t)^\alpha) v(t; t) dF(t) \\ &= \frac{\alpha}{1+\alpha} v(t; t) - \int_t^{\bar{t}} F(t)^\alpha \frac{1}{1+\alpha} F(t)^{1+\alpha} \frac{dv(t; t)}{dt} dt. \end{aligned}$$

Manipulating $B_2(\cdot; t)$, we find

$$\begin{aligned} B_2(\cdot; t) &= \int_t^{\bar{t}} (1 - F(t))^{1+\gamma} f(t) v(t; t) dt \\ &= \frac{1}{1+\gamma} v(t; t) + \frac{1}{1+\gamma} \int_t^{\bar{t}} (1 - F(t))^{1+\gamma} \frac{dv(t; t)}{dt} dt. \end{aligned}$$

Then

$$\begin{aligned} U_1(t) - U_2(t) &= \int_t^{\bar{t}} B_1(\cdot; t) dF(t) - B_2(\cdot; t) \\ &= \int_t^{\bar{t}} \frac{F(t)^{1+\gamma}}{(1+\gamma)} - F(t) - \frac{1}{1+\gamma} (1 - F(t))^{1+\gamma} + \frac{1}{1+\gamma} \frac{dv(t; t)}{dt} dt. \end{aligned}$$

When we apply Lemma 8 with $y = F(t)$ to the difference between $U_1(t)$ and $U_2(t)$, we find for $\gamma \in (0, 1)$ that the utility of the lowest type is strictly higher for FPSB than for SPSB. For $\gamma = 1$, by Lemma 8, $U_1(t) - U_2(t) = 0$. ■

4 Positive reserve price

Consider FPSB and SPSB with a reserve price $R > 0$. In order to keep the model tractable, we assume that the standard independent private values model holds, i.e., $v_i(t) = t_i$ for all i, t . Also, we restrict our attention to the case of two bidders.

This section mainly focuses on the existence of weakly separating Bayesian Nash equilibria, for which the following definition applies.

Definition 10 A weakly separating Bayesian Nash equilibrium is a Bayesian Nash equilibrium in which all types below a threshold type abstain from bidding, and all types above this type submit a bid according to a strictly increasing bid function.

4.1 First-price sealed-bid auction

In contrast to a situation without financial externalities, there exists no weakly separating Bayesian Nash equilibrium for FPSB. Proposition 11 shows that, if such an equilibrium would exist, R must be the threshold type. The equilibrium bid function can be constructed analogous to the equilibrium bid function for FPSB without reserve price. But then a contradiction is established, as a bidder with type R turns out to submit a bid below the reserve price.

Proposition 11 Let $v_i(t) = t_i$ for all i , t , and $n = 2$. There exists no weakly separating Bayesian Nash equilibrium of FPSB if $R > 0$.

Proof. The proof is by contradiction. Suppose for the moment that a weakly separating equilibrium does exist. Then it is easily derived that all bidders with a type below R abstain from bidding, and all types above R submit a bid according to a strictly increasing bid function, which we denote by h . Using similar arguments as in the proof of Proposition 1, it can be established that $h'(t) \geq 0$ if and only if $h(t) \leq \frac{t}{1+\alpha}$. Hence, for $t = R$, it holds that $h(R) \leq \frac{R}{1+\alpha}$. In other words, in a weakly separating equilibrium, a bidder with type R submits a bid strictly below R . This contradicts the fact that all submitted bids should exceed the reserve price R . ■

However, there is a symmetric equilibrium that involves pooling at the reserve price. Proposition 12 describes a Bayesian Nash equilibrium in which bidders with a type below a threshold type L do not bid, bidders with a type t above a threshold type H bid $g^R(t)$, where g^R is a strictly increasing function, and types in the interval $[L; H]$ submit a bid equal to R . More specifically, let

$$H = (1 + \alpha)R;$$

L the unique solution to

$$\frac{\alpha [F(H) - F(L)]R}{F(H) + F(L)} = L - R;$$

and

$$g^R(t) = F^{[1]}(t)^{\frac{1}{1+\alpha}} \int_{(1+\alpha)R}^t F^{[1]}(y)^{\alpha} f^{[1]}(y) dy + F^{[1]}((1+\alpha)R)^{\frac{1}{1+\alpha}} R.$$

This is an equilibrium, as L turns out to be indifferent between abstaining from bidding, and submitting a bid equal to the reserve price, and H turns out to be indifferent between bidding R (and therefore pool with all types in the interval $[L; H]$), and bidding marginally higher than R , and g^R is derived from the same differential equation as the bid function for FPSB without reserve price.

Proposition 12 Assume independent private values and two bidders. Let $B_1^R(\cdot; t)$, the bid of a bidder with value t , be given by

$$B_1^R(\cdot; t) = \begin{cases} g^R(t) & \text{if } t > H \\ R & \text{if } L \leq t \leq H \\ \text{"no bid"} & \text{if } t < L. \end{cases}$$

Then $B_1^R(\cdot; t)$ constitutes a symmetric Bayesian Nash equilibrium of FPSB if $R > 0$.¹²

¹²Note that $B_1^R(\cdot; t)$ is continuous at H : This must be the case in equilibrium. Suppose, on the contrary, that the bid function has a jump at H . Then a bidder with a type slightly higher than H has an incentive to deviate from the bid strategy to a bid of just above R .

Proof. Assume that threshold types L and H exist such that in equilibrium all types $t < L$ abstain from bidding, all types $t \in [L; H]$ bid R , and all types $t > H$ bid according to a strictly increasing bid function g^R .

A type L is indifferent between not bidding and bidding R . The utility of abstaining from bidding is equal to

$$Z^{\bar{t}} = \int_H^L g^R(t) dF(t) + (F(H) - F(L))R.$$

The utility when bidding R is equal to

$$Z^{\bar{t}} = \int_H^L g^R(t) dF(t) + b d[F(H) - F(L)]f'(R) + (L - R)g + F(L)(L - R).$$

Equating both expressions yields

$$\frac{[F(H) - F(L)]R}{F(H) + F(L)} = L - R. \quad (12)$$

L is uniquely determined from (12) as the LHS of (12) is strictly decreasing in L and the RHS of (12) is strictly increasing in L for $L \geq 0$:

A type H is indifferent between bidding R and bidding an infinitesimal \pm above R . The utility when bidding R is equal to

$$Z^{\bar{t}} = \int_H^L g^R(t) dF(t) + b d[F(H) - F(L)]f'(R) + (H - R)g + F(L)(H - R):$$

The utility when bidding $R + \pm$ when \pm converges to 0 is equal to

$$Z^{\bar{t}} = \int_H^L g^R(t) dF(t) + [F(H) - F(L)](H - R) + F(L)(H - R):$$

Equating both expressions, and some manipulation yields

$$H = (1 + \beta)R.$$

In order to complete the proof, we need to check whether types have no incentive to deviate from the proposed equilibrium. We only check if a type $t > H$ has no incentive to mimic another type $t^0 > H$, as by a standard argument, other deviations are not profitable. Incentive compatibility of types $t > H$ implies that g^R should follow from the same differential equation as derived in the proof of Proposition 1 with the boundary condition $g^R(H) = R$. Analogous to the proof

of Proposition 1, it can be established that $g^R(t)$ is strictly increasing for $t \leq H$ if and only if $g^R(t) < \frac{t}{1+\alpha}$. Now, for $t > H$:

$$\begin{aligned} g^R(t) &= F^{[1]}(t)^{1+\alpha} \int_H^t F^{[1]}(y)^{-\alpha} f^{[1]}(y) dy + F^{[1]}(H)^{1+\alpha} R \\ &< F^{[1]}(t)^{1+\alpha} \int_H^t F^{[1]}(y)^{-\alpha} f^{[1]}(y) t dy + F^{[1]}(H)^{1+\alpha} R \\ &= \frac{t}{1+\alpha} \int_H^t \frac{t}{1+\alpha} F^{[1]}(y)^{-\alpha} f^{[1]}(y) dy + F^{[1]}(H)^{1+\alpha} R \\ &= \frac{t}{1+\alpha} \cdot \frac{t}{1+\alpha} R + \frac{F^{[1]}(H)^{1+\alpha}}{F^{[1]}(t)} R \\ &= \frac{t}{1+\alpha} R + \frac{F^{[1]}(H)^{1+\alpha}}{F^{[1]}(t)} R \\ &< \frac{t}{1+\alpha} R + \frac{t}{1+\alpha} R = \frac{t}{1+\alpha} R \end{aligned}$$

■

To get an intuition why pooling at R occurs in equilibrium, consider a situation in which $R \leq \frac{t}{1+\alpha}$. The threshold level H , above which bidders bid according to a strictly increasing bid function, lies above $\frac{t}{1+\alpha}$, so that bidders either abstain from bidding, or bid R . Why is this an equilibrium? Suppose that one of the two bidders submits a bid $b \leq R$. Then the other bidder prefers losing to winning. This can be seen as follows. If she loses, then her utility is

$$u(b) = \frac{b}{1+\alpha} \leq \frac{t}{1+\alpha},$$

whereas winning gives her a utility of at most

$$u(t) = \frac{t}{1+\alpha} \leq \frac{t}{1+\alpha} = \frac{t}{1+\alpha}.$$

Low types are then willing to lose the opportunity of getting the object by abstaining from bidding. High types bid R , assuring themselves the object if the other bidder does not bid, but also making sure that if the other bids, to lose as often as possible.

4.2 Second-price sealed-bid auction

In contrast to FPSB, SPSB sometimes has a (weakly) separating Bayesian Nash equilibrium when a reserve price is imposed. This observation follows trivially when the reserve price is smaller than the lowest submitted equilibrium bid, which is strictly positive according to Proposition 4. However, also in nontrivial cases weakly separating Bayesian Nash equilibria exist. Proposition 13 gives a necessary and sufficient condition for the existence of a weakly separating Bayesian Nash equilibrium. If the equilibrium exists, types up to a threshold type \bar{b} abstain from bidding, and types above \bar{b} submit the same bid as in the case of no reserve price.

Proposition 13 Assume independent private values and two bidders. Let $R \geq [B_2(\cdot; \underline{t}); B_2(\cdot; \bar{t})]$. SPSB with a reserve price R has a weakly separating Bayesian Nash equilibrium if and only if $B_2(\cdot; R) \leq R$. If an equilibrium exists, then it is given by

$$B_2^R(\cdot; t) = \begin{cases} \frac{1}{2} B_2(\cdot; t) & \text{if } t \geq \bar{b} \\ \text{"no bid"} & \text{if } t < \bar{b} \end{cases}$$

where $B_2^R(\cdot; t)$ is the bid of a bidder with value t , and where \bar{b} is the unique solution to

$$\bar{b} (B_2(\cdot; \bar{b}) - R) (1 - F(\bar{b})) = F(\bar{b}) [R - \bar{b}].$$

Proof. Suppose there is an R for which a weakly separating equilibrium exists. Suppose that an indifference type \bar{b} exists, such that

$$B_2^R(\cdot; t) = \begin{cases} \frac{1}{2} B_2(\cdot; t) & \text{if } t \geq \bar{b} \\ \text{"no bid"} & \text{if } t < \bar{b} \end{cases}$$

is an equilibrium, where B_2 is the equilibrium bid function in the case of $R = 0$. \bar{b} is indifferent between submitting no bid, and submitting a bid equal to $B_2(\cdot; \bar{b})$. Hence, \bar{b} follows from the following equation

$$(1 - F(\bar{b})) \bar{b} R = F(\bar{b}) (\bar{b} - R) + (1 - F(\bar{b})) \bar{b} B_2(\cdot; \bar{b}),$$

which is equivalent to

$$\bar{b} R = \frac{F(\bar{b})}{(1 - F(\bar{b}))} (\bar{b} - R) + \bar{b} B_2(\cdot; \bar{b}). \quad (13)$$

For $t \geq \bar{b}$, $B_2^R(\cdot; t)$ follows from the same differential equation as derived in the proof of Proposition 4 with the same boundary condition $B_2^R(\cdot; \bar{t}) = \frac{\bar{t}}{1 + \bar{t}}$, so that indeed $B_2^R(\cdot; t) = B_2(\cdot; t)$ for all R and $t \geq \bar{b}$.

A weakly separating equilibrium exists if and only if $B_2(\cdot; \bar{b}) \leq R$, as all bids should be above R . We will show now that $B_2(\cdot; \bar{b}) \leq R$ is equivalent to the condition $B_2(\cdot; R) \leq R$, which completes the proof.

Define \bar{e} such that

$$B_2(\cdot; \bar{e}) = R. \quad (14)$$

As $B_2(\cdot; t)$ is strictly increasing in t , \bar{e} is uniquely determined. Consider the function h with

$$h(t) = \frac{F(t)}{1 - F(t)} (t - R) + \bar{e} B_2(\cdot; t)$$

for all t . Note that h is a strictly increasing function, with

$$h(\bar{b}) = \bar{e} R,$$

(which follows from (13)), and

$$h(\theta) = \frac{F(\theta)}{1 - F(\theta)}(\theta - R) + R. \quad (15)$$

Now, with (14), as B_2 is strictly increasing,

$$B_2(\cdot; R) \geq R \iff B_2(\cdot; \theta) = R \iff B_2(\cdot; R) \geq \theta - R.$$

Moreover, with (15), as h is strictly increasing,

$$\theta - R \geq h(\theta) - R = h(\theta) - R.$$

Finally, as B_2 is strictly increasing, and from (14),

$$\theta - R \geq B_2(\cdot; \theta) \geq R.$$

■

An intuition for the condition $B_2(\cdot; R) \geq R$ being necessary is the following. In a weakly separating Bayesian Nash equilibrium, a bidder with type R is always prepared to submit a bid of at least R . To see this, observe that for this bidder, in a weakly separating Bayesian Nash equilibrium, a bid equal to R yields the same revenue as abstaining from bidding. However, in equilibrium, each type that submits a bid, does so according to the equilibrium bid function for the situation with no reserve price. This implies that if $B_2(\cdot; R) < R$, a bidder with type R would submit a bid below the reserve price, which is not possible, so that a contradiction is established.

The intuition for the condition being sufficient is as follows. In a weakly separating Bayesian Nash equilibrium, each bidder who submits a bid, submits a bid as if there were no reserve price. Then, for the existence of a weakly separating equilibrium, it remains to be checked that $B_2(\cdot; \theta) \geq R$. If $B_2(\cdot; R) \geq R$, then there is a type $\theta - R$ for which $B_2(\cdot; \theta) = R$. As a reserve price does not affect equilibrium bidding of types that submit a bid, it follows that if type θ would submit a bid in equilibrium, she would submit a bid equal to R . However, type R is indifferent between bidding R and not submitting a bid, so that θ prefers not to submit a bid. Therefore, θ must exceed $\theta - R$, so that indeed $B_2(\cdot; \theta) \geq B_2(\cdot; \theta - R) = R$.

The necessary and sufficient condition $B_2(\cdot; R) \geq R$ implies that for small R a weakly separating Bayesian Nash equilibrium exists, but not for large R . As said, the existence of such an equilibrium is trivial in the case of small R . However, for R close to \bar{t} , $B_2(\cdot; R) < R$, as, by Proposition 4, $B_2(\cdot; \bar{t}) < \bar{t}$.

5 Concluding remarks

We have studied auctions in which losing bidders obtain financial externalities from the winning bidder. We have derived bidding equilibria for FPSB and

SPSB, and have established that the presence of a resale market does not affect equilibrium behavior. Also, we have shown that in the two-bidder case SPSB dominates FPSB in terms of expected auction revenue if $\lambda < 1$ and that both auctions are revenue equivalent if $\lambda = 1$. Moreover, we have studied equilibrium bidding for FPSB and SPSB when a reserve price is imposed. We have observed pooling at the reserve price for FPSB. For SPSB, we found a necessary and sufficient condition for the existence of a weakly separating Bayesian Nash equilibrium.

An interesting possibility for future research is to investigate what the effects are of asymmetric financial externalities in a private values environment. For instance, one may examine what happens in case only one of the bidders imposes a financial externality on the other bidders. Bulow et al. (1999) consider a situation in which two bidders bid for a common value object, and one of the bidders receives a fraction of the auction revenue. The bidder without toehold in the auction revenue faces a strong winner's curse, and therefore bids zero in equilibrium, even if the toehold of the other bidder in the auction revenue is infinitesimally small. Although the authors restrict their attention to a common value environment, their analysis shows that asymmetric financial externalities may have dramatic effects on the auction revenue.

Motivated by the observation that in SPSB, low signal bidders increase their bids when λ is increased (for λ not too large), also a model with asymmetries in the valuation function may be fruitful to study. One may imagine that with one bidder with a low value, and one bidder with a high value, the price in SPSB may be higher with financial externalities than without financial externalities, as the bidder with the low value has an incentive to push up the price when λ is strictly positive.

6 Appendix

Proof of Lemma 8. Define $\tilde{A}(y) = y + \frac{\lambda}{1+\lambda}(1-y)^{\frac{1+\lambda}{\lambda}} - (\frac{1}{1+\lambda})y^{1+\lambda}$. The first and second order derivatives of \tilde{A} are respectively given by

$$\begin{aligned}\tilde{A}'(y) &= 1 - (1-y)^{\frac{1}{\lambda}} - y^{\lambda}, \text{ and} \\ \tilde{A}''(y) &= \frac{1}{\lambda}(1-y)^{\frac{1}{\lambda}-1} - \lambda y^{\lambda-1}.\end{aligned}$$

Observe that

$$\begin{aligned}\tilde{A}(0) &= \tilde{A}(1) = 0, \\ \tilde{A}'(0) &= \tilde{A}'(1) = 0, \\ \lim_{y \rightarrow 0} \tilde{A}''(y) &= \frac{1}{\lambda}, \\ \tilde{A}''(1) &= -\lambda < 0.\end{aligned}$$

Hence, if y is close to 0, $\tilde{A}(y)$ must be below zero and concave, and similarly for y close to 1, $\tilde{A}(y)$ is negative and concave. Suppose now that, in contrast

to what is stated in the lemma, $\tilde{A}(y) > 0$ for some $y \in (0; 1)$. As \tilde{A} and all its derivatives are continuous functions on the interval $(0; 1)$, $\tilde{A}^{(0)}(y)$ must change sign at least four times, or, equivalently, $\tilde{A}^{(0)}(y) = 0$ for at least four values of y in $(0; 1)$. Define $\frac{1}{1} < \frac{1}{2}, \frac{1}{2} < \frac{1}{3} < \frac{1}{4} < \frac{1}{5}$, and $\frac{1}{5} < \frac{1}{6} < \frac{1}{7} < \frac{1}{8}$. Note that $\frac{1}{8} < 0$. Then,

$$\begin{aligned}\tilde{A}^{(0)}(y) &= \frac{1}{1}(1-y)^{\frac{1}{1}} + \frac{1}{2}y^{\frac{1}{2}} = 0 \\ \frac{1}{1} &= \frac{y^{\frac{1}{2}}}{(1-y)^{\frac{1}{1}}} = \frac{y}{(1-y)^{\frac{1}{2}}} \Rightarrow \\ \frac{1}{1} (1-y)^{\frac{1}{2}} &= y.\end{aligned}$$

The last expression has at most two solutions in the interval $[0; 1]$, as the left hand side is strictly convex in y , and the right hand side is a linear function in y . A contradiction is established, so the first part of the lemma must be true. The second part is trivial. \square

7 References

- Angeles de Frutos, M. (2000). "Asymmetric Price-Benefits Auctions," *Games Econom. Behavior* 33, 48-71.
- Benoit, J.-P., and Krishna, V. (2001). "Multiple-Object Auctions with Budget Constrained Bidders," *Rev. Econ. Stud.*, 68, 155-179.
- Börgers, T., and Dustmann, C. (2001). "Strange Bids: Bidding Behaviour in the United Kingdom's Third Generation Spectrum Auction," working paper, University College London.
- Bulow, J., Huang, M., and Klemperer, P. (1999). "Toeholds and Takeovers," *J. Polit. Econ.* 107, 427-454.
- Cramton, P., Gibbons, R., and Klemperer, P. (1987). "Dissolving a Partnership Efficiently," *Econometrica* 55, 615-632.
- Engelbrecht-Wiggans, R. (1994). "Auctions with Price-Proportional Benefits to Bidders," *Games Econom. Behavior* 6, 339-46.
- Engers, M., and McManus, B. (2000). "Charity Auctions," working paper, University of Virginia.
- Ettinger, D. (2000). "Auctions and Toeholds," working paper, CERAS.
- Goeree, J. K., and Turner, J. L. (2001). "All-Pay-All Auctions," working paper, University of Virginia.
- Goeree, J. K., and Turner, J. L. (2002). "How (Not) to Raise Money," working paper, University of Virginia.
- Haile, P. A. (1999). "Auctions with Resale," working paper, University of Wisconsin-Madison.
- Jehiel, P., and Moldovanu, B. (1996). "Strategic Nonparticipation," *RAND J. Econ.* 27, 84-98.
- Jehiel, P., and Moldovanu, B. (2000). "Auctions with Downstream Interaction Among Buyers," *RAND J. Econ.* 31, 768-791.

- Jehiel, P., Moldovanu, B., and Stacchetti, E. (1996). "How (Not) to Sell Nuclear Weapons," *Ame. Econ. Rev.* 86, 814-29.
- Jehiel, P., Moldovanu, B., and Stacchetti, E. (1999). "Multidimensional Mechanism Design for Auctions with Externalities," *J. Econ. Theory* 85, 258-83.
- Kittsteiner, T. (2001). "Dissolving a Partnership with Double Auctions in a Model with Interdependent Valuations," working paper, Mannheim University.
- Maasland, E., and Onderstal, S. (2002). "Optimal Auctions with Financial Externalities," working paper, Tilburg University.
- McAfee, R. P., and McMillan, J. (1992). "Bidding Rings." *Ame. Econ. Rev.* 82, 579-599.
- Milgrom, P. R., and Weber, R. J. (1982). "A Theory of Auctions and Competitive Bidding," *Econometrica* 50, 1089-1122.
- Myerson, R. B. (1981). "Optimal Auction Design," *Math. Oper. Res.* 6, 58-73.
- Van Damme, E. (1992). "Fair Division under Asymmetric Information," in R. Selten (ed.), *Rational Interaction - Essays in Honor of John C. Harsanyi*, Springer, Berlin - Heidelberg, 121-144.