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Cooperative Games Arising From Information Sharing Situations

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Abstract

Relations are established between information sharing (IS) situations and IS–games on one hand and information collecting (IC) situations and IC–games on the other hand. It is shown that IC–games can be obtained as convex combinations of so-called local games. Properties are described which IC–games possess if all related local games have the respective properties. Special attention is paid to the classes of convex IC–games and of k–concave IC–games. This last class turns out to consist of total big boss games. For the class of total big boss games a new solution concept is introduced: bi–monotonic allocation schemes.

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1 Introduction

In Brânzei, Tijs and Timmer (2000) information collecting (IC) situations and corresponding IC–games were introduced. They modelled situations where an action taker in an uncertain situation can improve his action choices by gathering information from players more informed about the situation. The problem of sharing the gains, when cooperating with informants, is tackled by constructing the corresponding IC–game and considering solutions developed for such games.

In Slikker, Norde, and Tijs (2000) information sharing (IS) situations were introduced and corresponding IS–games. They modelled situations where all players are action takers in an uncertain situation and where they can gain by sharing their information about the situation.

It turned out that the family of 0–normalized IC–games for a fixed action taker and fixed player set coincides with the cone of 0–normalized monotonic games where the action taker is a veto player (cf. Theorem 3.5 in Brânzei et al. (2000)). In Slikker et al. (2000) it was proved that the class of information sharing games coincides with the class of cooperative games for which a population monotonic allocation scheme exists.

In Section 2 of this paper we study relations between IC–situations and IS–situations and the corresponding games. We show that an n-person IS–game can be seen as the sum of n IC–games. Further we introduce local games corresponding to IC–situations and show that an IC–game is a suitable convex combination of the local games. Properties as convexity and k-concavity for the local games are inherited by the IC–games. For k–symmetric games a geometric method is described to discover whether the game is k-concave, convex or none of them. In Section 3 special attention is paid to total big boss games and bi–monotonic allocation schemes. It turns out that k-concave IC–games are total big boss games.

2 Information collecting and information sharing situations and games

We start with recalling (and modifying a bit) the notions IC–situation, IC–game, IS–situation and IS–game introduced in Brânzei et al. (2000) and Slikker et al. (2000).

An IC–situation is a tuple

$$<N, k, A_k, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, r_k>$$

Here $N = \{1, 2, ..., n\}$, A_k and Ω are non–empty finite sets, $k \in N$, μ is a positive probability distribution on Ω (i.e. $\mu(\omega) > 0$ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \mu(\omega) = 1$). For each $i \in N$, $\mathcal{I}_i$ is a partition of Ω and $r_k : \Omega \times A_k \longrightarrow \mathbb{R}$ is a real valued function on $\Omega \times A_k$.

The set N is called the set of players, player $k \in N$ is the action taker, Ω is the set of possible states (relevant for the action taker), μ is the prior probability distribution and $\mathcal{I}_i$ is the information partition of player i. Such an IC–situation corresponds to the following decision problem for player k. He has to choose an action a_k from his action set A_k, under uncertainty of the true state, leading to a reward $r_k(\omega, a_k)$ if ω is the true state. Before choosing his action he can collect information from all other players or a subset of players (where monetary compensations for delivering information can be made). If player k works alone his expected reward will be

$$v_k(\{k\}) = \sum_{I \in \mathcal{I}_k} \max_{a_k \in A_k} \sum_{\omega \in I} r_k(\omega, a_k)\mu(\omega).$$

Collecting information from $S \setminus \{k\} \subset N$, the expected payoff is:

$$v_k(S) = \sum_{I \in \mathcal{I}_S} \max_{a_k \in A_k} \sum_{\omega \in I} r_k(\omega, a_k)\mu(\omega). \tag{1}$$

Here $\mathcal{I}_S$ is the partition of Ω consisting of the non–empty sets of the form $\cap_{i \in S} J_i$, where $J_i \in \mathcal{I}_i$ for all $i \in S$. If we also define

$$v_k(S) = 0 \text{ for all } S \subset N \setminus \{k\} \tag{2}$$

we obtain the cooperative game $< N, v_k >$ which we call the IC-game corresponding to the above IC-situation.

Example 1. (Catch the monster; one trial) Consider the IC-situation

$$<N, k, A_k, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, r_k>$$

where $N = \{1, 2, 3, 4\}$, $k = 4$, $\Omega = A_4 = \{0, 1\}^3$, $\mu(\omega) = 1/8$ for each $\omega \in \Omega$, $\mathcal{I}_4 = \{\Omega\}$, $\mathcal{I}_i = \{\{x \in \Omega \mid x_i = 0\}, \{x \in \Omega \mid x_i = 1\}\}$ for $i \in \{1, 2, 3\}$, $r_4(\omega, a) = 160$ if $\omega = a$, $r_4(\omega, a) = 0$ otherwise.

This corresponds to a situation of an apartment with eight rooms, 000, 100, 010, 001, 110, 101, 011, and 111, where in one of the rooms a monster is hidden. Player 4 has to choose one room and if the monster is in that room he obtains a reward of 160; otherwise there is no reward. He can make use of the information of players 1,2,3 or not. Player $i \in \{1, 2, 3\}$ knows the i-th coordinate of the room number. One can calculate the corresponding IC-game $< N, v >$ and obtains $N = \{1, 2, 3, 4\}$, $v(S) = 0$ for all $S \subset N$ with $4 \notin S$, $v(\{4\}) = 20$, $v(\{4, i\}) = 40$ for $i \in \{1, 2, 3\}$, $v(\{4, i, j\}) = 80$ if $i, j \in \{1, 2, 3\}$ and $i \neq j$, and $v(N) = 160$.

Example 2. (Catch the monster; three trials) Take the situation as in Example 1 and modify it in such a way that player 4 can look in three rooms. This leads to the IC-situation

$$<N, 4, A_4', (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, r_4'>$$

where $A_4' = \{K \subset \Omega \mid |K| = 3\}$, $r_4'(\omega, K) = 160$ if $\omega \in K$, $r_4'(\omega, K) = 0$ otherwise. This leads to the IC-game $< N, v' >$ with $v'(S) = 0$ if $4 \notin S$, $v'(\{4\}) = 60$, $v'(\{i, 4\}) = 120$ for $i \in \{1, 2, 3\}$ and $v'(S) = 160$ if $|S| \geq 3$ and $4 \in S$.

Note that $< N, v >$ and $< N, v' >$ in Examples 1 and 2 are monotonic games with 4 as veto player.

An IS-situation is a tuple

$$<N, \{A_i \mid i \in N\}, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, \{r_i \mid i \in N\}>.$$

Here N, Ω, μ, and $\mathcal{I}_i$ have the same meaning as earlier. For *each* player $i \in N$ there is now a non-empty finite action set A_i and a reward function $r_i : \Omega \times A_i \longrightarrow \mathbb{R}$. Each player $i \in N$ has to choose an action $a_i \in A_i$ and obtains then a reward $r_i(\omega, a_i)$ depending on the chosen action and the true state ω.

If a group S of players decides to cooperate and share their information, then the total expected reward for S is given by

$$v(S) = \sum_{i \in S} \sum_{I \in \mathcal{I}_S} \max_{a_i \in A_i} \sum_{\omega \in I} r_i(\omega, a_i)\mu(\omega). \tag{3}$$

This leads to the IS-game $< N, v >$ where $v(S)$ is given by (3).

What to say about relations between IC-situations and IS-situations?

(i) Suppose we have an IC-situation $< N, k, A^*_k, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, r^*_k >$ leading to the IC-game v_k. Then this game is also the IS-game corresponding to the IS-situation

$$<N, \{A_i \mid i \in N\}, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, \{r_i \mid i \in N\}>$$

where $A_i = A^*_k$ if $i = k$, A_i is an arbitrary non-empty finite set for $i \neq k$, $r_i = r^*_k$ if $i = k$ and $r_i = 0$ if $i \in N \setminus \{k\}$.

So an IC-situation with k as action taker can be transformed into an IS-situation, where all action takers except k have a trivial reward function, the zero–function.

(ii) Suppose we have an IS-situation

$$<N, \{A_i \mid i \in N\}, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, \{r_i \mid i \in N\}>$$

with corresponding IS-game v.

Consider the n IC-situations

$$<N, k, A_k, < \Omega, \mu >, \{\mathcal{I}_i \mid i \in N\}, r_k> \text{ for } k \in N$$

with corresponding IC-games $v_1, v_2, ..., v_k, ..., v_n$. Then it follows from (1), (2) and (3) that $v = \sum_{i \in N} v_i$. So an IS-situation can be decomposed in n IC-situations and the IS-game is the sum of the n IC-games $v_1, v_2, ..., v_n$, where v_k is a monotonic k-veto game. In Slikker et al. (2000) it was proved that an IS-game is a game with a population monotonic allocation scheme (pmas) (cf. Sprumont (1990)). In fact, $[a_{i,S}]_{i \in N, S \in 2^N \setminus \{\emptyset\}}$ with $a_{i,S} = 0$ if $i \notin S$, and $a_{i,S} = v_i(S)$ is a *pmas* because

$$\sum_{i \in N} a_{i,S} = \sum_{i \in S} v_i(S) = v(S) \text{ and}$$

$$a_{i,S} \leq a_{i,T} \text{ for all } i \in S \subset T$$

because v_i is a monotonic game.

Example 3. (Avoid or catch) Consider the IS-situation

$$<N, (\Omega, \mu), \{A_i, \mathcal{I}_i, r_i \mid i \in N\}>$$

with $N = \{1,2\}$, $A_1 = A_2 = \Omega = \{0,1\}^2$, $\mu(0,0) = 1/10$, $\mu(0,1) = \mu(1,0) = \mu(1,1) = 3/10$. Further $\mathcal{I}_1 = \{\{00, 01\}, \{10, 11\}\}, \mathcal{I}_2 = \{\{00, 10\}, \{01, 11\}\}$ and

$$\begin{aligned} r_1(a,\omega) &= 20(|a_1 - \omega_1| + |a_2 - \omega_2|) && \text{for all } a = (a_1, a_2) \in A_1,\ \omega \in \Omega, \\ r_2(a,\omega) &= 40 - 10(|a_1 - \omega_1| + |a_2 - \omega_2|) && \text{for all } a = (a_1, a_2) \in A_2,\ \omega \in \Omega. \end{aligned}$$

One can think of the following situation. In an apartment with four rooms 00, 01, 10 and 11, a monster is hidden. The probability that the monster is in 00 is 1/10 and the probability that the monster is in one of the other rooms is 3/10. The players 1 and 2 have to visit a room. Player 1 likes a room as far away from the monster as possible, while player 2 likes a room as near as possible. Player 1 knows whether the monster is at the north side $N = \{00, 01\}$ or not, and player 2 knows whether the monster is at the west side $W = \{00, 10\}$ or not.

The corresponding IS-game $< N, v >$ is the sum of the two IC-games $< N, v_1 >$ and $< N, v_2 >$, where $N = \{1, 2\}$

$$\begin{aligned} &v_1(\{1\}) = 32, \quad v_1(\{2\}) = 0 \quad v_1(\{1,2\}) = 40 \\ &v_2(\{1\}) = 0, \quad v_2(\{2\}) = 36 \quad v_2(\{1,2\}) = 40. \end{aligned}$$

A population monotonic allocation scheme for $< N, v >$ is

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \{1\} & \{2\} & \{1,2\} \end{array} \\ \left[\begin{array}{ccc} v_1(\{1\}) & * & v_1(\{1,2\}) \\ * & v_2(\{2\}) & v_2(\{1,2\}) \end{array} \right] \end{array} = \left[\begin{array}{ccc} 32 & * & 40 \\ * & 36 & 40 \end{array} \right]$$

Now we introduce *local games* $v_\omega : 2^N \longrightarrow \mathbb{R}$ for each $\omega \in \Omega$, given an information collecting situation

$$IC_k = <N, k, A_k, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, r_k>.$$

The game $<N, v_\omega>$ is defined by $v_\omega(S) = 0$ if $k \notin S$, and for $k \in S$:

$$v_\omega(S) = \max_{a_k \in A_k} \sum_{\omega' \in I_S(\omega)} r_k(a_k, \omega')(\mu(I_S(\omega)))^{-1}\mu(\omega')$$

where $I_S(\omega)$ is that atom in $\mathcal{I}_S$ containing ω. The number $v_\omega(S)$ can be interpreted as the best expected reward for coalition S if ω is the true state. In that case S knows that the true state is one of the elements in $I_S(\omega)$ and each $\omega' \in I_S(\omega)$ has conditional probability

$$(\mu(I_S(\omega)))^{-1}\mu(\omega').$$

Note that $v_\omega(S) = v_{\omega'}(S)$ for all $\omega' \in I_S(\omega)$ and that, according to (1), for S with $k \in S$

$$\begin{aligned} v(S) &= \sum_{I \in \mathcal{I}_S} \max_{a_k \in A_k} \sum_{\omega' \in I} r_k(a_k, \omega')\mu(\omega') \\ &= \sum_{I \in \mathcal{I}_S} \mu(I) v_\omega(S) = \sum_{I \in \mathcal{I}_S} \sum_{\omega \in I} \mu(\omega) v_\omega(S) = \sum_{\omega \in \Omega} \mu(\omega) v_\omega(S). \end{aligned}$$

Also for $S \not\ni k$ we have $v(S) = 0 = \sum_{\omega \in \Omega} \mu(\omega) v_\omega(S)$. So we have proved

Proposition 1. *If* $< N, v >$ *is the* IC*-game corresponding to an information collecting situation* IC_k *and* $\{< N, v_\omega >| \ \omega \in \Omega\}$ *is the set of local games, then* $v = \sum_{\omega \in \Omega} \mu(\omega) v_\omega$.

The following proposition shows that local games can be of use in discovering special properties of an IC-game. Interesting properties of IC-games with k as action taker are k-concavity and k-convexity. A game $< N, v >$ which is monotonic and has k as veto player is called k–concave if

$$(CC_k) \qquad v(S) - v(S \setminus \{i\}) \geq v(T) - v(T \setminus \{i\})$$

for all $i \in N,$ and $S, T \in 2^N$ with $i, k \in S \subset T$ and k-convex if the reverse inequality holds:

$$(CV_k) \qquad v(S) - v(S \setminus \{i\}) \leq v(T) - v(T \setminus \{i\}).$$

Note that k-convex games are convex games (cf. Shapley (1971)) and have many interesting properties. A special section, section 3, will be devoted to k-concave IC-games.

Proposition 2. (Inheritance property) *Suppose the local games* v_ω *of an* IC*-situation are all* k*-concave* (*convex*), *then the corresponding* IC*-game* v *is* k*-concave* (*convex*).

Proof. We prove only that v is k-concave, given that all local games are k-concave. It follows from Proposition 1 that for all i, S, T with $i, k \in S \subset T$:

$$\begin{aligned} & v(S) - v(S \setminus \{i\}) - v(T) + v(T \setminus \{i\}) \\ = & \sum_{\omega \in \Omega} \mu(\omega)(v_\omega(S) - v_\omega(S \setminus \{i\}) - v_\omega(T) + v_\omega(T \setminus \{i\})) \geq 0 \end{aligned}$$

where the inequality follows from the fact that the local games v_ω are k-concave and $\mu(\omega) > 0$. ■

Example 4. (Treasure guessing) Consider the IC-situation $< N, k, A_k, (\Omega, \mu), \{\mathcal{I}_i \mid i \in N\}, r_k >$ where $N = \{1, 2, 3\}$, $k = 3$, $A_3 = \{n, g_a, g_b, g_c, g_d\}$, $\Omega = \{a, b, c, d\}$ and $\mu(\omega) = 1/4$ for all $\omega \in \Omega$. Further $\mathcal{I}_1 = \{\{a, b\}, \{c, d\}\}, \mathcal{I}_2 = \{\{a, b, c\}, \{d\}\}, \mathcal{I}_3 = \{\Omega\}$, and $r_k(\omega, n) = 0$ for all $\omega \in \Omega$,

$$r_k(\omega, g_x) = 40 \text{ if } \omega = x \text{ and } r_k(\omega, g_x) = -60 \text{ otherwise.}$$

One can think of a situation where a treasure with a value of 40 dollars is hidden in one of the places a, b, c, d (equal prior probabilities). Player 3 can guess where the treasure is, and if he guesses right he obtains the treasure. If he guesses wrong he has to pay 60 dollars. Another action for player 3 is not to guess (n) with resulting reward 0. It only makes sense to guess if player 3 knows where the treasure is. This implies that for the local games we have $v_a = v_b = 0$, $v_c(\{1,2,3\}) = 40$ and $v_c(S) = 0$ otherwise, $v_d(\{2,3\}) = v_d(\{1,2,3\}) = 40$ and $v_d(S) = 0$ otherwise. These local games are convex, so the IC-game v is also convex by Proposition 2 and $v = (v_a + v_b + v_c + v_d)/4$; $v(\{2,3\}) = 10$, $v(\{1,2,3\}) = 20$, $v(S) = 0$ otherwise.

For the special class of k-symmetric IC-games we can discover whether the game is convex, k-concave or none of them by looking at the graph of the so-called detection function as the proposition below shows.

Let us call a game $v \in MV_k$ (i.e. v is monotonic and k is a veto player) k-symmetric if

$$v(S) = v(T) \text{ for all } S, T \text{ with } k \in S,\ k \in T \text{ and } |S| = |T|.$$

Corresponding to such a k-symmetric game v we construct the detection function $B : [1, n] \longrightarrow \mathbb{R}$ as follows. Denote by b_r the value $v(S)$ of a coalition $S \ni k$ with $|S| = r$. Then B is defined by

$$B(x) = (x - r)b_{r+1} + (1 - x + r)b_r \text{ for } x \in [r, r+1]$$

where $r \in \{1, 2, ..., n-1\}$. Note that the graph of B coincides with the broken line in $\mathbb{R}^2$ obtained by connecting for $r \in \{1, 2, ..., n-1\}$ the points (r, b_r) and $(r+1, b_{r+1})$ with a line segment.

Proposition 3. *Let $v \in MV_k$ be k-symmetric and let B be the corresponding detection function. Then*

(i) *v is a convex game if and only if B is a convex function.*

(ii) *v is a k-concave game if and only if B is a concave function.*

(iii) *v satisfies the property*

$$(U_k) \qquad v(N) - v(S) \geq \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\})) \text{ for all } S \ni k$$

if and only if the graph of B lies below the line in $\mathbb{R}^2$ through the points $(n-1, b_{n-1})$ and (n, b_n).

The proof of the proposition is left to the reader.

Knowing that $v \in MV_k$ is a convex game discloses a lot of properties of the solutions of the game (cf. Shapley (1971)) for example, the Shapley value (cf. Shapley (1953)) lies in the core of the game. Also it is interesting to know whether $v \in MV_k$ is k-concave as the following section shows.

Example 5.

(i) Consider again Example 1. This leads to a 4-symmetric game with a convex detection function, so the game is convex.

(ii) Consider Example 2 (Catch the monster; 3 trials). This leads to a 4-symmetric game with a concave detection function, and so the game is 4-concave.

3 Total big boss games and bi–monotonic allocation schemes

In this section we pay attention to games $< N, v >$ with $v \in MV_k$ that also satisfy the k-concavity condition (CC_k). A subclass of IC-games with k as action taker has this property. The k-concavity condition says that for a player i the marginal contribution to a smaller coalition containing k is (weakly) larger than the marginal contribution to a larger coalition containing k. The first theorem below shows that the class of games $< N, v >$ with $v \in MV_k$ and v satisfying the k-concavity condition, coincides with the class of total big boss games with k as big boss, which we introduce now. Recall first (cf. Muto et al. (1988) and Tijs (1990)) that a game $< N, v >$ is a big boss game with k as big boss if $v \in MV_k$ and if v has the property (U_k), mentioned in Proposition 3.

Let us call a game $< N, v >$ with $v \in MV_k$ a *total big boss game with k as big boss*, if the game itself and each subgame $< T, v >$ containing k are big boss games. Stated otherwise, $v \in MV_k$ is a *total big boss game with k as big boss* if and only if

$$(TU_k) \qquad\qquad v(T) - v(S) \geq \sum_{i \in T \setminus S} M_i(T, v)$$

for all S, T containing k and $S \subset T$. Here $M_i(T, v) = v(T) - v(T \setminus \{i\})$.

Theorem 1. *Let $v \in MV_k$. Then v is a total big boss game with k as big boss if and only if v is k-concave.*

Proof. Assume first that v is k-concave. Take $k \in S \subset T$. Suppose $T \setminus S = \{i_1, i_2, ..., i_h\}$. Then

$$\begin{aligned} v(T) - v(S) &= \sum_{r=1}^{h} (v(S \cup \{i_1, i_2, ..., i_r) - v(S \cup \{i_1, i_2, ..., i_{r-1}\})) \\ &= \sum_{r=1}^{h} M_{i_r}(S \cup \{i_1, i_2, ..., i_r\}, v) \geq \sum_{r=1}^{h} M_{i_r}(T, v) = \sum_{i \in T \setminus S} M_i(T, v), \end{aligned}$$

where the inequality follows from (CC_k) applied h times. So k-concavity implies that v is a total big boss game with k as big boss.

Suppose now that v is a total big boss game with k as big boss. First we prove that

(a) $M_i(U, v) \geq M_i(U \cup \{j\}, v)$ for all $U \in 2^N$, $i, j, k \in N$ with $i, k \in U \subset N \setminus \{j\}$.

By (TU_k) we have

(b) $v(U \cup \{j\}) - v(U \setminus \{i\}) \geq M_j(U \cup \{j\}, v) + M_i(U \cup \{j\}, v)$.

On the other hand,

(c) $v(U \cup \{j\}) - v(U \setminus \{i\}) = (v(U \cup \{j\}) - v(U)) + (v(U) - v(U \setminus \{i\})) = M_j(U \cup \{j\}, v) + M_i(U, v)$.

Then (a) follows directly from (b) and (c). To prove that v is k-concave take $S,T \in 2^N$ with $i,k \in S \subset T$ and suppose that $T \setminus S = \{i_1, i_2, ..., i_h\}$. Apply (a) h-times and we obtain

$$M_i(S,v) \geq M_i(S \cup \{i_1\}, v) \geq M_i(S \cup \{i_1, i_2\}, v) \geq \cdots \geq M_i(T,v).$$

So $M_i(S,v) \geq M_i(T,v)$ which implies (CC_k). ■

From Muto et al. (1988) it follows that for a total big boss game with k as big boss we have for all $T \ni k$:

(3.1) The core $C(T,v)$ of the subgame $< T, v >$ is equal to $\{x \in \mathbb{R}^T \mid 0 \leq x_i \leq M_i(T,v) \text{ for all } i \in T \setminus \{k\}, \sum_{i \in T} x_i = v(T)\}$

(3.2) The τ-value $\tau(T,v)$ (cf. Tijs (1981)) and the nucleolus $Nu(T,v)$ (cf. Schmeidler (1969)) of $< T, v >$ coincide and are equal to the center z of the core $C(T,v)$ where $z_i = M_i(T,v)/2$ for all $i \in T \setminus \{k\}$ and $z_k = v(T) - \sum_{i \in T \setminus \{k\}} M_i(T,v)/2$.

Take $< N, v >$ in MV_k and denote by P_k the set $\{S \subset N \mid k \in S\}$. Call a scheme $[b_{i,S}]_{i \in S, S \in P_k}$ an allocation scheme if each column $[b_{i,S}]_{i \in S}$ corresponds to a core element of the subgame $< S, v >$.

We call such an allocation scheme $[b_{i,S}]_{i \in S, S \in P_k}$ a *bi–monotonic allocation scheme (bi–mas)* if for all $S,T \in P_k$ with $S \subset T$ we have $b_{i,S} \geq b_{i,T}$ for all $i \in S \setminus \{k\}$, and $b_{k,S} \leq b_{k,T}$. In a bi–mas the big boss is better off in larger coalitions, and the other players are worse off.

Let $[b_{i,S}]_{i \in S, S \in P_k}$ be defined by $b_{k,S} = v(S)$ and $b_{i,S} = 0$ if $i \in S \setminus \{k\}$. Then $[b_{i,S}]_{i \in S, S \in P_k}$ is a bi–mas for $< N, v >$. So each $v \in MV_k$ has a bi–mas. The next theorem shows that a total big boss game has many bi–monotonic allocation schemes. This theorem is comparable to a result by Sprumont (1990) which tells that for a convex game each core element is extendable to a population monotonic allocation scheme. We say that a bi–mas $[b_{i,S}]_{i \in S, S \in P_k}$ is an extension of the core element $x \in C(N,v)$ if $x_i = b_{i,N}$ for all $i \in N$.

Theorem 2. *Let $< N, v >$ be a total big boss game with k as big boss and let $x \in C(N,v)$. Then x is extendable to a bi–monotonic allocation scheme.*

Proof. Since $x \in C(N,v)$, by (3.1), we can find for each $i \in N \setminus \{k\}$ an $\alpha_i \in [0,1]$, such that $x_i = \alpha_i M_i(N,v)$, and then

$$x_k = v(N) - \sum_{i \in N \setminus \{k\}} \alpha_i M_i(N,v).$$

We will show that $[b_{i,S}]_{i \in S, S \in P_k}$, defined by $b_{i,S} = \alpha_i M_i(S,v)$ for all (i,S) with $i \in S \setminus \{k\}$ and $b_{k,S} = v(S) - \sum_{i \in S \setminus \{k\}} \alpha_i M_i(S,v)$, is a bi–mas. Then it is an extension of x.

Take $S,T \in P_k$ with $S \subset T$ and $i \in S \setminus \{k\}$. We have to prove that $b_{i,S} \geq b_{i,T}$ and $b_{k,S} \leq b_{k,T}$. First, $b_{i,S} = \alpha_i M_i(S,v) \geq \alpha_i M_i(T,v) = b_{i,T}$, where the inequality follows from the k-concavity of $< N, v >$. Second,

$$b_{k,T} \;=\; v(T) - \sum_{i \in T \setminus \{k\}} \alpha_i M_i(T,v)$$

$$
\begin{aligned}
&\geq (v(S) + \sum_{i \in T \setminus S} M_i(T,v)) - \sum_{i \in T \setminus \{k\}} \alpha_i M_i(T,v) \\
&= (v(S) - \sum_{i \in S \setminus \{k\}} a_i M_i(T,v)) + \sum_{i \in T \setminus S} (1-\alpha_i) M_i(T,v) \\
&\geq (v(S) - \sum_{i \in S \setminus \{k\}} \alpha_i M_i(S,v)) + \sum_{i \in T \setminus S} (1-\alpha_i) M_i(T,v) \\
&= b_{k,S} + \sum_{i \in T \setminus S} (1-\alpha_i) M_i(T,v) \geq b_{k,S}
\end{aligned}
$$

where the first inequality follows from (TU_k), the second inequality from the k-concavity and the third inequality from the monotonicity of the total big boss game $< N, v >$. So $b_{k,T} \geq b_{k,S}$. ■

Example 6. Consider again the IC-game in Example 2. This game $< N, v' >$ is a total big boss game. A bi–mas, consisting in each column S of the τ-value of $< S, v' >$ (see (3.2): $b_{i,S} = M_i(S, v')/2$ is given by

	$\{4\}$	$\{1,4\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
1	*	30	*	*	20	20	*	0
2	*	*	30	*	20	*	20	0
3	*	*	*	30	*	20	20	0
4	60	90	90	90	120	120	120	160

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