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Competitive environments and protective behaviour

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Abstract

The class of two-person competition games is introduced and analyzed. For any game in this class the set of Nash equilibria is convex, equilibrium strategies are exchangeable, and all Nash equilibria lead to the same payoff vector. Competition games are compared to other competitive environments such as unilaterally competitive games and rivalry games. Moreover, protective behaviour within competitive environments is analyzed. For matrix games it is known that protective strategies profiles exactly correspond to proper equilibria. It is shown that this result can be extended to the class of unilaterally competitive games.

Keywords: competitive environments, unilaterally competitive games, rivalry games, competition games, protective strategies.

JEL classification: C72

1 Introduction

In a matrix game, the set of Nash equilibria exhibits the following well-known characteristics: all Nash equilibria lead to the same payoff vector, the set of Nash equilibria is a convex set, and equilibrium strategies are exchangeable. Moreover, the set of proper equilibria (Myerson, 1978) can be alternatively characterized by the Dresher procedure (Dresher, 1961) or by the notion of protectiveness (Fiestras-Janeiro, Borm and van Megen, 1998). These characterizations do not hold in the more general class of bimatrix games.

In the literature several classes of bimatrix games have been considered in which the Nash equilibrium set retains (most of) the first three characteristics. We mention the class of almost strictly competitive games (Aumann, 1961), the class of strictly competitive games (Friedman, 1983), the class of unilaterally competitive games (Kats and Thisse, 1992), and perhaps less known the class of rivalry games (Rauhut, Schmitz and Zachow, 1979). Any strictly competitive game is unilaterally competitive; any unilaterally competitive game is a rivalry game; and, any rivalry game is almost strictly competitive. For every game in any of these classes, all

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Nash equilibria have identical payoff vectors and the set of Nash equilibria is a convex set. The property of exchangeability is valid in all these classes except for the class of almost strictly competitive games. Here, only Nash equilibria which are also twisted equilibria (Aumann, 1961) are exchangeable.

In this paper we introduce the class of competition games. This class is in between the class of rivalry games and the class of almost strictly competitive games. For this class, the three properties considered above are valid. We then focus on the possible relation between proper equilibria and protective strategy profiles in competitive environments a la Fiestras-Janeiro et al. (1998). It turns out that the set of protective strategy profiles coincides with the set of proper equilibria in the class of unilaterally competitive games. In the class of rivalry games, protective strategy profiles are (perfect) equilibria but not necessarily proper equilibria. In the process we analyze relations between the set of equilibria of bimatrix games (A, B) and the equilibria in the related matrix games $(A, -A)$ and $(B^t, -B^t)$. These relations will be used in the proofs of the main theorems.

The paper is organized as follows. Basic definitions are provided in Section 2. In Section 3 we define the different competitive environments under consideration and describe the relationship between them. Section 4 is devoted to the relation between protective strategy profiles and proper Nash equilibria.

2 Preliminaries

A *bimatrix game* (A, B) is a two person game $(\Delta(S_1), \Delta(S_2), \pi_1, \pi_2)$ in strategic form, where A and B are two $m \times n$ matrices, $S_1 = \{e_1, \ldots, e_m\}$ and $S_2 = \{f_1, \ldots, f_n\}$ are the pure strategy sets of player 1 and player 2, respectively, and the payoff functions π_1 and π_2 are defined as[1]

$$\pi_1(p, q) = pAq \text{ and } \pi_2(p, q) = pBq$$

for every pair of mixed strategies $p \in \Delta(S_1)$ and $q \in \Delta(S_2)$. A bimatrix game (A, B) where $B = -A$ is called a *matrix game* and it is usually denoted by A.

Let us consider a bimatrix game (A, B). A combination $(p, q) \in \Delta(S_1) \times \Delta(S_2)$ is called a *strategy profile*. For any $p \in \Delta(S_1)$, the set

$$B_2(p) = \{\bar{q} \in \Delta(S_2) \mid pB\bar{q} = \max_{q \in \Delta(S_2)} pBq\}$$

is the *set of best replies* of player 2 against the strategy p of player 1; the set

$$W_2(p) = \{\bar{q} \in \Delta(S_2) \mid pA\bar{q} = \min_{q \in \Delta(S_2)} pAq\}$$

gives us the *set of worst replies* of player 2 (from the perspective of player 1) with respect to the strategy p of player 1. With the obvious modifications one defines the sets $B_1(q)$ and $W_1(q)$ for any $q \in \Delta(S_2)$.

[1] We write pAq in stead of p^tAq and pBq in stead of p^tBq.

We say that $\bar{p} \in \Delta(S_1)$ is a *completely mixed strategy* if $\bar{p}_i (= \bar{p}(e_i)) > 0$ for all $i = 1, \ldots, m$. Analogously, we define completely mixed strategies for player 2.

A strategy profile $(\bar{p}, \bar{q})$ is called a *Nash equilibrium* if

$$\bar{p}A\bar{q} \geq pA\bar{q} \text{ for all } p \in \Delta(S_1) \quad \text{and} \quad \bar{p}B\bar{q} \geq \bar{p}Bq \text{ for all } q \in \Delta(S_2).$$

Hence, $(\bar{p}, \bar{q})$ is a Nash equilibrium for (A, B) if and only if $\bar{p} \in B_1(\bar{q})$ and $\bar{q} \in B_2(\bar{p})$.

A strategy profile $(\bar{p}, \bar{q})$ is called a *twisted equilibrium* (Aumann, 1961) if

$$\bar{p}A\bar{q} \leq \bar{p}Aq \text{ for all } q \in \Delta(S_2) \quad \text{and} \quad \bar{p}B\bar{q} \leq pB\bar{q} \text{ for all } p \in \Delta(S_1).$$

Hence, $(\bar{p}, \bar{q})$ is a twisted equilibrium for (A, B) if and only if $\bar{p} \in W_1(\bar{q})$ and $\bar{q} \in W_2(\bar{p})$. $\mathrm{E}(A, B)$ will denote the set of Nash equilibria and $\mathrm{TE}(A, B)$ the set of twisted equilibria of (A, B). Notice that the twisted equilibria of (A, B) exactly correspond to the Nash equilibria of the bimatrix game $(-B, -A)$.

The following example shows that $\mathrm{E}(A, B) \cap \mathrm{TE}(A, B)$ can be empty.

Example 2.1. Consider the bimatrix game (A, B) defined by

$$A = \begin{pmatrix} 3 & 5 \\ 3 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

Then, $\mathrm{E}(A, B) = \{(e_1, f_2)\}$ but (e_1, f_2) is not a twisted equilibrium since $W_1(f_2) = \{e_2\}$. Consequently, $\mathrm{E}(A, B) \cap \mathrm{TE}(A, B) = \emptyset$. $\square$

Next, we recall the definitions of the main concepts that we will use later on. Let (A, B) be a bimatrix game.

A strategy profile (p, q) is a *proper equilibrium* (Myerson, 1978) if there exist sequences $\left\{\left(p^k, q^k\right)\right\}_{k \in \mathbb{N}}$ of strategies profiles and $\left\{\varepsilon^k\right\}_{k \in \mathbb{N}}$ of real numbers such that $\lim_{k \to \infty} \varepsilon^k = 0$, $\lim_{k \to \infty} \left(p^k, q^k\right) = (p, q)$, and for all $k \in \mathbb{N}$,

(i) $\varepsilon^k > 0$ and $\left(p^k, q^k\right)$ is a completely mixed strategy profile,

(ii) for all $e_i, e_j \in S_1$ such that $e_i A q^k < e_j A q^k$ we have $p_i^k \leq \varepsilon^k p_j^k$,

and

(iii) for all $f_r, f_s \in S_2$ such that $p^k B f_r < p^k B f_s$ we have $q_r^k \leq \varepsilon^k q_s^k$.

We will denote by $\mathrm{PROP}(A, B)$ the set of proper equilibria of the bimatrix game (A, B).

Next, let $p \in \Delta(S_1)$ be a strategy of player 1. We recursively define $a^r(p) \in \mathbb{R}$ and $S_2^r(p) \subset S_2$, $r \in \mathbb{N}$, by

(i) $a^1(p) = \min\{pAf_j \mid f_j \in S_2\}$ and
$S_2^1(p) = \{f_j \in S_2 \mid pAf_j = a^1(p)\}$.

(ii) for $r > 1$,

$$\begin{aligned} a^r(p) &= \min\left\{pAf_j \mid f_j \in S_2 \setminus \bigcup_{k=1}^{r-1} S_2^k(p)\right\} \text{ and} \\ S_2^r(p) &= \{f_j \in S_2 \mid pAf_j = a^r(p)\}. \end{aligned}$$

Analogously, we define $b^r(q) \in \mathbb{N}$ and $S_1^r(q) \subset S_1$ for any $q \in \Delta(S_2)$ and $r \in \mathbb{N}$.

Now, consider $\bar{p}, \hat{p} \in \Delta(S_1)$. We say that $\bar{p}$ *protectively dominates* $\hat{p}$ if there exists $l \in \mathbb{N}$, such that

(i) $a^r(\bar{p}) = a^r(\hat{p})$ and $S_2^r(\bar{p}) = S_2^r(\hat{p})$ for all $r < l$

and

(ii) $a^l(\bar{p}) > a^l(\hat{p})$ or both $a^l(\bar{p}) = a^l(\hat{p})$ and $S_2^l(\bar{p}) \subsetneq S_2^l(\hat{p})$.

A mixed strategy for player 1 is called *protective* in (A, B) if there does not exist a mixed strategy which protectively dominates it. Similarly, protective strategies for player 2 can be defined. By $\mathrm{PROT}_1(A, B)$ and by $\mathrm{PROT}_2(A, B)$ we denote the set of protective strategies for player 1 and 2, respectively. Moreover, $\mathrm{PROT}(A, B) := \mathrm{PROT}_1(A, B) \times \mathrm{PROT}_2(A, B)$ consists of all protective strategy profiles.

The following result can be found in Fiestras-Janeiro et al. (1998).

Theorem 2.1. For every matrix game A, $\mathrm{PROT}(A) = \mathrm{PROP}(A)$.

3 Competitive environments

First, we will recall the definitions of some two-person competitive environments, i.e. bimatrix games with the features of matrix games, from the literature.

A bimatrix game (A, B) is called *strictly competitive* (Friedman, 1983) if for all strategy profiles (p, q) and $(\bar{p}, \bar{q})$ we have

$$pAq \geq \bar{p}A\bar{q} \text{ if and only if } pBq \leq \bar{p}B\bar{q}.$$

A bimatrix game (A, B) is called *unilaterally competitive* (Kats and Thisse, 1992) if for all strategy profiles (p, q) and $(\bar{p}, \bar{q})$ we have

$$pAq \geq \bar{p}Aq \text{ if and only if } pBq \leq \bar{p}Bq \tag{1}$$

and

$$pAq \geq pA\bar{q} \text{ if and only if } pBq \leq pB\bar{q}. \tag{2}$$

Rauhut et al. (1979) introduced the perhaps less well known class of rivalry games. A bimatrix game (A, B) is called a *rivalry game* if for all strategy profile (p, q) we have that $B_1(q) = W_1(q)$ and $B_2(p) = W_2(p)$.

Finally, a bimatrix game (A, B) is called *almost strictly competitive* (Aumann, 1961) if $E(A, B) \cap TE(A, B) \neq \emptyset$ and $\{(pAq, pBq)|(p, q) \in \mathrm{E}(A, B)\} = \{(pAq, pBq)|(p, q) \in \mathrm{TE}(A, B)\}$.

Kats and Thisse (1992) investigated the relationships between three of the four classes of games discussed above. The class of strictly competitive games is a proper subset of the class of unilaterally competitive games; the class of unilaterally competitive games is a proper subset of the class of almost strictly competitive games. The position of the class of rivalry games is in between unilaterally competitive games and almost strictly competitive games. It is readily shown that any unilaterally competitive game is a rivalry game and any rivalry game is almost strictly competitive.

In general, a rivalry game is not necessarily unilaterally competitive[2] as the following example illustrates.

Example 3.1. Consider the bimatrix game (A, B) defined by

$$A = \begin{pmatrix} 8 & 0 & 2 \\ 12 & 4 & 8 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 8 & 2 \\ 0 & 4 & 0 \end{pmatrix}.$$

The sets of best and worst replies for agent 1 against any $q \in \Delta(S_2)$ and for agent 2 against any $p \in \Delta(S_1)$ are given by

$$B_1(q) = W_1(q) = \{e_2\} \quad \text{and} \quad B_2(p) = W_2(p) = \{f_2\}.$$

Hence, this game is a rivalry game, but it is not unilaterally competitive because,

$$e_2 B f_1 = 0 = e_2 B f_3$$

while

$$e_2 A f_1 = 12 > 8 = e_2 A f_3. \qquad \square$$

An almost strictly competitive game, in general, is not necessarily a rivalry game.

Example 3.2. Consider the 2×2 bimatrix game (A, B) given by

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}.$$

This game is almost strictly competitive because $E(A, B) = TE(A, B) = \{(e_1, f_1)\}$. However, $B_1(f_2) = \{e_1\}$ while $W_1(f_2) = \Delta(S_1)$. Hence, (A, B) is not a rivalry game. $\square$

[2]It can be shown that any 2×2 rivalry game (A, B) is unilaterally competitive.

We now propose a new competitive environment. A bimatrix game (A, B) is called a *competition game* if $E(A,B) = TE(A,B)$.

Clearly, any rivalry game is a competition game and any competition game is almost strictly competitive. The class of competition games is larger than the class of rivalry games. Example 3.2 is a competition game which is not a rivalry game. Moreover, the class of almost strictly competitive games is larger than the class of competition games as the following example illustrates.

Example 3.3. Consider the 2×2 bimatrix game (A, B) given by

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}.$$

This game is almost strictly competitive since its set of Nash equilibria is given by the set $E(A,B) = \Delta(S_1) \times \{f_1\}$ and its set of twisted equilibria is given by

$$TE(A,B) = \{((p, 1-p), f_1) \mid p \in [1/2, 1]\}.$$

Each Nash equilibrium and each twisted equilibrium leads to the payoff vector $(1, 2)$. Clearly, (A, B) is not a competition game. □

The proof of the following proposition is straightforward and therefore omitted.

Proposition 3.1. In the class of competition games, all Nash equilibria lead to the same payoff vector, the set of Nash equilibria is convex and equilibrium strategies are exchangeable.

The set of Nash equilibria of a competition game (A, B) corresponds to the "intersection" of the set of Nash equilibria of the related matrix games A and B^t.

Proposition 3.2. Let (A, B) be a competition game. Then,

$$(p, q) \in \mathrm{E}(A, B) \Leftrightarrow (p, q) \in \mathrm{E}(A) \text{ and } (q, p) \in \mathrm{E}(B^t).$$

Proof. We will show first the $(\Rightarrow)$ part. Let $(p, q) \in \mathrm{E}(A, B)$. Then

$$pA\bar{q} \geq pAq \geq \bar{p}Aq \text{ for every } \bar{p} \in \Delta(S_1) \text{ and every } \bar{q} \in \Delta(S_2)$$

where the first inequality holds because (p, q) is a twisted equilibrium, and the second holds because (p, q) is a Nash equilibrium. Hence, $(p, q) \in \mathrm{E}(A)$. Analogously, one shows that $(q, p) \in \mathrm{E}(B^t)$.

Next, we will show the $(\Leftarrow)$ part. Let $(p, q) \in \mathrm{E}(A)$ and $(q, p) \in \mathrm{E}(B^t)$. Hence, $pAq \geq \bar{p}Aq$ for every $\bar{p} \in \Delta(S_1)$ and $pBq = qB^tp \geq \bar{q}B^tp = pB\bar{q}$ for every $\bar{q} \in \Delta(S_2)$. It follows that $(p, q) \in \mathrm{E}(A, B)$. □

In Figure 1 we summarize the relations between the different types of competitive environments.

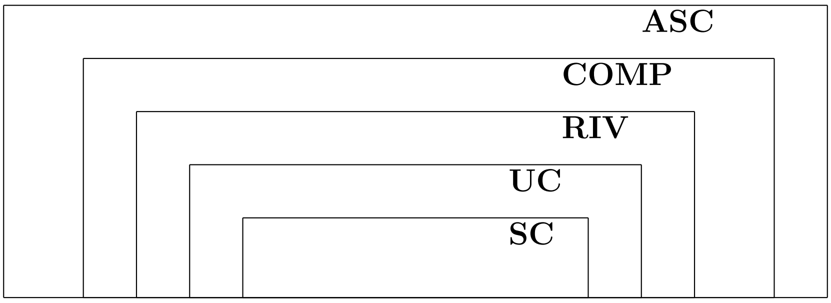

Figure 1: Relationships among the competitive environments.

4 Protective behaviour in competitive environments

In this section we analyze the relations between protective strategy profiles and proper equilibria in competitive environments. We will first show that protective strategy profiles lead to Nash equilibria in the class of rivalry games. In the proof of this result we use the following lemma.

Lemma 4.1. Let (A, B) be a rivalry game. Then[3],

(a) $(\bar{p}, \bar{q}) \in \mathrm{E}(A)$ if and only if $(\bar{q}, \bar{p}) \in \mathrm{E}(B^t)$.

(b) $\mathrm{E}(A, B) = \mathrm{E}(A)$.

Proof. First, we will show (a). Let $(\bar{p}, \bar{q})$ be a Nash equilibrium for the matrix game A. Then,

$$\bar{p}A\bar{q} \geq pA\bar{q} \text{ for all } p \in \Delta\,(S_1) \quad \text{and} \quad \bar{p}A\bar{q} \leq \bar{p}Aq \text{ for all } q \in \Delta(S_2).$$

This implies that $\bar{p}$ is a best reply to $\bar{q}$ and $\bar{q}$ is a worst reply to $\bar{p}$ in the game (A, B). Since (A, B) is a rivalry game, it also holds $\bar{p}$ is a worst reply to $\bar{q}$ and $\bar{q}$ is a best reply to $\bar{p}$. Hence,

$$pB\bar{q} \geq \bar{p}B\bar{q} \text{ for all } p \in \Delta(S_1) \quad \text{and} \quad \bar{p}B\bar{q} \geq \bar{p}Bq \text{ for all } q \in \Delta(S_2).$$

It immediately follows that $(\bar{q}, \bar{p})$ is a Nash equilibrium for the matrix game B^t. By similar reasoning, we obtain that if $(\bar{q}, \bar{p}) \in \mathrm{E}(B^t)$, then $(\bar{p}, \bar{q}) \in \mathrm{E}(A)$.

Part (b) is a direct consequence of part (a) and Proposition 3.2. □

Theorem 4.2. For any rivalry bimatrix game (A, B), $\mathrm{PROT}(A, B) \subset \mathrm{E}(A, B)$[4].

Proof. Let (A, B) be a rivalry bimatrix game. Let $(\hat{p}, \hat{q}) \in \mathrm{PROT}(A, B)$ and $(\bar{p}, \bar{q}) \in \mathrm{E}(A, B)$. By definition of a protective strategy for player 1, $\hat{p}$ is optimal in the matrix game A and hence, $(\hat{p}, \tilde{q})$ is a Nash equilibrium of the matrix game A for some $\tilde{q} \in \Delta(S_2)$. Since (A, B) is a rivalry game and by Lemma 4.1, we know that $(\hat{p}, \tilde{q})$ is also a Nash equilibrium of (A, B).

[3] In fact, it can be shown that for a rivalry game (A, B) we have $\mathrm{PERF}(A, B) = \mathrm{PERF}(A)$, where $\mathrm{PERF}(A, B)$ denotes the set of perfect equilibria (Selten, 1975) of the game (A, B).

[4] Using the equivalence between perfect equilibria and undominated Nash equilibria (van Damme, 1991) it in fact can be shown that every protective strategy profile is perfect.

Since the set of Nash equilibria of (A,B) satisfies the property of exchangeability, we find that $(\hat{p},\bar{q}) \in E(A,B)$. Similarly, we obtain that $(\bar{p},\hat{q}) \in E(A,B)$. Applying the property of exchangeability again, it follows that $(\hat{p},\hat{q}) \in E(A,B)$.

□

The following example shows that, within rivalry games, a protective strategy profile need not be a proper equilibrium.

Example 4.1. Consider the 2×4 bimatrix game given by

$$A = \begin{pmatrix} 4 & 8 & 2 & 0 \\ 8 & 4 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 16 & 8 & 2 & 32 \\ 8 & 16 & 4 & 32 \end{pmatrix}.$$

One readily verifies that (A,B) is a rivalry game. Using the method of Borm (1992) one can check that $\mathrm{E}(A,B) = \mathrm{conv}\{e_1,e_2\} \times \{f_4\}$, $\mathrm{PROP}(A,B) = \{(\frac{1}{2}e_1 + \frac{1}{2}e_2, f_4)\}$ and $\mathrm{PROT}(A,B) = \{(e_1,f_4)\}$. Note that $\mathrm{PROP}(A) = \{(e_1,f_4)\}$ and $\mathrm{PROP}(A) \neq \mathrm{PROP}(A,B)$. □

We will show that in the class of unilaterally competitive games, the set of proper equilibria and the set of protective strategy profiles coincide. For this we use the following two Lemmas.

Lemma 4.3. In a unilaterally competitive bimatrix game (A,B),

$$\mathrm{PROP}(A,B) = \mathrm{PROP}(A).$$

Proof. Let (A,B) be a unilaterally competitive bimatrix game. First, we will show that $\mathrm{PROP}(A,B) \subset \mathrm{PROP}(A)$. Let $(\bar{p},\bar{q}) \in \mathrm{PROP}(A,B)$ and let $\{(\bar{p}^k,\bar{q}^k)\}_{k\in\mathbb{N}}$ and $\{\varepsilon^k\}_{k\in\mathbb{N}}$ be a pair of sequences satisfying the conditions in the definition for $(\bar{p},\bar{q})$ to be a proper equilibrium in (A,B). For every $e_i, e_j \in S_1$ such that $e_iA\bar{q}^k < e_jA\bar{q}^k$, it holds $\bar{p}_i^k \leq \varepsilon^k\bar{p}_j^k$. Next, take $f_i, f_j \in S_2$ such that $\bar{p}^k(-A)f_i < \bar{p}^k(-A)f_j$. Then, it holds $\bar{p}^kAf_i > \bar{p}^kAf_j$ and since (A,B) is unilaterally competitive it holds $\bar{p}^kBf_i < \bar{p}^kBf_j$ and hence $\bar{q}_i^k \leq \varepsilon^k\bar{q}_j^k$. Then, $\{(\bar{p}^k,\bar{q}^k)\}_{k\in\mathbb{N}}$ and $\{\varepsilon^k\}_{k\in\mathbb{N}}$ also satisfy the conditions for $(\bar{p},\bar{q})$ to be a proper equilibrium in the matrix game A.

Next, we will prove $\mathrm{PROP}(A,B) \supset \mathrm{PROP}(A)$. Let $(\bar{p},\bar{q}) \in \mathrm{PROP}(A)$ and let $\{(\bar{p}^k,\bar{q}^k)\}_{k\in\mathbb{N}}$ and $\{\varepsilon^k\}_{k\in\mathbb{N}}$ be a pair of sequences satisfying the conditions in the definition for $(\bar{p},\bar{q})$ to be a proper equilibrium of A. For every $e_i, e_j \in S_1$ such that $e_iA\bar{q}^k < e_jA\bar{q}^k$, it holds $\bar{p}_i^k \leq \varepsilon^k\bar{p}_j^k$. Next, take $f_i, f_j \in S_2$ such that $\bar{p}^kBf_i < \bar{p}^kBf_j$. Using the fact that (A,B) is unilaterally competitive we have $\bar{p}^kAf_i > \bar{p}^kAf_j$, or equivalently $\bar{p}^k(-A)f_i < \bar{p}^k(-A)f_j$ and hence $\bar{q}_i^k \leq \varepsilon^k\bar{q}_j^k$. Then, $\{(\bar{p}^k,\bar{q}^k)\}_{k\in\mathbb{N}}$ and $\{\varepsilon^k\}_{k\in\mathbb{N}}$ also satisfy the conditions for $(\bar{p},\bar{q})$ to be a proper equilibrium for (A,B). □

Lemma 4.4. Let (A,B) be a unilaterally competitive game. Then,

$$\mathrm{PROT}(A,B) = \mathrm{PROT}_1(A) \times \mathrm{PROT}_2(A).$$

Proof. Note that, by definition, we only need to show that $\mathrm{PROT}_2(A,B) = \mathrm{PROT}_2(A)$. For this, it suffices to show that the set of protectively dominated strategies for player 2 in the bimatrix game (A,B) and in the matrix game A coincide. To avoid confusion we will add the underlying game in our notations and write e.g. $S_1^r(q;(A,B))$ instead of $S_1^r(q)$ and $b^r(q;(A,B))$ instead of $b^r(q)$.

First, we will show by induction to r that $S_1^r(q;A) = S_1^r(q;(A,B))$ for every $q \in \Delta(S_2)$ and $r \in \mathbb{N}$. Let $q \in \Delta(S_2)$.

Consider $r = 1$. Then,

$$\begin{aligned} e_i \in S_1^1(q;(A,B)) &\Leftrightarrow e_iBq \leq e_jBq, \text{ for every } e_j \in S_1 \\ &\Leftrightarrow e_iAq \geq e_jAq, \text{ for every } e_j \in S_1 \\ &\Leftrightarrow e_i(-A)q \leq e_j(-A)q, \text{ for every } e_j \in S_1 \\ &\Leftrightarrow e_i \in S_1^1(q;A) \end{aligned}$$

where the second equivalence follows from the fact that (A,B) is unilaterally competitive.

Let us assume that $S_1^r(q;A) = S_1^r(q;(A,B))$ for $r = 1,\ldots,t-1$. Let us take $e_i \in S_1^t(q;(A,B))$. Then, $e_jBq < e_iBq$ and $e_j(-A)q < e_i(-A)q$ for every $e_j \in \cup_{r=1}^{t-1} S_1^r(q;(A,B)) = \cup_{r=1}^{t-1} S_1^r(q;A)$. Moreover,

$$\begin{aligned} &e_iBq = e_kBq \quad \text{and hence} \quad e_i(-A)q = e_k(-A)q, \text{ for every } e_k \in S_1^t(q;(A,B)) \\ &e_iBq < e_kBq \quad \text{and hence} \quad e_i(-A)q < e_k(-A)q, \text{ for every } e_k \in S_1^r(q;(A,B)) \text{ with } r > t. \end{aligned}$$

We may conclude that $e_i \in S_1^t(q;A)$. In a similar way one can prove that $S_1^t(q;A) \subset S_1^t(q;(A,B))$. Note that $b^r(q;(A,B)) = e_iBq$ and $b^r(q;A) = -e_iAq$ for every $e_i \in S_1^r(q)$ and $r \in \mathbb{N}$.

Now, let $q \in \Delta(S_2)$ be a protectively dominated strategy for player 2 in (A,B). Take $\bar{q} \in \Delta(S_2)$ such that $\bar{q}$ protectively dominates q. By definition, there exists an $l \in \mathbb{N}$ such that

(i) For $r < l$, $S_1^r(\bar{q};(A,B)) = S_1^r(q;(A,B))$ and $e_iB\bar{q} = e_jBq$ for every $e_i, e_j \in S_1^r(\bar{q};(A,B))$.

and

(ii) Either $e_iB\bar{q} > e_jBq$ for all $e_i \in S_1^l(\bar{q};(A,B))$ and $e_j \in S_1^l(q;(A,B))$, or
$e_iB\bar{q} = e_jBq$ for all $e_i \in S_1^l(\bar{q};(A,B))$, $e_j \in S_1^l(q;(A,B))$, and $S_1^l(\bar{q};(A,B)) \subsetneq S_1^l(q;(A,B))$.

Using the fact that (A,B) is unilaterally competitive game and $S_1^r(q;(A,B)) = S_1^r(q;A)$ for all $r \in \mathbb{N}$ we find

(i') For $r < l$, $S_1^r(\bar{q};A) = S_1^r(q;A)$ and $e_i(-A)\bar{q} = e_j(-A)q$ for every $e_i, e_j \in S_1^r(\bar{q};A)$.

and

(ii') Either $e_i(-A)\bar{q} > e_j(-A)q$ for all $e_i \in S_1^l(\bar{q};A)$ and $e_j \in S_1^l(q;A)$, or
$e_i(-A)\bar{q} = e_j(-A)q$ for all $e_i \in S_1^l(\bar{q};A)$ and $e_j \in S_1^l(q;A)$, and $S_1^l(\bar{q};A) \subsetneq S_1^l(q;A)$.

Hence, $\bar{q}$ protectively dominates q in A. Similarly, one derives that every protectively dominated strategy for player 2 in A is also protectively dominated in (A,B). □

From Lemma 4.3, Lemma 4.4, and the result in Fiestras-Janeiro et al. (1998), one obtains

Theorem 4.5. For any unilaterally competitive game (A,B)

$$\mathrm{PROP}(A,B) = \mathrm{PROT}(A,B).$$

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