On the first entrance time distribution of the M/D/∞ queue: a combinatorial approach

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Abstract

In this paper we will derive an exact expression for the cumulative distribution function (cdf) of the first time that the number of customers in an M/D/∞ queue reaches a given level K. This problem is equivalent to finding the cdf of the first entrance time into state K of the process $\{N(t) - N(t-D)\}$, with $\{N(t)\}$ a Poisson process and $D > 0$ the constant service time. The main difficulty arises from the fact that this process is non-Markovian. The motivation for this problem stems from a logistic model that assumes a producer must satisfy every demand within a constant lead time D. We start with the simple case $K = 2$. Next we derive the cdf at integer multiples of D for general K, by using a combinatorial result on lattice path counting. From this analysis we infer the cdf at arbitrary time points. It turns out that the tail of this cdf can be closely approximated by an exponential function, and we exploit this fact to obtain good and efficient approximations for the expected first entrance times.

M/D/∞, DELIVERY, FIRST ENTRANCE TIMES, LATTICE PATH COUNTING

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1 Introduction

Throughout this paper we will be concerned with the M/D/∞ queueing system: Markovian arrivals, deterministic service times and infinitely many servers. Let $\{N(t), t \geq 0\}$ be a Poisson arrival process with parameter λ and let A_i ($i = 0, 1, \ldots$) denote the interarrival time between the $(i-1)^{\text{th}}$ and i^{th} arrival, i.e. $\{A_i\}$ is a sequence of i.i.d. exponential random variables with mean $\frac{1}{\lambda}$. Define $S_i := \sum_{j=1}^{i} A_j$ (the arrival epoch of the i^{th} customer), and let $D > 0$ be the constant service time. Now note that, since the number of servers is infinite, any customer leaves the system exactly time D after arriving. Therefore the number of

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customers in the system at any given time is just the number of customers that arrived during the last D time units. Let $X(t)$ denote the number of customers in the system at time t $(t \geq 0)$, then

$$X(t) = \begin{cases} N(t) & (0 \leq t \leq D); \\ N(t) - N(t-D) & (t > D). \end{cases} \tag{1}$$

Since $N(t)$ is Poisson(λt) distributed, it follows immediately from (1) that the distribution of $X(t)$ is given by

$$\Pr\{X(t) = k\} = \begin{cases} e^{-\lambda t}\dfrac{(\lambda t)^k}{k!} & (k = 0, 1, \ldots;\ 0 \leq t \leq D); \\ e^{-\lambda D}\dfrac{(\lambda D)^k}{k!} & (k = 0, 1, \ldots;\ t > D). \end{cases} \tag{2}$$

Define the first entrance times of the process $\{X(t)\}$,

$$T_K \ := \ \inf\{t \geq 0 : X(t) \geq K\} \quad (K = 1, 2, \ldots).$$

The rest of the paper will focus on the distribution function $F_{T_K}(t) := \Pr\{T_K \leq t\}$ and the mean $E(T_K)$.

The application that motivated this research is the following. Consider a producer of durable consumer goods (e.g. cars), where demands for the good occur according to a Poisson process $\{N(t)\}$. The producer is subject to a service contract which obliges him to satisfy the demand within a constant lead time of D, the so-called delay limit. The producer now has the choice between two types of delivery. Firstly, at any point in time he can do a batch delivery (e.g. a shipment) that accommodates all waiting demand, incurring a cost of $a_B + b_B i$ if the batch size is i. Secondly, when a demand has reached its delay limit of D, besides doing an immediate batch delivery he can also do an immediate - relatively costly - individual delivery (e.g. by plane) for that demand only, incurring a cost of b_I $(> b_B)$. The problem is to determine at which point in time the producer should initiate a batch delivery, in order to minimize the average delivery costs. Now we restrict ourselves to the following policy: do a batch delivery when the number of demands waiting for delivery reaches the level K for the first time.

It is easily seen that the number of waiting demands at time t, starting with an empty system at time 0 and before a batch delivery is done, is given by $X(t)$ in (1). Therefore, the time to a batch delivery for the given policy is just T_K. Define $g(K)$ as the average costs per unit of time as a function of the policy parameter K. Then, since a batch delivery regenerates the system, we can apply the Renewal Reward Theorem to obtain

$$g(K) \ = \ \frac{a_B + b_B K + b_I E(N(T_K - D))}{E(T_K)}. \tag{3}$$

Minimizing (3) with respect to K gives the optimal policy within the subclass of policies considered here (this is not necessarily the global optimal policy).

In fact, this model is a continuous time version of an original discrete time model extensively studied in [Berg et al. 1995]. In that model the time axis is discretized into intervals of length $\frac{D}{n}$, where n is the dimension of the state space that determines the accuracy of the model.

2 Some preliminary results

First note that $T_1 = A_1$, and hence

$$F_{T_1}(t) = 1 - e^{-\lambda t}\ (t \geq 0), \quad E(T_1) = \frac{1}{\lambda}. \tag{4}$$

In general, we have that

$$\begin{aligned} \bar{F}_{T_K}(t) &:= \Pr\{T_K > t\} \\ &= \Pr\{X(s) < K,\ 0 \leq s \leq t\} \\ &= \Pr\{N(S_i) < K,\ i = 1, \ldots, N(t)\} \\ &= \Pr\{S_K > S_1 + D, \ldots, S_{N(t)} > S_{N(t)-K+1} + D\}. \end{aligned} \tag{5}$$

Obviously,

$$\bar{F}_{T_K}(t) = \Pr\{N(t) < K\} = \sum_{k=0}^{K-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \quad (0 \leq t \leq D). \tag{6}$$

Conditioning on $\{N(t) = k\}$ and $S_1, \ldots, S_k$, and using the fact that

$$f_{S_1,\ldots,S_k|N(t)=k}(t_1, \ldots, t_k) = \frac{k!}{t^k} \quad (0 \leq t_1 \leq \cdots \leq t_k \leq t),$$

it follows from (5) that

$$\begin{aligned} \bar{F}_{T_K}(t) &= \sum_{k=0}^{K-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \sum_{k=K}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \int\limits_{\substack{t_1,\ldots,t_k: \\ 0 \leq t_1 \leq \cdots \leq t_k \leq t}} \frac{k!}{t^k} \cdot \\ &\quad \cdot \Pr\{S_i > S_{i-K+1} + D,\ i = K, \ldots, k \mid N(t) = k;\ S_1 = t_1, \ldots, S_k = t_k\}\, dt_k \cdots dt_1 \\ &= \sum_{k=0}^{K-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \sum_{k=K}^{\lceil \frac{t}{D} \rceil (K-1)} e^{-\lambda t} \lambda^k \int\limits_{\substack{t_1,\ldots,t_k: \\ 0 \leq t_1 \leq \cdots \leq t_k \leq t; \\ t_i > t_{i-K+1} + D, i = K, \ldots, k}} 1\ dt_k \cdots t_1 \quad (t > D). \end{aligned} \tag{7}$$

Note that the maximal number of arrivals in $[0, t]$ for which probability (5) is nonzero consists of $K - 1$ arrivals in each of the intervals $[(i-1)D, iD)$ $(i = 1, \ldots, \lfloor \frac{t}{D} \rfloor)$ and $[\lfloor \frac{t}{D} \rfloor D, t)$, or a total of $\lceil \frac{t}{D} \rceil (K-1)$ arrivals. In the next section we will elaborate (7) for $K = 2$, but this turns out to be extremely difficult for $K > 2$ (if at all possible). Therefore, we will use an alternative combinatorial approach in section 4.

Next we turn to the mean of T_K. To this end we define the discrete r.v. N_K by

$$\begin{aligned} \{N_K = k\} &:\Longleftrightarrow \{T_K = S_k\} \\ &\Longleftrightarrow \{S_i > S_{i-K+1} + D,\ i = K, \ldots, k-1;\ S_k \leq S_{k-K+1} + D\} \\ &\Longleftrightarrow \{\sum_{j=i-K+2}^{i} A_j > D,\ i = K, \ldots, k-1;\ \sum_{j=k-K+2}^{k} A_j \leq D\}, \end{aligned} \tag{8}$$

so that N_K is the index of the first customer that increases the number of customers in the system to K. Note that $\Pr\{N_K = k\} = 0$ for $k < K$. It easily follows from (8) that N_K is a stopping time for the sequence $\{A_i;\ i = 1, 2, \ldots\}$ for any K, and hence we can apply Wald's theorem to obtain

$$E(T_K) = E(S_{N_K}) = E(\sum_{i=1}^{N_K} A_i) = E(N_K)E(A_1) = \frac{1}{\lambda}E(N_K). \tag{9}$$

It remains to find $E(N_K)$, and we can write

$$\begin{aligned} E(N_K) &= \sum_{k=1}^{\infty} \Pr\{N_K > k\} \\ &= K + \sum_{k=K}^{\infty} \Pr\{A_2 + \ldots + A_K > D, \ldots, A_{k-K+2} + \ldots + A_k > D\} \\ &= K + \sum_{k=K}^{\infty} \Pr\{\min_{i=K,\ldots,k} \sum_{j=i-K+2}^{i} A_j > D\}. \end{aligned} \tag{10}$$

Again, (10) is hard to elaborate except for $K = 2$.

In terms of the delivery application, it is important to note that the number of individual deliveries in a cycle equals $N(T_K - D) = N_K - K$. Hence the cost function in (3) reduces to

$$g(K) = \lambda \left(b_I - \frac{(b_I - b_B)K - a_B}{E(N_K)} \right) = \lambda b_I - \frac{(b_I - b_B)K - a_B}{E(T_K)}, \tag{11}$$

and this only requires the computation of $E(T_K)$.

3 The case $K = 2$

For $K = 2$ it is possible to simplify (7), and we will use the following lemma.

Lemma 1 *For any $j = 0, 1, \ldots$ and $k = j + 1, j + 2, \ldots$ we have*

$$I(j,k) := \int\limits_{\substack{t_{j+1},\ldots,t_k: \\ t_{i-1}+D \le t_i \le t-(k-i)D, i=j+1,\ldots,k}} dt_k \cdots dt_{j+1} = \frac{\left(t - t_j - (k-j)D\right)^{k-j}}{(k-j)!}. \tag{12}$$

Proof.
We use induction on j. For $j = k - 1$ (12) trivially holds. Suppose that (12) holds for $j = j' + 1$. It follows that

$$I(j',k) = \int_{t_{j'+1}=t_{j'}+D}^{t-(k-j'-1)D} I_2(j'+1,k)\, dt_{j'+1}$$

$$= \int_{t_{j'+1}=t_{j'}+D}^{t-(k-j'-1)D} \frac{(t - t_{j'+1} - (k-j'-1)D)^{k-j'-1}}{(k-j'-1)!}\, dt_{j'+1}$$

$$= \int_{t_{j'+1}=0}^{t-t_{j'}-(k-j')D} \frac{t_{j'+1}^{k-j'-1}}{(k-j'-1)!}\, dt_{j'+1}$$

$$= \frac{(t - t_{j'} - (k-j')D)^{k-j'}}{(k-j')!},$$

and hence (12) holds for $j = j'$. □

Theorem 1 *(i)*

$$\bar{F}_{T_2}(t) = \sum_{k=0}^{\lceil \frac{t}{D} \rceil} e^{-\lambda t} \frac{\Big(\lambda(t-(k-1)D)\Big)^k}{k!} \quad (t \geq 0). \tag{13}$$

(ii)

$$E(T_2) = \frac{1}{\lambda} \frac{2 - e^{-\lambda D}}{1 - e^{-\lambda D}}. \tag{14}$$

Proof.
It follows from lemma 1 with $j = 1$ that

$$\int_{\substack{t_1,\ldots,t_k: \\ 0 \leq t_1 \leq \cdots \leq t_k \leq t; \\ t_i > t_{i-1}+D, i=2,\ldots,k}} 1\, dt_k \cdots t_1 = \int_{\substack{t_1,\ldots,t_k: \\ 0 \leq t_1 \leq t-(k-1)D; \\ t_{i-1}+D \leq t_i \leq t-(k-i)D, i=2,\ldots,k}} 1\, dt_k \cdots dt_1$$

$$= \int_{t_1=0}^{t-(k-1)D} \frac{\Big(t - t_1 - (k-1)D\Big)^{k-1}}{(k-1)!}\, dt_1$$

$$= \frac{\Big(t-(k-1)D\Big)^k}{k!} \quad (t \geq (k-1)D). \tag{15}$$

Substituting (15) into (7) yields (13).
(ii) Integrating (12) over t gives

$$E(T_2) = \int_{t=0}^{\infty} \left(e^{-\lambda t} + \sum_{k=1}^{\lceil \frac{t}{D} \rceil} e^{-\lambda t} \frac{\Big(\lambda(t-(k-1)D)\Big)^k}{k!} \right) dt$$

$$= \int_{t=0}^{\infty} e^{-\lambda t}\, dt + \sum_{k=1}^{\infty} \int_{t=(k-1)D}^{\infty} e^{-\lambda t} \frac{\Big(\lambda(t-(k-1)D)\Big)^k}{k!}\, dt$$

$$= \frac{1}{\lambda} + \sum_{k=1}^{\infty} \int_{u=0}^{\infty} e^{-\lambda(u+(k-1)D)} \frac{(\lambda u)^k}{k!}\, du$$

$$
\begin{aligned}
&= \frac{1}{\lambda} + \sum_{k=1}^{\infty} (e^{-\lambda D})^{k-1} \frac{1}{\lambda} \int_{u=0}^{\infty} \frac{\lambda^{k+1} u^k e^{-\lambda u}}{k!}\, du \\
&= \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{k=1}^{\infty} (e^{-\lambda D})^{k-1} \\
&= \frac{1}{\lambda} \frac{2 - e^{-\lambda D}}{1 - e^{-\lambda D}}.
\end{aligned}
$$

A second proof of (14) exploits relation (9). It follows from (8) that $N_2 - 1$ is geometrically distributed with parameter $1 - e^{-\lambda D}$, so that $E(N_2 - 1) = \frac{1}{1-e^{-\lambda D}}$. Applying (9) then yields

$$
E(T_2) = \frac{1}{\lambda} E(N_2) = \frac{1}{\lambda}\left(1 + \frac{1}{1 - e^{-\lambda D}}\right) = \frac{1}{\lambda} \frac{2 - e^{-\lambda D}}{1 - e^{-\lambda D}}.
$$

A third proof of (14) is by conditioning on the arrival epoch of the second customer, which leads to

$$
\begin{aligned}
ET_2 &= \frac{1}{\lambda} + \int_{t=0}^{D} t \lambda e^{-\lambda t} dt + e^{-\lambda D}(D + ET_2) \\
&= \frac{1}{\lambda} + \frac{1}{\lambda}(1 - e^{-\lambda D}) + e^{-\lambda D} ET_2, \qquad (16)
\end{aligned}
$$

and solving (16) for ET_2. □

4 A combinatorial approach

In this section we will use a combinatorial approach to derive $\bar{F}_{T_K}(nD)$ $(n = 1, 2, \ldots)$, the distribution function of T_K at integer multiples of D. To this end we divide the time interval $[0, nD]$ into n periods of length D. Define

$$
\begin{aligned}
I_i &:= \text{time interval } [(i-1)D, iD) \quad (i = 1, 2, \ldots); \\
N_i(t) &:= \text{number of arrivals in } [(i-1)D, (i-1)D + t) \\
&= N((i-1)D + t) - N((i-1)D) \quad (0 \le t < D;\ i = 1, \ldots, n); \\
\mathbf{N}(t) &:= (N_1(t), \ldots, N_n(t)) \quad (0 \le t < D); \\
R_{ij} &:= \inf\{0 \le t < D : N_i(t) = j\} \quad (i = 1, \ldots, n;\ j = 1, 2, \ldots); \\
M(t) &:= N_1(t) + \ldots + N_n(t) \quad (0 \le t < D); \\
R_k &:= \inf\{0 \le t < D : M(t) = k\} \quad (k = 1, 2, \ldots); \\
X_{ki} &:= N_i(R_k) \quad (k = 1, 2, \ldots;\ i = 1, \ldots, n); \\
\mathbf{X}_k &:= (X_{k1}, \ldots, X_{kn}) \quad (k = 1, 2, \ldots).
\end{aligned}
$$

Note that X_{ki} is the number of arrivals in I_i upto the time of the k^{th} arrival of $\{M(t)\}$, so that $\sum_{i=1}^{n} X_{ki} = k$. Conditioning on $\mathbf{N}(D)$ we have

$$\bar{F}_{T_K}(nD) = \sum_{\substack{k_i=0,\dots,K-1;\\ i=1,\dots,n}} \left(\prod_{i=1}^{n} e^{-\lambda D} \frac{(\lambda D)^{k_i}}{k_i!} \right) \Pr\{X(t) < K,\ 0 \le t \le nD \mid \mathbf{N}(D) = \mathbf{k}\}. \quad (17)$$

Next observe that every arrival in I_{i-1} corresponds to a departure in I_i. Therefore,

$$\begin{aligned}
& \Pr\{X(t) < K,\ 0 \le t \le nD \mid \mathbf{N}(D) = \mathbf{k}\} \\
= \;& \Pr\{X((i-1)D + t) < K,\ 0 \le t \le D,\ i = 1, \dots, n \mid \mathbf{N}(D) = \mathbf{k}\} \\
= \;& \Pr\{N_{i-1}(D) - N_{i-1}(t) + N_i(t) < K, 0 \le t \le D, i = 1, \dots, n \mid \mathbf{N}(D) = \mathbf{k}\} \\
= \;& \Pr\{N_i(t) - N_{i-1}(t) < K - k_{i-1},\ 0 \le t \le D,\ i = 1, \dots, n \mid \mathbf{N}(D) = \mathbf{k}\} \\
= \;& \Pr\{X_{ki} - X_{k,i-1} < K - k_{i-1},\ k = 1, \dots, \sum_{i=1}^{n} k_i,\ i = 1, \dots, n \mid \mathbf{N}(D) = \mathbf{k}\}, \quad (18)
\end{aligned}$$

with $N_0(t) = X_{k0} = k_0 := 0$. Probability (18) is completely determined by the sample paths of the n-dimensional finite discrete stochastic process $\{\mathbf{X}_k;\ k = 1, \dots, \sum_{i=1}^{n} k_i\}$. In the appendix we prove that every sample path of $\{\mathbf{X}_k\}$ is equally likely. This implies that probability (18) is equal to the number of paths from $(0, \dots, 0)$ to $(k_1, \dots, k_n)$ that satify the conditions

$$x_i - x_{i-1} < K - k_{i-1} \quad (i = 1, \dots, n) \quad (19)$$

for every point $(x_1, \dots, x_n)$ on the path, divided by the total number of paths from $(0, \dots, 0)$ to $(k_1, \dots, k_n)$.

To illustrate this point, we first consider the case $n = 2$. In this case probability (18) reduces to

$$\Pr\{X_{k2} - X_{k1} < K - k_1,\ k = 1, \dots, k_1 + k_2 \mid N_1(D) = k_1, N_2(D) = k_2\}. \quad (20)$$

Now every sample path of $\{(X_{k_1}, X_{k2})\}$ corresponds to a lattice path from $(0,0)$ to (k_1, k_2) (see Figure 1). More specifically, a horizontal step corresponds to an arrival in I_1, or a departure in I_2, and a vertical step to an arrival I_2. Therefore, the number of customers in the system at the point (x_1, x_2) equals $k_1 - x_1 + x_2$, and this must be smaller than K for any x_1 and x_2, or $x_2 - x_1 < K - k_1$. Since every sample path of $\{(X_{k_1}, X_{k2})\}$ is equally likely, probability (20) equals the number of lattice paths that remain below the line $x_2 - x_1 = K - k_1$, divided by the total number of paths $\binom{k_1+k_2}{k_1}$ (see Figure 1). We can count the number of paths remaining below this line by using the so-called principle of reflection (see e.g. [Feller 1968], p.72, Lemma).

Proposition 1 *The number of minimal paths from (a_1, a_2) to (b_1, b_2) which touch or cross the line l is equal to the number of paths from (a_1', a_2') to (b_1, b_2), with (a_1', a_2') the mirror image of (a, b) with respect to l.*

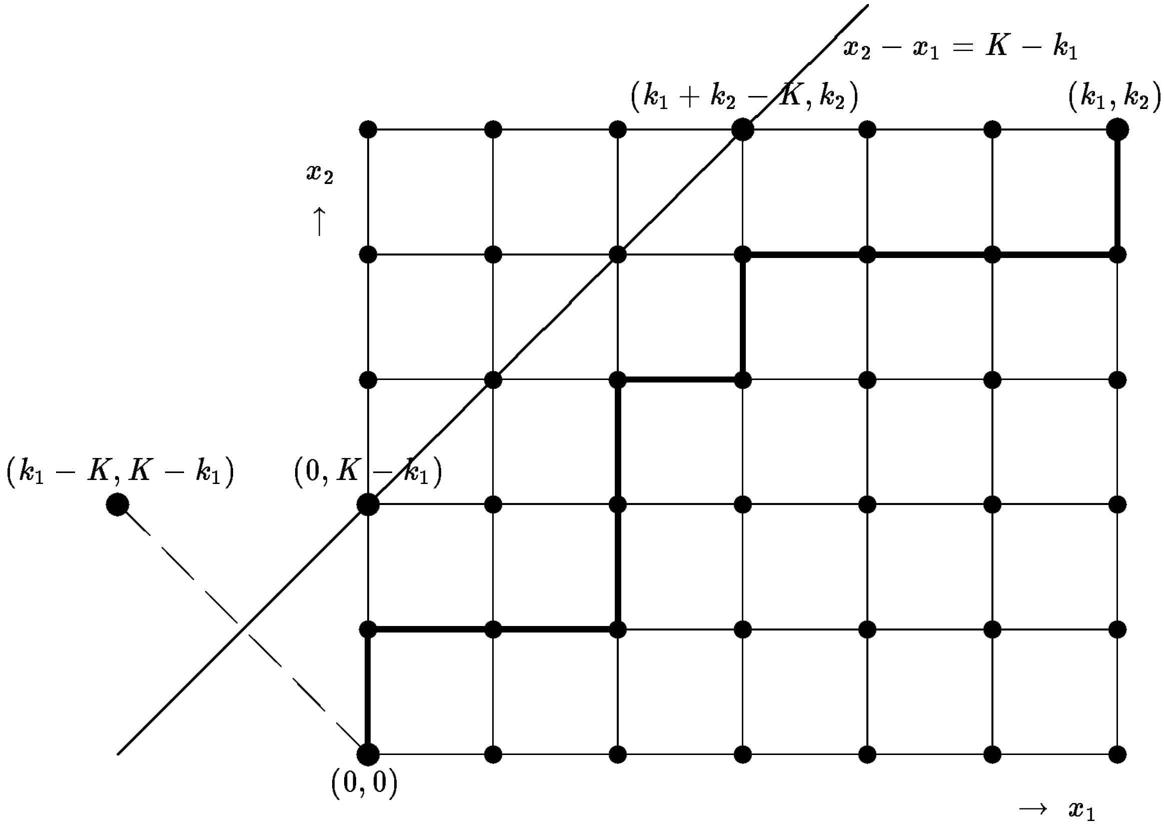

Figure 1: An example of a sample path of $\{(X_{k1}, X_{k2})\}$ ($K = 8$, $k_1 = 6$,$k_2 = 5$)

So according to this principle, the number of paths from $(0,0)$ to (k_1, k_2) which touch or cross the line $x_2 - x_1 = K - k_1$ equals the number of paths from $(k_1 - K, K - k_1)$ to (k_1, k_2), or $\binom{k_1+k_2}{K}$ (see Figure 1). Hence the number of paths remaining below this line equals $\binom{k_1+k_2}{k_1} - \binom{k_1+k_2}{K}$ (if $k_1 + k_2 \geq K$; if $k_1 + k_2 < K$ then all paths automatically remain below this line), and it follows from (18) and (20) that

$$\Pr\{T_K > 2D \mid N_1(D) = k_1, N_2(D) = k_2\} = \begin{cases} 1 - \dfrac{\binom{k_1+k_2}{K}}{\binom{k_1+k_2}{k_1}} & \text{if } k_1 + k_2 \geq K; \\ 1 & \text{if } k_1 + k_2 < K. \end{cases} \tag{21}$$

Returning to the case of general n, we can use the following result from combinatorics, which can be seen as a generalisation of the principle of reflection; see e.g. [McMahon 1915] (p. 133), [Mohanty 1979] (p. 39, Theorem 3), or [Böhm et al. 1993] (Proposition 1).

Proposition 2 *The number of paths from* $\mathbf{a} := (a_1, \ldots, a_n)$ *to* $\mathbf{b} := (b_1, \ldots, b_n)$ *such that every point on the path satisfies* $x_1 \geq x_2 \geq \cdots \geq x_n$ *is given by*

$$\left(\sum_{i=1}^{n}(b_i - a_i)\right)!\ \det(C_n(\mathbf{a}, \mathbf{b})),$$

with

$$(C_n(\mathbf{a},\mathbf{b}))_{ij} = \begin{cases} \dfrac{1}{(b_i - a_j - i + j)!} & \textit{if } b_i - a_j \geq i - j \\ 0 & \textit{if } b_i - a_j < i - j \end{cases} \qquad (i, j = 1, \dots, n).$$

Now using the transformation

$$\begin{aligned} y_1 &= x_1; \\ y_2 &= x_2 - (K - 1 - k_1) = x_2 + k_1 - K + 1; \\ y_3 &= x_3 - (K - 1 - k_1) - (K - 1 - k_2) = x_3 + k_1 + k_2 - 2(K - 1); \\ &\dots \\ y_n &= x_n - (K - 1 - k_1) - \dots - (K - 1 - k_{n-1}) = \\ &= x_n + \sum_{i=1}^{n-1} k_i - (n-1)(K-1), \end{aligned} \tag{22}$$

the conditions (17) reduce to $y_1 \geq y_2 \geq \cdots \geq y_n$. As a result, we can now apply the proposition by setting

$$a_i := \sum_{j=1}^{i-1} k_j - (i-1)(K-1), \; b_i := \sum_{j=1}^{i} k_j - (i-1)(K-1) \quad (i = 1, \dots, n). \tag{23}$$

It follows that probability (18) is equal to

$$\frac{\left(\sum_{i=1}^{n} k_i\right)! \; \det(C_n(\mathbf{a},\mathbf{b}))}{\binom{k_1 + \cdots + k_n}{k_1, \dots, k_n}} = \left(\prod_{i=1}^{n} k_i!\right) \det(C_n(\mathbf{a},\mathbf{b})), \tag{24}$$

with

$$(C_n(\mathbf{a},\mathbf{b}))_{ij} = \begin{cases} \dfrac{1}{\left(\sum_{l=j}^{i} k_l - (i-j)K\right)!} & (i = 1, \dots, n; \; j = 1, \dots, i); \\ \dfrac{1}{K!} & (i = 1, \dots, n-1; \; j = i+1); \\ \dfrac{1}{\left((j-i)K - \sum_{l=i+1}^{j-1} k_l\right)!} & (i = 1, \dots, n-2; \; j = i+2, \dots, n), \end{cases} \tag{25}$$

where $(C_n(\mathbf{a},\mathbf{b}))_{ij} = 0$ if the argument of the factorial is negative.

Finally, combining (17), (18) and (24) we find

$$\bar{F}_{T_K}(nD) = e^{-n\lambda D} \sum_{\substack{k_i = 0, \dots, K-1; \\ i = 1, \dots, n}} (\lambda D)^{\sum_{i=1}^{n} k_i} \det(C_n(\mathbf{a},\mathbf{b})), \tag{26}$$

with $\mathbf{a}$ and $\mathbf{b}$ defined as in (23). For numerical purposes, (26) requires the computation of K^n determinants of order n.

5 The complete distribution function

In this section we will derive an expression for $\bar{F}_{T_K}(nD+t) = \Pr\{T_K > nD+t\}$. Define

$$\begin{aligned} J_i(t) &:= [(i-1)D, (i-1)D+t) \quad (i=1,2,\ldots;\ 0 < t \le D); \\ K_i(t) &:= [(i-1)D+t, iD) \quad (i=1,2,\ldots;\ 0 < t \le D); \\ E_n &:= \{\mathbf{N}(t) = \mathbf{l},\ \mathbf{N}(D) - \mathbf{N}(t) = \mathbf{m},\ N_{n+1}(t) = l_{n+1}\} \end{aligned}$$

(we omit the dependency of E_n on $\mathbf{l}$ and $\mathbf{m}$ for ease of notation). Note that $I_i = J_i(t) \cup K_i(t)$ $(i = 1,2,\ldots;\ 0 < t \le D)$, i.e. we chop every interval I_i into a left hand part of length t and a right hand part of length $D-t$. Now conditional on E_n, since customers that arrive in $J_i(t)$ $(K_i(t))$ depart in $J_{i+1}(t)$ $(K_{i+1}(t))$, we can decompose the process $\{X(s)\}$ on $[0, nD+t)$ into two independent parts: one on $\cup_{i=1}^{n+1} J_i(t)$ and one on $\cup_{i=1}^{n} K_i(t)$. More formally, we have

$$\begin{aligned} &\Pr\{X(s) < K;\ 0 \le s \le nD+t \mid E_n\} \\ =\ &\Pr\{X(s) < K;\ s \in \left(\cup_{i=1}^{n+1} J_i(t)\right) \bigcup \left(\cup_{i=1}^{n} K_i(t)\right) \mid E_n\} \\ =\ &\Pr\{X(s) < K;\ s \in \cup_{i=1}^{n+1} J_i(t) \mid E_n\} \cdot \Pr\{X(s) < K;\ s \in \cup_{i=1}^{n} K_i(t) \mid E_n\}. \end{aligned} \tag{27}$$

Both probabilities on the right hand side of (27) can be computed in a similar fashion as probability (18). Consider the left hand intervals $J_i(t)$ $(i = 1,\ldots,n+1)$. Given E_n, the number of customers at the start of $J_i(t)$ is equal to $l_{i-1} + m_{i-1}$, during $J_i(t)$ there are l_{i-1} departures and l_i arrivals, and hence the number of customers at the end of $J_i(t)$ is equal to $m_{i-1} + l_i$. Therefore, the left hand probability in (27) can be written as

$$\begin{aligned} &\Pr\{X(s) < K;\ s \in \cup_{i=1}^{n+1} J_i(t) \mid E_n\} \\ =\ &\Pr\{l_{i-1} + m_{i-1} + X_{ki} - X_{k,i-1} < K;\ k = 1,\ldots,\sum_{i=1}^{n+1} l_i,\ i = 1,\ldots,n \mid E_n\} \\ =\ &\Pr\{X_{ki} - X_{k,i-1} < K - l_{i-1} - m_{i-1};\ k = 1,\ldots,\sum_{i=1}^{n+1} l_i,\ i = 1,\ldots,n \mid E_n\}. \end{aligned} \tag{28}$$

Since every sample path of $\{\mathbf{X}_k\}$ is equally likely (see the Appendix), we see that probability (28) equals the number of paths from $(0,\ldots,0)$ to $(l_1,\ldots,l_{n+1})$ that satify the conditions

$$x_i - x_{i-1} < K - (l_{i-1} + m_{i-1}) \quad (i = 1,\ldots,n+1) \tag{29}$$

for every point $(x_1,\ldots,x_{n+1})$ on the path, divided by the total number of paths from $(0,\ldots,0)$ to $(l_1,\ldots,l_{n+1})$. The transformation

$$y_i = x_i + \sum_{j=1}^{i-1}(l_j + m_j) - (i-1)(K-1) \quad (i = 1,\ldots,n+1),$$

enables us to apply Proposition 1, and it follows that (28) equals

$$\frac{(\sum_{i=1}^{n+1} l_i)!\ \det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l))}{\binom{l_1+\cdots+l_{n+1}}{l_1,\ldots,l_{n+1}}} = \left(\prod_{i=1}^{n+1} l_i!\right) \det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l)), \tag{30}$$

with (for $i=1,\ldots,n+1$)

$$a_i^l := \sum_{j=1}^{i-1}(l_j+m_j)-(i-1)(K-1),\ b_i^l := \sum_{j=1}^{i}(m_{j-1}+l_j)-(i-1)(K-1). \tag{31}$$

Analogously, using the transformation

$$y_i = x_i + \sum_{j=1}^{i}(m_{j-1}+l_j)-(i-1)(K-1) \quad (i=1,\ldots,n), \tag{32}$$

it follows that the right hand probability in (27) equals

$$\frac{(\sum_{i=1}^{n} m_i)!\ \det(C_n(\mathbf{a}^r,\mathbf{b}^r))}{\binom{m_1+\cdots+m_n}{m_1,\ldots,m_n}} = \left(\prod_{i=1}^{n} m_i!\right) \det(C_n(\mathbf{a}^r,\mathbf{b}^r)), \tag{33}$$

with (for $i=1,\ldots,n$)

$$a_i^r := \sum_{j=1}^{i}(m_{j-1}+l_j)-(i-1)(K-1),\ b_i^r := \sum_{j=1}^{i}(l_j+m_j)-(i-1)(K-1). \tag{34}$$

Conditioning on E_n and using (27), (30) and (33) we obtain

$$\begin{aligned}
&\bar{F}_{T_K}(nD+t)\\
&= \sum_{\substack{l_i=0,\ldots,K-1-m_{i-1};\ i=1,\ldots,n+1\\ m_i=0,\ldots,K-1-l_i;\ i=1,\ldots,n}} \left(\prod_{i=1}^{n+1} e^{-\lambda t}\frac{(\lambda t)^{l_i}}{l_i!}\right)\cdot\left(\prod_{i=1}^{n} e^{-\lambda(D-t)}\frac{(\lambda(D-t))^{m_i}}{m_i!}\right)\cdot\\
&\qquad\cdot\left(\prod_{i=1}^{n+1} l_i!\right)\det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l))\cdot\left(\prod_{i=1}^{n} m_i!\right)\det(C_n(\mathbf{a}^r,\mathbf{b}^r))\\
&= e^{n\lambda D+\lambda t}\sum_{\substack{l_i=0,\ldots,K-1-m_{i-1};\ i=1,\ldots,n+1\\ m_i=0,\ldots,K-1-l_i;\ i=1,\ldots,n}} (\lambda t)^{\sum_{i=1}^{n+1} l_i}(\lambda(D-t))^{\sum_{i=1}^{n} m_i}\det(C_{n+1}(\mathbf{a}^l,\mathbf{b}^l))\det(C_n(\mathbf{a}^r,\mathbf{b}^r)),
\end{aligned} \tag{35}$$

with $\mathbf{a}^l$ and $\mathbf{b}^l$ defined as in (31), and $\mathbf{a}^r$ and $\mathbf{b}^r$ as in (34). In computing $E(T_K)$ from (35) we use the following lemma.

Lemma 2 *Define* $f(l,m) := \int_0^D e^{-\lambda t}(\lambda t)^l(\lambda(D-t))^m$, *then for* $l,m=0,1,\ldots$

$$f(l,m) = \frac{(-1)^m}{\lambda}\left(\sum_{i=0}^{m}(l+m-i)!\binom{m}{i}(-\lambda D)^i - e^{-\lambda D}\sum_{i=0}^{l}(l+m-i)!\binom{l}{i}(\lambda D)^i\right). \tag{36}$$

It follows from (35) and (36) that

$$E(T_K) = \sum_{n=0}^{\infty} e^{-n\lambda D} \sum_{\substack{l_i=0,\dots,K-1-m_{i-1};\\ i=1,\dots,n+1\\ m_i=0,\dots,K-1-l_i;\, i=1,\dots,n}} f(\sum_{i=1}^{n+1} l_i, \sum_{i=1}^{n} m_i) \det(C_{n+1}(\mathbf{a}^l, \mathbf{b}^l)) \det(C_n(\mathbf{a}^r, \mathbf{b}^r)). \quad (37)$$

This expression is obviously not suited for numerical purposes, since the number of terms in the second summation grows exponentially with n. However, in the next section we will see that (35) can be closely approximated by an exponential function, and already for small values of n. In this way we obtain good and efficient approximations for $E(T_K)$.

6 Numerical results

We will conclude the paper with some computational results. In all the calculations to follow we will assume w.l.o.g. that $D = 1$, so that λ denotes the mean number of arrivals during the unit service time. Tables (1) and (2) below give $\bar{F}_{T_K}(t)$ at $t = 1$ (from (6)), $t = 2, 3, 4$ (from (26)) and $t = 2\frac{1}{2}$ (from (35)) for $K = 2, \dots, 10$, for $\lambda = 3$ and $\lambda = 5$, respectively.

K	$\bar{F}_{T_K}(1)$	$\bar{F}_{T_K}(2)$	$\bar{F}_{T_K}(2\frac{1}{2})$	$\bar{F}_{T_K}(3)$	$\bar{F}_{T_K}(4)$	$\bar{F}_{T_K}(5)$
2	0.19195	0.02851	0.01061	0.00401	0.00057	0.00008
3	0.42319	0.12332	0.06642	0.03612	0.01067	0.00315
4	0.64723	0.30597	0.21187	0.14729	0.07119	0.03440
5	0.81526	0.53164	0.43307	0.35318	0.23494	0.15627
6	0.91608	0.73160	0.65793	0.59177	0.47881	0.38741
7	0.96649	0.86820	0.82561	0.78511	0.71000	0.64207
8	0.98810	0.94379	0.92366	0.90396	0.86581	0.82926
9	0.99620	0.97889	0.97081	0.96279	0.94695	0.93138
10	0.99890	0.99293	0.99010	0.98728	0.98165	0.97605

Table 1: $\bar{F}_{T_K}(t)$ for different values of K and t ($\lambda = 3$)

The computation time for $\bar{F}_{T_{10}}(5)$ already lies in the order of hours on a 486 PC. However, tables (3) and (4) reveal that the tail of the distribution function can be approximated by an exponential function, even for moderate values of K and t (we omit the index T_K for ease of notation).

Based on this observation we propose the following approximation for $\bar{F}_{T_K}(t)$:

$$\hat{\bar{F}}_{T_K}(t) = C_1(K) e^{-C_2(K)t} \quad (t \geq 3D), \quad (38)$$

with

$$C_2(K) := \ln \bar{F}_{T_K}(3D) - \ln \bar{F}_{T_K}(4D), \quad C_1(K) := \bar{F}_{T_K}(4D) e^{4DC_2(K)}. \quad (39)$$

K	$\bar{F}_{T_K}(1)$	$\bar{F}_{T_K}(2)$	$\bar{F}_{T_K}(2\frac{1}{2})$	$\bar{F}_{T_K}(3)$	$\bar{F}_{T_K}(4)$	$\bar{F}_{T_K}(5)$
2	$4.0428 \cdot 10^{-2}$	$1.0669 \cdot 10^{-3}$	$1.6483 \cdot 10^{-4}$	$2.6563 \cdot 10^{-5}$	$6.7237 \cdot 10^{-7}$	$1.7099 \cdot 10^{-8}$
3	0.12465	$8.9173 \cdot 10^{-3}$	$2.2954 \cdot 10^{-3}$	$6.1420 \cdot 10^{-4}$	$4.3114 \cdot 10^{-5}$	$3.0349 \cdot 10^{-6}$
4	0.26503	$4.0090 \cdot 10^{-2}$	$1.5332 \cdot 10^{-2}$	$6.0199 \cdot 10^{-3}$	$9.1783 \cdot 10^{-4}$	$1.4011 \cdot 10^{-4}$
5	0.44049	0.11738	$6.0597 \cdot 10^{-2}$	$3.1756 \cdot 10^{-2}$	$8.6782 \cdot 10^{-3}$	$2.3726 \cdot 10^{-3}$
6	0.61596	0.25112	0.16183	0.10505	$4.4191 \cdot 10^{-2}$	$1.8592 \cdot 10^{-2}$
7	0.76218	0.42496	0.32106	0.24327	0.13963	$8.0141 \cdot 10^{-2}$
8	0.86663	0.60381	0.50913	0.42969	0.30608	0.21803
9	0.93191	0.75522	0.68461	0.62073	0.51033	0.41956
10	0.96817	0.86385	0.81924	0.77694	0.69882	0.62855

Table 2: $\bar{F}_{T_K}(t)$ for different values of K and t ($\lambda = 5$)

K	$\ln \bar{F}(1) - \ln \bar{F}(2)$	$\ln \bar{F}(2) - \ln \bar{F}(3)$	$\ln \bar{F}(3) - \ln \bar{F}(4)$	$\ln \bar{F}_{T_K}(4) - \ln \bar{F}(5)$
2	1.94395	1.96111	1.94999	1.94955
3	1.23306	1.22782	1.21968	1.21967
4	0.74921	0.73106	0.72713	0.72723
5	0.42755	0.40900	0.40765	0.40771
6	0.22487	0.21211	0.21181	0.21183
7	0.10725	0.10059	0.10056	0.10057
8	0.04588	0.04312	0.04312	0.04313
9	0.01752	0.01658	0.01659	0.01659
10	0.00599	0.00572	0.00572	0.00572

Table 3: $\ln \bar{F}_{T_K}(n) - \ln \bar{F}_{T_K}(n+1)$ converges to a constant ($\lambda = 3$)

Using (38) we approximate $E(T_K)$ by

$$\hat{E}(T_K) = \int_0^{3D} \bar{F}_{T_K}(t)\, dt + \int_{3D}^{\infty} \hat{\bar{F}}_{T_K}(t) = \sum_{n=0}^{2} \int_0^{D} \bar{F}_{T_K}(nD+t)\, dt + \frac{C_1(K)}{C_2(K)} e^{-3DC_2(K)}, \quad (40)$$

where the three integrals are computed as in (37). In tables (5) and (6) we compare approximation (40) with the simulation value and its 95% confidence interval after 10^6 runs for $\lambda = 3$ and $\lambda = 5$, respectively.

The approximation performs very well, and always falls within the 95% confidence interval. However, the width of the confidence interval increases strongly with K, due to the strongly increasing variance of T_K (see e.g. $K = 9$ and $K = 10$ in table 5).

We now apply the results to the delivery application by evaluating the cost function (11) for the case $a_B = 5$, $b_B = 1$ and $b_I = 3$ (see the last column of tables 5 and 6). We find

K	$\ln \bar{F}(1) - \ln \bar{F}(2)$	$\ln \bar{F}(2) - \ln \bar{F}(3)$	$\ln \bar{F}(3) - \ln \bar{F}(4)$	$\ln \bar{F}_{T_K}(4) - \ln \bar{F}(5)$
2	3.63476	3.69301	3.67645	3.67178
3	2.63753	2.67542	2.65648	2.65367
4	1.88869	1.89606	1.88082	1.87955
5	1.32246	1.30735	1.29727	1.29684
6	0.89725	0.87152	0.86589	0.86582
7	0.58420	0.55781	0.55518	0.55521
8	0.36135	0.34020	0.33921	0.33924
9	0.21022	0.19612	0.19584	0.19585
10	0.11401	0.10603	0.10597	0.10598

Table 4: $\ln \bar{F}_{T_K}(n) - \ln \bar{F}_{T_K}(n+1)$ converges to a constant ($\lambda = 5$)

K	$E_{\text{sim}}(T_K)$	$\hat{E}(T_K)$	$\hat{g}(K)$
2	0.68417±0.00053	0.68413	10.47
3	1.1139±0.0013	1.1139	8.10
4	1.7607±0.0036	1.7599	7.30
5	2.9134±0.0117	2.9131	7.28
6	5.2519±0.0436	5.2513	7.67
7	10.537±0.193	10.541	8.15
8	23.824±1.054	23.850	8.54
9	60.952±7.082	61.012	8.79
10	175.51±59.73	175.76	8.91

Table 5: $E_{\text{sim}}(T_K)$, $\hat{E}(T_K)$ and $\hat{g}(K)$ ($\lambda = 3$)

K	$E_{\text{sim}}(T_K)$	$\hat{E}(T_K)$	$\hat{g}(K)$
2	0.40147±0.00016	0.40136	17.50
3	0.61082±0.00029	0.61104	13.36
4	0.84772±0.00053	0.84789	11.47
5	1.1497±0.0011	1.1499	10.65
6	1.5859±0.0025	1.5853	10.60
7	2.2789±0.0062	2.2786	11.05
8	3.4727±0.0169	3.4724	11.83
9	5.6748±0.0507	5.6790	12.71
10	10.051±0.174	10.055	13.51

Table 6: $E_{\text{sim}}(T_K)$, $\hat{E}(T_K)$ and $\hat{g}(K)$ ($\lambda = 5$)

that the optimal control limit K^* equals 5 for $\lambda = 3$ and 6 for $\lambda = 5$ (recall that K is the number of waiting demands triggering a batch delivery).

Appendix: Proof of sample path result

In this appendix we will prove that every sample path of the process $\{\mathbf{X}_k\}$ (defined in section 4) has equal probability. In order to do so, we need the following results.

Lemma 3 *Let $\{U_{ij};\ j = 1, \ldots, j_i\}$ $(i = 1, \ldots, n)$ be n finite sequences of mutually independent and identically distributed random variables with a uniform distribution over (a, b). Then*
(i)

$$\Pr\{U_{rs} = \min_{\substack{i=1,\ldots,n;\\ j=1,\ldots,j_i}} U_{ij}\} \quad = \quad \frac{1}{\sum_{i=1}^{n} j_i} \quad (r = 1, \ldots, n;\ s = 1, \ldots, j_r);$$

(ii)

$$\Pr\{\min_{\substack{i=1,\dots,n;\\ j=1,\dots,j_i}} U_{ij} = \min_{j=1,\dots,j_r} U_{rj}\} \;=\; \frac{j_r}{\sum_{i=1}^{n} j_i} \quad (r = 1,\dots,n).$$

Proof.
(i) Define

$$\begin{aligned} E_{rs} &:= \{U_{rs} = \min_{\substack{i=1,\dots,n;\\ j=1,\dots,j_i}} U_{ij}\} \quad (r = 1,\dots,n;\ s = 1,\dots,j_r); \\ E_r &:= \{\min_{\substack{i=1,\dots,n;\\ j=1,\dots,j_i}} U_{ij} = \min_{j=1,\dots,j_r} U_{rj}\} \quad (r = 1,\dots,n). \end{aligned}$$

Conditioning on U_{rs} yields

$$\begin{aligned} \Pr\{E_{rs}\} &= \int_a^b \frac{1}{b-a} \Pr\{U_{ij} \geq x,\ i = 1,\dots,n,\ j = 1,\dots,j_i \mid U_{rs} = x\}\, dx \\ &= \int_a^b \frac{1}{b-a} \left(\frac{b-x}{b-a}\right)^{\sum_{i=1}^{n} j_i - 1} dx \\ &= \frac{1}{\sum_{i=1}^{n} j_i} \quad (r = 1,\dots,n;\ s = 1,\dots,j_r). \end{aligned}$$

(ii) Since $E_r = \bigcup_{s=1}^{j_r} E_{rs}$ and $\bigcap_{s=1}^{j_r} E_{rs} = \emptyset$, it follows that

$$\Pr\{E_r\} = \Pr\{\bigcup_{s=1}^{j_r} E_{rs}\} = \sum_{s=1}^{j_r} \Pr\{E_{rs}\} = \frac{j_r}{\sum_{i=1}^{n} j_i} \quad (r = 1,\dots,n). \quad \Box$$

Theorem 2 *For $x_i \leq k_i$ $(i = 1,\dots,n)$ with $\sum_{i=1}^{n} x_i = k$, and any $t \geq 0$, we have*

$$\Pr\{\mathbf{X}_{k+1} = \mathbf{x} + \mathbf{e}_r \mid \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} \;=\; \frac{k_r - x_r}{\sum_{i=1}^{n}(k_i - x_i)} \quad (r = 1,\dots,n).$$

Proof.
Define $f(x) := \frac{d}{dx}\Pr\{R_k \leq x \mid \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\}$. By conditioning with respect to $f(x)$ and using Lemma (3) it follows that

$$\begin{aligned} &\Pr\{\mathbf{X}_{k+1} = \mathbf{x} + \mathbf{e}_r \mid \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} \\ =\ & \int_0^t f(u) \Pr\{\mathbf{X}_{k+1} = \mathbf{x} + \mathbf{e}_r \mid R_k = u;\ \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} \\ =\ & \int_0^t f(u) \Pr\{\min_{\substack{i=1,\dots,n;\\ j=x_i+1,\dots,k_i}} R_{ij} = \min_{j=x_r+1,\dots,k_r} R_{rj} \mid R_k = u;\ \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} \\ =\ & \frac{k_r - x_r}{\sum_{i=1}^{n}(k_i - x_i)}, \end{aligned}$$

since

$$\Pr\{R_{ij} \leq x \mid R_k = u;\ \mathbf{X}_k = \mathbf{x};\ \mathbf{N}(t) = \mathbf{k}\} = \Pr\{U \leq x\} \quad (u \leq x \leq t)$$

with U uniformly distributed over $[u, t]$. □

Corollary 1 *For $x_i \leq k_i$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^n x_i = k$, and any $t \geq 0$, we have*

$$\Pr\{\mathbf{X}_k = \mathbf{x} \mid \mathbf{N}(t) = \mathbf{k}\} \quad = \quad \frac{\prod_{i=1}^n \binom{k_i}{x_i}}{\binom{\sum_{i=1}^n k_i}{k}}.$$

Proof.
We use induction on k. For $k = 1$ the corollary reduces to

$$\Pr\{\mathbf{X}_1 = \mathbf{e}_r \mid \mathbf{N}(t) = \mathbf{k}\} = \frac{k_r}{\sum_{i=1}^n k_i} \quad (r = 1, \ldots, n),$$

and this is true by Theorem (2) with $k = 0$ and $\mathbf{x} = \mathbf{0}$. Conditioning on $\mathbf{X}_{k-1}$ and then using the induction hypothesis together with Theorem (2) yields

$$\begin{aligned}
&\Pr\{\mathbf{X}_k = \mathbf{x} \mid \mathbf{N}(t) = \mathbf{k}\} \\
= \;& \sum_{r=1}^n \Pr\{\mathbf{X}_{k-1} = \mathbf{x} - \mathbf{e}_r \mid \mathbf{N}(t) = \mathbf{k}\} \Pr\{\mathbf{X}_k = \mathbf{x} \mid \mathbf{X}_{k-1} = \mathbf{x} - \mathbf{e}_r;\ \mathbf{N}(t) = \mathbf{k}\} \\
= \;& \sum_{r=1}^n \frac{\binom{k_r}{x_r - 1} \prod_{\substack{i=1 \\ i \neq r}}^n \binom{k_i}{x_i}}{\binom{\sum_{i=1}^n k_i}{k-1}} \cdot \frac{k_r - x_r + 1}{\sum_{i=1}^n (k_i - x_i) + 1} \\
= \;& \sum_{r=1}^n \frac{x_r \prod_{i=1}^n \binom{k_i}{x_i}}{\binom{\sum_{i=1}^n k_i}{k-1} \left(\sum_{i=1}^n k_i - k + 1\right)} \\
= \;& \frac{\prod_{i=1}^n \binom{k_i}{x_i}}{\binom{\sum_{i=1}^n k_i}{k}},
\end{aligned}$$

since $\sum_{r=1}^n x_r = k$. □

The following theorem proves that every sample path $\{\mathbf{X}_k;\ k=1,\ldots,\sum_{i=1}^{n} k_i\}$ is equally likely.

Theorem 3 *For $x_{ki} \le k_i$ $(i=1,\ldots,n)$ with $\sum_{i=1}^{n} x_{ki} = k$ and $\mathbf{x}_k - \mathbf{x}_{k-1} \in \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ $(k=1,\ldots,\sum_{i=1}^{n} k_i)$, and any $t \ge 0$, we have*

$$\Pr\{\mathbf{X}_k = \mathbf{x}_k;\ k=1,\ldots,\sum_{i=1}^{n} k_i \mid \mathbf{N}(t) = \mathbf{k}\} \quad = \quad \binom{k_1+\cdots+k_n}{k_1,\ldots,k_n}^{-1}.$$

Proof.
Since $\mathbf{x}_k - \mathbf{x}_{k-1} = \mathbf{e}_r$ iff the k^{th} event is an arrival in I_r it follows that

$$\begin{aligned}
&\Pr\{\mathbf{X}_k = \mathbf{x}_k;\ k=1,\ldots,\sum_{i=1}^{n} k_i \mid \mathbf{N}(t) = \mathbf{k}\} \\
= &\prod_{k=1}^{\sum_{i=1}^{n} k_i} \Pr\{\mathbf{X}_k = \mathbf{x}_k \mid \mathbf{X}_{k-1} = \mathbf{x}_{k-1};\ \mathbf{N}(t) = \mathbf{k}\} \\
= &\prod_{k=1}^{\sum_{i=1}^{n} k_i} \frac{\sum_{r=1}^{n} I_{\{\mathbf{x}_k - \mathbf{x}_{k-1} = \mathbf{e}_r\}}(k_r - x_{kr})}{\sum_{i=1}^{n}(k_i - x_{ki})},
\end{aligned} \tag{41}$$

where in the first equality we use the Markov property

$$\begin{aligned}
&\Pr\{\mathbf{X}_k = \mathbf{x}_k \mid \mathbf{X}_1 = \mathbf{x}_1,\ldots,\mathbf{X}_{k-1} = \mathbf{x}_{k-1};\ \mathbf{N}(t) = \mathbf{k}\} \\
= &\Pr\{\mathbf{X}_k = \mathbf{x}_k \mid \mathbf{X}_{k-1} = \mathbf{x}_{k-1};\ \mathbf{N}(t) = \mathbf{k}\}.
\end{aligned}$$

Now observe that in the nominator of (41) every factor $k_i - j$ $(j = 0,\ldots,k_i - 1)$ occurs exactly once for all $i = 1,\ldots,n$, so that the product of these $\sum_{i=1}^{n} k_i$ factors is just $k_1!\cdots k_n!$. The denominator of (41) obviously equals $(k_1+\cdots+k_n)!$, and hence the desired result follows. □

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