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Compensations in information collecting situations[1]

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Abstract

How to compensate people who provide relevant information to a decision-maker who faces uncertainty? This paper suggests some compensation rules. These are studied both in a cooperative and a noncooperative environment. *JEL codes:* C71, C72, D89.

Key Words: information collecting, games, core, Shapley value, equilibrium in dominant strategies.

1 Introduction

When dealing with decision-making under uncertainty, an agent can often consult people who can give him information to reduce the uncertainty. But how should he compensate his informants? This will be the main question of this paper.

Information collecting situations are introduced in Brânzei, Tijs and Timmer (2000a,b). In these situations a single decision-maker faces uncertainty, that is, if all possible states of the world are represented by the set Ω then this decision-maker does not know which state is the true one. Because his reward depends on both the action he takes and the true state, consulting agents who can have extra information is desirable.

The information of agent i is represented by a finite partition $\mathcal{I}_i$ of the set Ω. An element of such a partition is called an event. If two states are in the

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same element of the partition then this agent cannot distinguish them. Let $I_i(\omega)$ denote that element of $\mathcal{I}_i$ in which the state $\omega \in \Omega$ lies. If ω is the true state then agent i knows that the event $I_i(\omega)$ happens.

Concerning the true state we distinguish two situations. First, we consider situations where the true state ω has obtained. The informants tell the decision-maker that $I_i(\omega)$ happens. So, they provide 'local' information concerning the true state. Based on this information the decision-maker takes an action and he shares the resulting reward in some way with the informants. This is the topic of the next section.

Second, if the true state has to obtain then the decision-maker starts again by contacting the informants. He asks each of them to share their information partition ('global' information) with him now and after the true state obtains to tell in which part of their partition the true state lies. In exchange he pays them for this information before the true state obtains. This payoff is based on his expected reward obtained by using the global information. Finally, if the true state obtains, the informants provide their local information to the decision-maker, as was agreed. This information is used by the decision-maker to take an action. We discuss this kind of situations in section 3.

In the sections 2 and 3 about local and global information, respectively, suggestions for compensations are provided. These compensation rules are based on the idea that an informant should receive (a part of) the marginal contribution generated by his information.

Following this, section 4 studies information collecting (IC) situations in a game-theoretical context. We define both cooperative and noncooperative games related to IC situations. The cooperative (IC) games are already partly studied in Brânzei, Tijs and Timmer (2000a,b). We show that one of the compensation rules of section 2 coincides with the Shapley value (Shapley (1953)) of a particular game. For the noncooperative game we show that dividing the total reward according to the compensation rule corresponds to the payoff of an equilibrium in dominant strategies. An example shows that this need not correspond to the payoff of a strong equilibrium.

The final section contains a brief study of the compensation rule in (cooperative) IC games.

2 Starting cooperation after the true state obtains

An adventurer searches an island for a big treasure with lots of gold and jewels. His treasure map says that this treasure can be found in one of the locations 1, 2, 3, or 4 but it fails to specify which one. The adventurer has no information about the true location and gets one chance to guess the location of the treasure. Fortunately he can consult three wise men. These men are located at the north, southeast and southwest of the island, respectively. Each of them has some information about the true location of the treasure. Hence, it will be good for the adventurer to visit these wise men.

This is an example of a situation where a single decision-maker has to take an action in an uncertain situation. Such situations can be modeled by a tuple

$$< N, k, (\Omega, \mathcal{F}, \mu), \{\mathcal{I}_i\}_{i \in N}, A_k, r_k > .$$

Let agent k be the action taker. The set N is the set of all agents including k who have some information about the true state ω. Ω is the set of all possible states, $\mathcal{F}$ is a σ-algebra on Ω and μ a probability measure on Ω. The information of player i is represented by $\mathcal{I}_i$, a finite partition of Ω into $\mathcal{F}$-measurable sets with positive measure. The action taker chooses an action a from the set A_k and if ω is the true state then his reward equals $r_k(\omega, a)$.

If players in S share information then the global information of S is given by

$$\mathcal{I}_S = \{\cap_{i \in S} I_i \,|I_i \in \mathcal{I}_i,\ \cap_{i \in S} I_i \neq \emptyset\}\ ,$$

the coarsest common refinement of the information partitions $\mathcal{I}_i$, $i \in S$.

Let $I_S(\omega)$ be that element of the partition $\mathcal{I}_S$ that contains ω. If ω obtains then S knows the event $I_S(\omega)$. The reward of each coalition S with $k \in S$ is given by

$$R_{I(\omega)}(S) = (\mu(I_S(\omega)))^{-1} \sup_{a \in A_k} \int_{I_S(\omega)} r_k(\omega', a) d\mu.$$

The reward function $R_{I(\omega)}$ indicates that ω has obtained and therefore we only consider $I_S(\omega)$.

Example 2.1 The example at the beginning of this section can be modeled by the tuple

$$< N, k, (\Omega, \mathcal{F}, \mu), \{\mathcal{I}_i\}_{i \in N}, A_k, r_k >$$

where $N = \{1, 2, 3, 4\}$ is the set of wise men $\{1, 2, 3\}$ together with the adventurer (decision-maker) $k = 4$. The treasure is hidden in one of the locations in $\Omega = \{1, 2, 3, 4\}$. Let $\mathcal{F}$ be the set containing all subsets of Ω. All locations are equally likely for the adventurer and therefore $\mu(E) = |E|/4$ for all $E \in \mathcal{F}$.

The adventurer has no information about the true location, so $\mathcal{I}_4 = \{\Omega\}$. The wise men have the following information partitions: $\mathcal{I}_1 = \{\{1\}, \{2, 3, 4\}\}$, $\mathcal{I}_2 = \{\{2\}, \{1, 3, 4\}\}$ and $\mathcal{I}_3 = \{\{3\}, \{1, 2, 4\}\}$. The adventurer has to guess the location of the treasure. Thus $A_4 = \{1, 2, 3, 4\}$. If $\omega \in \Omega$ is the true location of the treasure then the reward $r_4(\omega, a)$ equals 1 if $\omega = a$, that is, the adventurer finds the treasure, and it equals 0 otherwise.

One way for agent k to collect information is to invite the other agents one by one and to have them reveal their information $I_i(\omega)$. In this *sequential approach* to collecting information, the decision-maker, agent k, starts with his own information $I_k(\omega)$ that gives the expected reward

$$R_{I(\omega)}(\{k\}) = (\mu(I_k(\omega)))^{-1} \sup_{a \in A_k} \int_{I_k(\omega)} r_k(\omega', a) d\mu.$$

Let $\sigma(1) \in N \setminus \{k\}$ be the first agent to be invited by agent k. The marginal contribution of this agent, that is, the increase in reward obtained by using the information of this agent, is

$$R_{I(\omega)}(\{k, \sigma(1)\}) - R_{I(\omega)}(\{k\}).$$

Let $P_\sigma(i) = \{j \in N \setminus \{k\} \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$ be the set of agents that are invited before agent i. Now the marginal contribution of agent i in the order σ when ω is the true state is

$$m_i^\sigma(\omega) = R_{I(\omega)}(\{k, i\} \cup P_\sigma(i)) - R_{I(\omega)}(\{k\} \cup P_\sigma(i)).$$

Agent k might announce what the order of invitations is. In exchange for their information, agent k pays the informants a fraction of their marginal contribution $\alpha_i m_i^\sigma(\omega)$ with $0 \leq \alpha_i \leq 1$. The remainder of the expected reward will be for agent k himself.

Example 2.2 The adventurer announces that he will first visit wise man 1, then 3 and finally 2. Thus he announces the order σ with $\sigma(1) = 1$, $\sigma(2) = 3$ and $\sigma(3) = 2$. Furthermore, he gives each wise man two-third of this marginal contribution. If the treasure is located at $\omega = 4$ then

$$\frac{2}{3}m_1^\sigma(4) = \frac{2}{3}(R_{I(4)}(\{1,4\}) - R_{I(4)}(\{4\})) = 2/3(1/3 - 1/4) = 1/18,$$

$\frac{2}{3}m_3^\sigma(4) = 1/9$ and $\frac{2}{3}m_2^\sigma(4) = 1/3$. From $R_{I(4)}(N) = 1$ there remains $1/2$ for the adventurer.

Suppose now that instead of announcing a fixed order σ, the decision-maker does not tell which order σ will be used. He offers each informant a fraction of his average marginal contribution $\sum_{\sigma \in \Pi_k(N)} m_i^\sigma(\omega)/(n-1)!$, where $\Pi_k(N)$ is the set of all orders σ on $N \setminus \{k\}$. Once again, all that remains of the expected reward $R_{I(\omega)}(N)$ is for agent k.

Example 2.3 If $\omega = 4$ then the adventurer needs all three wise men to find the treasure, no matter in what order he visits them. One can easily calculate that $m_{\sigma(1)}^\sigma(4) = 1/12$, $m_{\sigma(2)}^\sigma(4) = 1/6$ and $m_{\sigma(3)}^\sigma(4) = 1/2$ for any order σ on $\{1,2,3\}$. It readily follows that the average marginal contribution is $\sum_{\sigma \in \Pi_k(N)} m_i^\sigma(4)/3! = 1/4$ for any wise man i. If these men receive two-third of this contribution then any wise man receives $1/6$ while the adventurer gets $1/2$.

An alternative way of collecting information is the *one-shot approach*. The name refers to the simultaneous revelation of information. The decision-maker invites a group S of informants, $S \subset N \setminus \{k\}$, to share their information with him. This information sharing results in the expected reward $R_{I(\omega)}(\{k\} \cup S)$. Agent k pays each of the informants a fraction of their marginal contribution to the other selected agents and agent k. Thus if i is one of the selected agents in S then he receives a fraction of

$$R_{I(\omega)}(\{k\} \cup S) - R_{I(\omega)}(\{k\} \cup S \setminus \{i\}).$$

Naturally, if an agent is not selected to share his information with the decision-maker then he receives zero.

Example 2.4 The adventurer wants to visit the wise men 2 and 3. He pays them two-third of their marginal contribution. Hence, wise man 2 receives

$$\frac{2}{3}(R_{I(4)}(\{2,3,4\}) - R_{I(4)}(\{3,4\})) = 2/3(1/2 - 1/3) = 1/9,$$

if the treasure is located at $\omega = 4$. Wise man 3 also receives $1/9$. Since wise man 1 is not selected, he receives zero. The adventurer receives the remainder $R_{I(4)}(\{2,3,4\}) - 1/9 - 1/9 = 5/18$.

3 Starting cooperation before the true state obtains

In the previous section we discussed information collecting situations where the true state ω had already obtained, that is, where each coalition S of agents knows that the event $I_S(\omega)$ happens. But one can also think of situations where the decision-maker has to decide with whom to cooperate before the state ω obtains. Consequently, compensations for the informants will be based on expected rewards. In this section we take a closer look at this kind of situations. The following example is taken from Brânzei, Tijs and Timmer (2000a).

Example 3.1 A fair die is thrown and agent 3 has the possibility to guess the outcome or not to participate in this gamble. If this player makes a correct guess then he receives 60, if he makes a wrong guess he has to pay 18. In case he decides not to participate in this gamble his payoff is 0. Without extra information this is not an attractive gamble for agent 3. There are however two possible informants, the agents 1 and 2. Agent 1 is told whether the outcome of the die is low or high, and agent 2 is told whether the outcome is odd or even.

We can model this as a tuple

$$< N, k, (\Omega, \mathcal{F}, \mu), \{\mathcal{I}_i\}_{i \in N}, A_k, r_k >$$

with $N = \{1, 2, 3\}$, agent $k = 3$ is the action taker, $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F}$ contains all the subsets of Ω and $\mu(E) = |E|/6$ for each $E \in \mathcal{F}$. The information partitions are $\mathcal{I}_1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$, $\mathcal{I}_2 = \{\{1, 3, 5\}, \{2, 4, 6\}\}$, and $\mathcal{I}_3 = \{\Omega\}$. Agent 3's action set is $A_3 = \{1, 2, 3, 4, 5, 6, n\}$, where n stands for not participating in the gamble. The reward function is given by $r_3(\omega, n) = 0$ for all $\omega \in \Omega$ and for $a \in \{1, 2, ..., 6\}$ we have $r_3(\omega, a) = 60$ if $a = \omega$ and $r_3(\omega, a) = -18$ otherwise.

Working on his own yields the action-taker k an expected reward of

$$R_{\mathcal{I}}(\{k\}) = \sum_{I \in \mathcal{I}_k} \sup_{a \in A_k} \int_I r_k(\omega, a) d\mu.$$

The subscript $\mathcal{I}$ in the reward function $R_{\mathcal{I}}$ indicates that ω has not obtained and therefore we use the whole information partition. This reward can be improved upon by collecting information from the informants. For this, we can use the same approaches as in the previous section.

First, there is the sequential approach to collecting information. The decision-maker invites the informants one by one to share their information with him. Each informant receives a fraction of his marginal contribution. This marginal contribution may be interpreted as the marginal value of his information given the information that is already known. If σ is the order in which the decision-maker k invites the informants then the marginal contribution of agent i is

$$m_i^\sigma = R_{\mathcal{I}}(\{i, k\} \cup P_\sigma(i)) - R_{\mathcal{I}}(\{k\} \cup P_\sigma(i)).$$

Notice that

$$m_i^\sigma = \int_\Omega m_i^\sigma(\omega) d\mu.$$

The decision-maker k keeps the remainder of $R_{\mathcal{I}}(N)$ after he paid all the informants a fraction of their marginal contribution.

Instead of announcing a fixed order σ and paying the informants accordingly, agent k can also pay the informants a fraction of their average marginal contribution

$$\sum_{\sigma \in \Pi_k(N)} m_i^\sigma/(n-1)!.$$

After these payments are done, the remainder of $R_{\mathcal{I}}(N)$ will be for the decision-maker.

Example 3.2 Consider the situation in example 3.1. Agent 3 decides to pay the agents 1 and 2 half of their marginal contribution. Thus agent 1 receives half of

$$\sum_{\sigma \in \Pi_k(N)} m_1^\sigma/2! = (8+26)/2 = 17$$

and agent 2 also receives 17/2. The remaining 17 of $R_{\mathcal{I}}(N) = 34$ is for agent 3.

Second, there is the one-shot approach. The decision-maker invites a group S of agents to share their information with him. In exchange an informant $i \in S$ receives a fraction of his marginal contribution

$$R_{\mathcal{I}}(\{k\} \cup S) - R_{\mathcal{I}}(\{k\} \cup S \setminus \{i\}).$$

Agents that are not selected receive nothing. The decision-maker collects all that remains of $R_{\mathcal{I}}(\{k\} \cup S)$.

Example 3.3 With only two informants being present, there is not much to select for agent 3 in the previous example. Assume he selects both the informants and gives them half of their marginal contribution

$$R_{\mathcal{I}}(N) - R_{\mathcal{I}}(N \setminus \{i\}) = 34 - 8 = 26,$$

$i = 1, 2$. In this case the informants both receive 13 and agent 3 keeps the remaining 8.

Besides working with expected values as the former two approaches do, we may also keep the uncertainty as it is. This way we arrive at an IC-situation with *random payoffs*. Timmer, Borm and Tijs (2000) introduce a model to analyse this kind of situations. In their model $X(S) : \Omega \to \mathbb{R}_+$ is the random payoff[4] to coalition S of agents. In case of an information collecting situation, these payoffs can be defined as follows. If the decision-maker k is not present in the coalition S of agents, then $X(S) = 0$, that is, this coalition S receives the payoff zero for sure. Otherwise, if the decision-maker is present, then we define

$$X(S)(\omega) = R_{I(\omega)}(S)$$

for all $\omega \in \Omega$. The payoff $X(S)$ takes the value $X(S)(\omega)$ with probability $\mu(\omega)$. Notice that the expected value of $X(S)$ equals $R_{\mathcal{I}}(S)$.

The example below illustrates this model with random payoffs.

[4]In Timmer et al. (2000) this payoff is denoted by $R(S)$. To avoid confusion, we changed it to $X(S)$.

Example 3.4 Consider once again the situation in example 3.1. We will calculate the random payoffs corresponding to this IC situation.

Without agent 3 the informants cannot earn anything, so the corresponding random payoffs are $X(\{1\}) = X(\{2\}) = X(\{1,2\}) = 0$. If agent 3 does not consult the informants then the best thing for him to do is not to participate in this gamble, no matter what the outcome of the die would be. Hence, $X(\{3\})(\omega) = 0$ for all $\omega \in \Omega$, or equivalently $X(\{3\}) = 0$. In a similar fashion we obtain $X(\{1,3\}) = X(\{2,3\}) = 8$. Finally

$$X(N)(\omega) = \begin{cases} 21, & \omega \in \{1,3,4,6\} \\ 60, & \omega \in \{2,5\} \end{cases}$$

is the random payoff in case all the agents work together.

4 Cooperative and noncooperative games

In this section we introduce two types of games arising from information collecting situations. First we consider cooperative information collecting (IC) games. Depending on whether or not ω is obtained at the moment of deciding with whom to cooperate we arrive at respectively local IC games and (global) IC games. This is done in the following subsection. Hereafter we turn our attention to noncooperative games. The decision-maker has to decide whom to work with and the informants should decide whether to offer their information or not. Once again depending upon ω being obtained or not, we arrive at so-called local entry games or (global) entry games. These noncooperative games will be introduced in subsection 4.2.

4.1 Information collecting games

A cooperative game (with transferable utility) is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ is the finite set of players and v is a function that assigns to any coalition $S \subset N$ a real number that represents the worth of this coalition. By convention $v(\emptyset) = 0$.

Local information collecting (IC) games are cooperative games that correspond to IC situations where ω has already obtained, as discussed in section 2. These local IC games are introduced and studied in Brânzei, Tijs and Timmer (2000b). Let

$$< N, k, (\Omega, \mathcal{F}, \mu), \{\mathcal{I}_i\}_{i \in N}, A_k, r_k >$$

be an IC situation and assume that $\omega \in \Omega$ has obtained. The corresponding local IC game $(N, v_{\omega,k})$ is defined as follows. The player set N is the set of all the informants and the decision-maker. Let $S \subset N$ be a coalition of players. If the decision-maker k is not a member of this coalition then $v_{\omega,k}(S) = 0$. Otherwise

$$v_{\omega,k}(S) = R_{I(\omega)}(S) = (\mu(I_S(\omega)))^{-1} \sup_{a \in A_k} \int_{I_S(\omega)} r_k(\omega', a) d\mu.$$

If ω has not obtained then we arrive at (global) IC games, which are introduced in Brânzei, Tijs and Timmer (2000a). Given an IC situation the corresponding

IC game (N, v_k) is defined by $v_k(\emptyset) = 0$, $v_k(S) = 0$ if $k \notin S$ and

$$v_k(S) = R_{\mathcal{I}}(S) = \sum_{I \in \mathcal{I}_S} \sup_{a \in A_k} \int_I r_k(\omega, a) d\mu$$

if $k \in S$. Closely related to this game is the game $(N \setminus \{k\}, v_{-k,\alpha})$ with

$$v_{-k,\alpha}(S) = \alpha(v_k(S \cup \{k\}) - v_k(\{k\}))$$

for all $S \subset N \setminus \{k\}$. The player set of this game is the set of all informants and the value $v_{-k,\alpha}(S)$ of a group S of informants is a fraction α of its contribution to the decision-maker k. This game is useful in determining the payoff of an informant if ω has to obtain and if the decision-maker uses the sequential approach without announcing a fixed order σ. It turns out that this payoff coincides with the Shapley value ϕ (Shapley (1953)) of the game $(N \setminus \{k\}, v_{-k,\alpha})$.

Theorem 4.1 *If $\alpha_i = \alpha$ for all $i \in N \setminus \{k\}$ then $\alpha_i \sum_{\sigma \in \Pi_k(N)} m_i^\sigma/(n-1)! = \phi_i(v_{-k,\alpha})$.*

Proof. Let $\alpha_i = \alpha$ for all $i \in N \setminus \{k\}$ and let $\sigma \in \Pi_k(N)$. Then

$$\begin{aligned} m_i^\sigma(v_{-k,\alpha}) &= v_{-k,\alpha}(\{i\} \cup P_\sigma(i)) - v_{-k,\alpha}(P_\sigma(i)) \\ &= \alpha(v_k(\{i,k\} \cup P_\sigma(i)) - v_k(\{k\} \cup P_\sigma(i))) \\ &= \alpha m_i^\sigma. \end{aligned}$$

Therefore, the payoff to informant i equals

$$\alpha_i \sum_{\sigma \in \Pi_k(N)} \frac{m_i^\sigma}{(n-1)!} = \sum_{\sigma \in \Pi_k(N)} \frac{m_i^\sigma(v_{-k,\alpha})}{(n-1)!} = \phi_i(v_{-k,\alpha}),$$

where the last equality follows from the definition of the Shapley value. ■

The marginal contribution of informant i to $S \setminus \{i\}$, $i \in S$, is

$$R_{\mathcal{I}}(S) - R_{\mathcal{I}}(S \setminus \{i\}).$$

An IC situation satisfies the *decreasing marginals* property if each informant i contributes more to a smaller group than to a larger group, that is,

$$R_{\mathcal{I}}(S) - R_{\mathcal{I}}(S \setminus \{i\}) \geq R_{\mathcal{I}}(T) - R_{\mathcal{I}}(T \setminus \{i\}) \tag{1}$$

for all $i \in N \setminus \{k\}$ and all S, T with $i, k \in S \subset T \subset N$. Denote by DM_k the set of IC situations that satisfy (1). Define $M_i(S, v_k) = v_k(S) - v_k(S \setminus \{i\})$ as the marginal contribution of player $i \in S$ to $S \setminus \{i\}$. Notice that $M_i(S, v_k) \geq 0$. IC situations in DM_k lead to games (N, v_k) that satisfy

$$M_i(S, v_k) \geq M_i(T, v_k) \tag{2}$$

for all $i \in N$ and $i, k \in S \subset T \subset N$. This is the so-called *k-concavity* property. Brânzei, Tijs and Timmer (2000b) show that (2) is equivalent to the *total union* property, defined by

$$v_k(T) - v_k(S) \geq \sum_{i \in T \setminus S} M_i(T, v_k) \tag{3}$$

for all $S \subset T \subset N$ with $k \in S$. If (N, v_k) is the IC game corresponding to an IC situation in DM_k, then

$$C(N, v_k) = \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{l} 0 \leq x_i \leq M_i(N, v_k),\ i \in N \setminus \{k\}; \\ \sum_{i \in N} x_i = v_k(N) \end{array} \right\} \quad (4)$$

is the core of this game (cf. Muto, Nakayama, Potters and Tijs (1988)).

4.2 Entry games

Entry games are noncooperative games in strategic form derived from information collecting situations. A game in strategic form is a tuple $(N, (X_i, u_i)_{i \in N})$ where N is the player set. Each player $i \in N$ has a set X_i of strategies and a payoff function u_i that assigns to each vector $x = (x_i)_{i \in N}$ of strategies a real number $u_i(x)$.

Let

$$< N, k, (\Omega, \mathcal{F}, \mu), \{\mathcal{I}_i\}_{i \in N}, A_k, r_k >$$

be an IC situation and assume that $\omega \in \Omega$ has obtained. The corresponding local entry game $E^{\alpha}_{k,\omega}$ for $\alpha \in [0,1]^{N \setminus \{k\}}$ is the tuple $(N, (X_i, u_i)_{i \in N})$ with N the set of informants and the decision-maker k. This decision-maker selects a group of agents with whom he wants to cooperate. His strategy set is $X_k = 2^{N \setminus \{k\}}$. An informant i can choose whether to 'enter' or not, where 'enter' stands for providing information to the decision-maker. Hence, $X_i = \{0, 1\}$, $i \in N \setminus \{k\}$, where 1 means 'enter' and 0 means 'do not enter'.

Let $x = (x_i)_{i \in N}$ be a strategy profile with $x_i \in X_i$. Those informants that are selected by player k and are willing to cooperate, form together with the decision-maker the set of players $S(x)$ that will cooperate:

$$S(x) = \{i \in x_k \mid x_i = 1\} \cup \{k\}.$$

The payoff to informant i equals

$$u_i(x) = \begin{cases} 0 & ,\ i \notin S(x), \\ \alpha_i[R_{I(\omega)}(S(x)) - R_{I(\omega)}(S(x) \setminus \{i\})] & ,\ i \in S(x). \end{cases}$$

If informant i is not selected by k or if he does not want to enter, then he receives a payoff of zero. Otherwise he receives the fraction α_i of his marginal contribution to the other players in $S(x) \setminus \{i\}$.

The decision-maker receives all that remains of the total reward:

$$u_k(x) = R_{I(\omega)}(S(x)) - \sum_{i \in S(x) \setminus \{k\}} u_i(x).$$

It may also be the case that ω has not yet obtained. Now we define a corresponding (global) entry game E^{α}_{k} with $\alpha \in [0,1]^{N \setminus \{k\}}$. This game is represented by the tuple $(N, (X_i, u_i)_{i \in N})$ where N and X_i, $i \in N$, have the same meaning as before. The payoff functions change because ω is not known yet. For an informant i we have

$$u_i(x) = \begin{cases} 0 & ,\ i \notin S(x) \\ \alpha_i[R_{\mathcal{I}}(S(x)) - R_{\mathcal{I}}(S(x) \setminus \{i\})] & ,\ i \in S(x). \end{cases}$$

Once again, if informant i is not selected by the decision-maker or if he does not want to enter then his payoff will be zero. Otherwise, he receives the fraction α_i of his marginal contribution to $S(x) \setminus \{i\}$ in case ω has not yet obtained. The remainder

$$u_k(x) = R_{\mathcal{I}}(S(x)) - \sum_{i \in S(x) \setminus \{k\}} u_i(x)$$

goes to the decision-maker k.

Notice that, using IC games, there is an easier way to write the payoff function. Let (N, v_k) be an IC game that corresponds to the same IC situation as the game E_k^{α}. Recall that

$$M_j(S, v_k) = v_k(S) - v_k(S \setminus \{j\}) = R_{\mathcal{I}}(S) - R_{\mathcal{I}}(S \setminus \{j\}).$$

Then we can write for all informants i

$$u_i(x) = \begin{cases} 0 & , \ i \notin S(x), \\ \alpha_i M_i(S(x), v_k) & , \ i \in S(x) \end{cases}$$

and

$$u_k(x) = v_k(S(x)) - \sum_{i \in S(x) \setminus \{k\}} \alpha_i M_i(S(x), v_k)$$

for the decision-maker. Similar things can be done for local IC games and local entry games.

Let $\bar{x}$ be the strategy profile where the decision-maker k selects all the informants, i.e. $\bar{x}_k = N \setminus \{k\}$, and every informant wants to enter, i.e. $\bar{x}_i = 1$ for all $i \in N \setminus \{k\}$. This strategy profile is an equilibrium in dominant strategies (cf. Van Damme (1987)) of the game E_k^{α} if this game corresponds to an IC situation in DM_k.

Theorem 4.2 *For all $\alpha \in [0,1]^{N \setminus \{k\}}$ and all games E_k^{α} corresponding to an IC situation in DM_k, $\bar{x}$ is an equilibrium in dominant strategies.*

Proof. Let $\alpha \in [0,1]^{N \setminus \{k\}}$ and let $i \in N \setminus \{k\}$. First, take $x_{-i} \in \Pi_{j \in N \setminus \{i\}} X_j$ and $x_i \in X_i$. We show that $u_i(x_{-i}, \bar{x}_i) \geq u_i(x_{-i}, x_i)$.

If $i \notin x_k$ then $u_i(x_{-i}, \bar{x}_i) = 0 = u_i(x_{-i}, x_i)$. Otherwise, if $i \in x_k$ then define $S = \{j \in x_k | x_j = 1\}$. Now

$$u_i(x_{-i}, \bar{x}_i) = \alpha_i M_i(S \cup \{k\}, v_k) \begin{cases} = u_i(x_{-i}, x_i), & x_i = 1 \\ \geq u_i(x_{-i}, x_i) = 0, & x_i = 0. \end{cases}$$

Next, let $x_{-k} \in \Pi_{j \in N \setminus \{k\}} X_j$ and $x_k \in X_k$. Define $\bar{S} = \{i \in \bar{x}_k | x_i = 1\}$. Note that $\bar{S} \supset S$ because $\bar{x}_k = N \setminus \{k\} \supset x_k$. We derive the following.

$$\begin{aligned} u_k(x_{-k}, x_k) &= v_k(S \cup \{k\}) - \sum_{i \in S} \alpha_i M_i(S \cup \{k\}, v_k) \\ &\leq \ v_k(S \cup \{k\}) - \sum_{i \in S} \alpha_i M_i(S \cup \{k\}, v_k) + \sum_{i \in \bar{S} \setminus S} (1 - \alpha_i) M_i(\bar{S} \cup \{k\}, v_k) \\ &\leq \ v_k(S \cup \{k\}) - \sum_{i \in S} \alpha_i M_i(\bar{S} \cup \{k\}, v_k) + \sum_{i \in \bar{S} \setminus S} (1 - \alpha_i) M_i(\bar{S} \cup \{k\}, v_k) \end{aligned}$$

$$
\begin{aligned}
&= v_k(S \cup \{k\}) + \sum_{i \in \bar{S} \setminus S} M_i(\bar{S} \cup \{k\}, v_k) - \sum_{i \in \bar{S}} \alpha_i M_i(\bar{S} \cup \{k\}, v_k) \\
&\leq v_k(\bar{S} \cup \{k\}) - \sum_{i \in \bar{S}} \alpha_i M_i(\bar{S} \cup \{k\}, v_k) \\
&= u_k(x_{-k}, \bar{x}_k).
\end{aligned}
$$

The first inequality follows from $M_i(T, v_k) \geq 0$ for all i and all $T \supset \{i, k\}$. The second and the third inequality follow from (2) and (3), respectively. ■

One may wonder whether $\bar{x}$ is a strong equilibrium, that is, whether

$$
u(\bar{x}_{-S}, x_S) \leq u(\bar{x})
$$

for all coalitions S. The following example shows that this is not true.

Example 4.1 Consider the situation in example 3.1 but assume now that both the informants are told whether the outcome is lower than 4 or higher than 3. The corresponding IC game is the game (N, v_3) with $v_3(\emptyset) = v_3(\{1\}) = v_3(\{2\}) = v_3(\{1,2\}) = 0$, $v_3(\{3\}) = 0$, $v_3(\{1,3\}) = v_3(\{2,3\}) = v_3(N) = 8$. This game belongs to DM_3.

Now consider the corresponding entry game E_k^α with $\alpha \in [0,1]^{\{1,2\}}$. Strategy profile $\bar{x}$ with $\bar{x}_1 = \bar{x}_2 = 1$ and $\bar{x}_3 = \{1,2\}$ is not a strong equilibrium. We will show why not. The payoffs corresponding to this strategy profile are

$$
u_1(\bar{x}) = u_2(\bar{x}) = 0, \; u_3(\bar{x}) = v_3(N) = 8
$$

because $M_i(N, v_3) = 0$ for $i = 1, 2$; both informants have the same information and therefore their marginal contribution equals zero. Suppose that the informants 1 and 2 deviate together to the strategies $x_1 = 0$ and $x_2 = 1$. Now

$$
0 = u_1(x_1, x_2, \bar{x}_3) = u_1(\bar{x})
$$

because 1 decided not to enter but

$$
\begin{aligned}
u_2(x_1, x_2, \bar{x}_3) &= \alpha_2 M_2(\{2,3\}, v_3) \\
&= \alpha_2(v_3(\{2,3\}) - v_3(\{3\})) \\
&= \alpha_2 \cdot 8 \\
&> 0 = u_2(\bar{x})
\end{aligned}
$$

if $\alpha_2 > 0$. Hence, $\bar{x}$ is not a strong equilibrium.

To conclude this section we show that core-elements of a game associated to an IC situation in DM_k are closely related to equilibria in the related entry game E_k^α.

Theorem 4.3 *Let $B \in DM_k$ be an IC situation with related IC game (N, v_k) and entry games E_k^α, $\alpha \in [0,1]^{N \setminus \{k\}}$. Then for all $x \in C(v_k)$ there is an $\alpha \in [0,1]^{N \setminus \{k\}}$ such that x is a vector of equilibrium payoffs corresponding to an equilibrium in dominant strategies of the game E_k^α.*

Proof. Let $x \in C(v_k)$. According to (4) we have $0 \leq x_i \leq M_i(N, v_k)$, $i \neq k$, and $x_k = v_k(N) - \sum_{i \in N \setminus \{k\}} x_i$. Because of this there exists an $\alpha \in [0,1]^{N \setminus \{k\}}$ such that $x_i = \alpha_i M_i(N, v_k)$ for all $i \in N \setminus \{k\}$. Let $\bar{x}$ be the strategy profile defined by $\bar{x}_i = 1$, $i \neq k$, and $\bar{x}_k = N \setminus \{k\}$. According to theorem 4.2 $\bar{x}$ is an equilibrium in dominant strategies of the game E_k^α. The equilibrium payoffs are $u_i(\bar{x}) = x_i$, for $i \neq k$, and $u_k(\bar{x}) = x_k$. ■

5 Marginal based allocation rules

In the previous section we often used the following rule to divide the total reward over the decision-maker k and the informants $i \in N \setminus \{k\}$. An informant i receives $\alpha_i M_i(N, v_k)$, a fraction of his marginal contribution to $N \setminus \{i\}$. After we have done so for all the informants, the remainder goes to the decision-maker, who thus receives $v_k(N) - \sum_{i \in N \setminus \{k\}} \alpha_i M_i(N, v_k)$. We call these rules that depend upon $\alpha \in [0,1]^{N \setminus \{k\}}$ *marginal based allocation rules.* The remainder of this section is devoted to studying properties of these rules.

Let (N, v_k) be an IC game corresponding to an IC situation with decision-maker k. Denote by

$$A(v_k) = \left\{ x \in \mathbb{R}^N \,\middle|\, \begin{array}{l} \exists \alpha \in [0,1]^{N \setminus \{k\}} : x_i = \alpha_i M_i(N, v_k), \\ i \in N \setminus \{k\};\ \sum_{j \in N} x_j = v_k(N) \end{array} \right\}$$

the set of all marginal based allocation rules in this game. This set contains the core of (N, v_k), that is, it is a so-called core catcher.

Theorem 5.1 *$A(v_k) \supset C(v_k)$ for all IC games (N, v_k).*

Proof. Let (N, v_k) be an IC game. Then $v_k(S) = 0$ if $k \notin S$. Let $x \in C(v_k)$. For $i \neq k$ we have $0 = v_k(\{i\}) \leq x_i$ and

$$x_i = \sum_{j \in N} x_j - \sum_{j \in N \setminus \{i\}} x_j \leq v_k(N) - v_k(N \setminus \{i\}) = M_i(N, v_k).$$

Hence there exists an $\alpha_i \in [0,1]$ such that $x_i = \alpha_i M_i(N, v_k)$. The core-condition $\sum_{j \in N} x_j = v_k(N)$ implies $x_k = v_k(N) - \sum_{i \in N \setminus \{k\}} \alpha_i M_i(N, v_k)$. ■

If the IC game (N, v_k) is a big boss game, that is, it satisfies

$$v_k(N) - v_k(S) \geq \sum_{i \in N \setminus S} M_i(N, v_k) \tag{5}$$

for all $S \subset N$ with $k \in S$, then $A(v_k)$ and $C(v_k)$ coincide.

Theorem 5.2 *$A(v_k) = C(v_k)$ for all IC games (N, v_k) that satisfy (5).*

Proof. In theorem 5.1 we have shown $A(v_k) \supset C(v_k)$. It remains to show $A(v_k) \subset C(v_k)$.

Let $x \in A(v_k)$ and let $\alpha^x \in [0,1]^{N \setminus \{k\}}$ be such that $x_i = \alpha_i^x M_i(N, v_k)$ for $i \neq k$. First, x satisfies $\sum_{j \in N} x_j = v_k(N)$. Second, we show that $\sum_{j \in S} x_j \geq v_k(S)$ for all nonempty coalitions S.

Let S be a nonempty subset of N. If $k \notin S$ then

$$\sum_{j \in S} x_j = \sum_{j \in S} \alpha_j^x M_j(N, v_k) \geq 0 = v_k(S).$$

On the other hand, if $k \in S$ then

$$\begin{aligned} \sum_{j \in S} x_j &= \sum_{j \in S \setminus \{k\}} \alpha_j^x M_j(N, v_k) + v_k(N) - \sum_{j \in N \setminus \{k\}} \alpha_j^x M_j(N, v_k) \\ &= v_k(N) - \sum_{j \in N \setminus S} \alpha_j^x M_j(N, v_k) \\ &\geq v_k(N) - \sum_{j \in N \setminus S} M_j(N, v_k) \\ &\geq v_k(S) \end{aligned}$$

with the first inequality following from $M_j(N, v_k) \geq 0$ and the second one from (5). We conclude that $x \in C(v_k)$. ■

The theorems 5.1 and 5.2 have some similarities with the results in Monderer, Samet and Shapley (1992). There the set of weighted values is a core catcher and it coincides with the core if the game is convex. Our results are that the set $A(v_k)$ of marginal based allocation rules is a core catcher and it coincides with the core in case v_k is a big boss game.

An IC game (N, v_k) is a so-called total big boss game if it satisfies

$$v_k(T) - v_k(S) \geq \sum_{i \in T \setminus S} M_i(T, v_k)$$

for all $S \subset T \subset N$ with $k \in S$. Hence, (5) is satisfied and $A(v_k) = C(v_k)$ according to theorem 5.2. One particular core-element is given by Muto et al. (1988) who show that for total big boss games the τ-value (cf. Tijs (1981)) and the nucleolus (cf. Schmeidler (1969)) coincide and are equal to the marginal based allocation rule with $\alpha_i = 1/2$ for all $i \in N \setminus \{k\}$.

Sprumont (1990) introduces population monotonic allocation schemes, or *pmas* in short. Define P_k to be the set $\{S \subset N \mid k \in S\}$ of coalitions containing the decision-maker k. A scheme $[a_{i,S}]_{i \in S, S \in P_k}$ is a pmas if

$$\sum_{i \in S} a_{i,S} = v_k(S) \text{ and } a_{i,S} \leq a_{i,T}$$

for all $S, T \in P_k$ with $S \subset T$. Brânzei, Tijs and Timmer (2000b) define bi-monotonic allocation schemes (*bi-mas*) for k-concave IC games. So, (2) is satisfied. A scheme $[b_{i,S}]_{i \in S, S \in P_k}$ is a bi-mas if

$$(b_{i,S})_{i \in S} \in C(S, v_k), \; b_{k,S} \leq b_{k,T} \text{ and } b_{i,S} \geq b_{i,T}$$

for all $i \in S \setminus \{k\}$ and $S, T \in P_k$ with $S \subset T$. Hence, in a bi-mas the decision-maker k gains if more informants cooperate, while the informants are better off with fewer informants cooperating. Brânzei, Tijs and Timmer (2000b) show that for a game (N, v_k) corresponding to an IC situation in DM_k we can extend any $x \in C(v_k)$ to a bi-mas. That is, there exists a bi-mas $[b_{i,S}]_{i \in S, S \in P_k}$ such that $x_j = b_{j,N}$ for all $j \in N$. A similar result concerning pmas holds for IC games (N, v_k) that satisfy

$$v_k(T) - v_k(S) \geq \sum_{i \in T \setminus S} M_i(N, v_k) \tag{6}$$

for all $S \subset T \subset N$ with $k \in S$.

Theorem 5.3 *For all IC games (N, v_k) that satisfy (6) we can extend each core-element x to a pmas.*

Proof. Let (N, v_k) be an IC game satisfying (6) and let $x \in C(v_k)$. Because (6) implies (5), it follows from theorem 5.2 that $x_i = \alpha_i M_i(N, v_k)$, $i \in N \setminus \{k\}$, and $x_k = v_k(N) - \sum_{i \in N \setminus \{k\}} \alpha_i M_i(N, v_k)$ for an $\alpha \in [0,1]^{N \setminus \{k\}}$. Define the allocation scheme $[a_{i,S}]_{i \in S, S \in P_k}$ by $a_{i,S} = \alpha_i M_i(N, v_k)$ for all $i \in S \setminus \{k\}$, $S \in P_k$, and

$$a_{k,S} = v_k(S) - \sum_{i \in S \setminus \{k\}} \alpha_i M_i(N, v_k)$$

for all $S \in P_k$. Now $x_j = a_{j,N}$ for all $j \in N$. Further, it is obvious that $\sum_{i \in S} a_{i,S} = v_k(S)$ for all $S \in P_k$. Next, let $S, T \in P_k$ with $i \in S \subset T$. Then

$$a_{i,S} = \alpha_i M_i(N, v_k) = a_{i,T}$$

for all $i \in S \setminus \{k\}$ and

$$\begin{aligned} a_{k,S} &= v_k(S) - \sum_{i \in S \setminus \{k\}} \alpha_i M_i(N, v_k) \\ &= v_k(S) + \sum_{i \in T \setminus S} \alpha_i M_i(N, v_k) - \sum_{i \in T \setminus \{k\}} \alpha_i M_i(N, v_k) \\ &\leq v_k(T) - \sum_{i \in T \setminus \{k\}} \alpha_i M_i(N, v_k) \\ &= a_{k,T}, \end{aligned}$$

where the inequality follows from (6). Hence, $[a_{i,S}]_{i \in S, S \in P_k}$ is a pmas. ■

To conclude this section we want to discuss a special kind of consistency of marginal based allocation rules specific for IC situations. Hence, we call it *IC consistency.* Consider an IC situation with corresponding IC game (N, v_k). The total reward $v_k(N)$ will be divided according to a marginal based allocation rule for some $\alpha \in [0,1]^{N \setminus \{k\}}$. Suppose that the group S of informants takes their payoffs $(\alpha_i M_i(N, v_k))_{i \in S}$ and leaves. Just before leaving they gave their information to the decision-maker k. The *reduced IC situation* that arises is the following:

$$< N \setminus S, k, (\Omega, \mathcal{F}, \mu), \{\mathcal{I}_i^*\}_{i \in N \setminus S}, A_k, r_k^* >,$$

where $\mathcal{I}_i^* = \mathcal{I}_i$ for all $i \neq k$, $\mathcal{I}_k^* = \mathcal{I}_{\{k\} \cup S}$ and $r_k^* = r_k - \sum_{i \in S} \alpha_i M_i(N, v_k)$. Let $(N \setminus S, v_k^*)$ be the corresponding IC game. We say that a marginal based allocation rule is *IC consistent* if for all $S \subset N \setminus \{k\}$

$$\alpha_i M_i(N \setminus S, v_k^*) = \alpha_i M_i(N, v_k) \tag{7}$$

for all $i \in N \setminus S$, $i \neq k$, an informant receives the same after the group S has left, and if

$$v_k^*(N \setminus S) - \sum_{i \in N \setminus (S \cup \{k\})} \alpha_i M_i(N \setminus S, v_k^*) = v_k(N) - \sum_{i \in N \setminus \{k\}} \alpha_i M_i(N, v_k) \quad (8)$$

are satisfied. The latter equality says that the payoff of the decision-maker is not influenced by the absence of the group S of informants.

Theorem 5.4 *The marginal based allocation rules are IC consistent.*

Proof. Let $\alpha \in [0,1]^{N \setminus \{k\}}$ and let $S \subset N \setminus \{k\}$ be a group of informants that has left. Notice that

$$v_k^*(Q) = v_k(Q \cup S) - \sum_{i \in S} \alpha_i M_i(N, v_k)$$

for all $Q \subset N \setminus S$. Let $i \in N \setminus S$, $i \neq k$. Now

$$\begin{aligned} \alpha_i M_i(N \setminus S, v_k^*) &= \alpha_i [v_k^*(N \setminus S) - v_k^*(N \setminus (S \cup \{i\}))] \\ &= \alpha_i [v_k(N) - v_k(N \setminus \{i\})] \\ &= \alpha_i M_i(N, v_k). \end{aligned}$$

Equation (7) is satisfied. Using this, we obtain

$$\begin{aligned} & v_k^*(N \setminus S) - \sum_{i \in N \setminus (S \cup \{k\})} \alpha_i M_i(N \setminus S, v_k^*) \\ = \quad & (v_k(N) - \sum_{i \in S} \alpha_i M_i(N, v_k)) - \sum_{i \in N \setminus (S \cup \{k\})} \alpha_i M_i(N, v_k) \\ = \quad & v_k(N) - \sum_{i \in N \setminus \{k\}} \alpha_i M_i(N, v_k). \end{aligned}$$

We conclude that (8) is also satisfied. ■

An interesting question for future research is whether the marginal based allocation rules can be axiomatized with the help of IC consistency.

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