Center
for
Economic Research

No. 2000-108

ON CONVEXITY FOR NTU-GAMES

By Ruud Hendrickx, Peter Borm and Judith Timmer

November 2000

ISSN 0924-7815

On Convexity for NTU-Games

Ruud Hendrickx[1,2] Peter Borm[1] Judith Timmer[1,3]

Abstract

For cooperative games with transferable utility, convexity has turned out to be an important and widely applicable concept. Convexity can be defined in a number of ways, each having its own specific attractions. Basically, these definitions fall into two categories, namely those based on a supermodular interpretation and those based on a marginalistic interpretation. For games with non-transferable utility, however, the literature only offers two kinds of convexity, ordinal and cardinal convexity, which both extend the supermodular interpretation. In this paper, we introduce and analyse three new types of convexity for NTU-games that generalise the marginalistic interpretation of convexity.

[1]CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

[2]Corresponding author. E-mail: ruud@kub.nl.

[3]This author acknowledges financial support from the Netherlands Organisation for Scientific Research (NWO) through project 613-304-059.

1 Introduction

The notion of convexity for cooperative games with transferable utility (TU-games) was introduced by Shapley (1971) and is one of the most analysed properties in cooperative game theory. Many economic and combinatorial situations give rise to convex (or concave) cooperative games, such as airport games (cf. Littlechild and Owen (1973)), bankruptcy games (cf. Aumann and Maschler (1985)) and sequencing games (cf. Curiel et al. (1989)).

Convexity for TU-games can be defined in a number of equivalent ways. One of these is by means of the *supermodularity property*, which has its origins outside the field of game theory. Vilkov (1977) and Sharkey (1981) have extended this property towards cooperative games with non-transferable utility (NTU-games) to define *ordinal* and *cardinal convexity*, respectively. The supermodular interpretation of convexity also plays an important role in the context of effectivity functions (cf. Abdou and Keiding (1991)).

Economically more appealing than this supermodular interpretation of convexity are the definitions of convexity that are based on the concept of *marginal contributions.* In cooperative games with stochastic payoffs, this marginalistic interpretation of convexity has already been succesfully applied (cf. Timmer et al. (2000) and Suijs (2000)). For NTU-games, however, such an extension has not yet been made. In this paper, we define three new types of convexity for NTU-games, which are based on three corresponding marginalistic convexity properties for TU-games.

Although all five convexity properties for NTU-games coincide within the subclass of TU-games, this is not the case in general. In this paper we analyse the relations between these convexity concepts. In addition, we investigate the convexity properties in special classes of NTU-games, such as hyperplane games, 1-corner games and bargaining games.

Convex TU-games have some nice properties. In this paper we focus on three of these: for convex TU-games, the Shapley value belongs to the core, semi-convexity is satisfied and the bargaining set coincides with the core. We find that semi-convexity can be extended to NTU-games in such a way that it is satisfied if either of the five convexity notions holds. It is shown that the first property can be extended to NTU-games that satisfy the three marginalistic convexity properties in some classes of NTU-games. We show that the third property cannot be extended for general NTU-games.

This paper is organised as follows. In Section 2, we introduce some notation and basic definitions. In Section 3, three new types of convexity for NTU-games are introduced. In Section 4, we investigate how the various types of convexity are related and in Section 5, we look at some specific classes of games. Finally, in Section 6, we relate the different types of convexity to various solution concepts for NTU-games.

2 Notation and Basic Definitions

The set of all real numbers is denoted by $\mathbb{R}$, the set of nonnegative reals by $\mathbb{R}_+$ and the set of nonpositive reals by $\mathbb{R}_-$. For a finite set N, we denote its power set by $2^N = \{S \,|\, S \subset N\}$ and its number of elements by $|N|$. By $\mathbb{R}^N$ we denote the set of all real-valued functions on N. An element of $\mathbb{R}^N$ is denoted by a vector $x = (x_i)_{i \in N}$. For $S \subset N, S \neq \emptyset$, we denote the restriction of x on S by $x_S = (x_i)_{i \in S}$. For $x, y \in \mathbb{R}^N$, $y \geq x$ denotes $y_i \geq x_i$ for all $i \in N$ and $y > x$ denotes $y_i > x_i$ for all $i \in N$.

A *cooperative game with transferable utility*, or *TU-game*, is described by a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, assigning to every coalition $S \subset N$ of players a value $v(S)$, representing the total payoff to this group of players when they cooperate. By convention, $v(\emptyset) = 0$.

An *allocation* of $v(S)$ is a vector $x \in \mathbb{R}^S$ such that $\sum_{i \in S} x_i \leq v(S)$, with x_i representing the payoff to player $i \in S$. An allocation x of $v(S)$ is called *Pareto efficient* if $\sum_{i \in S} x_i = v(S)$. The core $C(V)$ is the set of Pareto efficient allocations of $v(N)$ for which it holds that no coalition $S \subset N$ has an incentive to split off:

$$C(v) = \{x \in \mathbb{R}^N \,|\, \forall_{S \subset N} : \sum_{i \in S} x_i \geq v(S), \sum_{i \in N} x_i = v(N)\}.$$

A TU-game (N, v) is called *superadditive* if for all coalitions $S, T \subset N$ such that $S \cap T = \emptyset$ we have

$$v(S) + v(T) \leq v(S \cup T).$$

An *ordering* of the players in N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(i)$ denotes which player in N is at position i. The set of all $n!$ permutations of N is denoted by $\Pi(N)$. The *marginal vector* of a TU-game (N, v) corresponding to the order $\sigma \in \Pi(N)$ is defined by

$$m^{\sigma}_{\sigma(k)}(v) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$$

for all $k \in \{1, \ldots, n\}$.

A *cooperative game with non-transferable utility*, or *NTU-game*, is described by a pair (N, V), where $N = \{1, \ldots, n\}$ is the set of players and V is the payoff map assigning to each coalition $S \subset N, S \neq \emptyset$ a subset $V(S)$ of $\mathbb{R}^S$ such that, for all $i \in N$,

$$V(\{i\}) = (-\infty, 0]$$

and for all $S \subset N, S \neq \emptyset$ we have

$V(S)$ is nonempty, closed and convex,

$V(S)$ is comprehensive, i.e., $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$,

$V(S) \cap \mathbb{R}^S_+$ is bounded.

In addition, we assume that (N, V) is *monotonic*: for all $S \subset T \subset N, S \neq \emptyset$ and for all $x \in V(S)$ there exists a $y \in V(T)$ such that $y_S \geq x$. Note that we do not define $V(\emptyset)$. For all $S \subset N, S \neq \emptyset$ we define $V^{\circ}(S) = V(S) \times 0^{N \setminus S}$ and $V^{\circ}(\emptyset) = 0^N$. The class of NTU-games with player set N is denoted by NTU^N. For ease of notation, we sometimes use V rather than (N, V) to denote an NTU-game.

NTU-games generalise TU-games. Every TU-game (N, v) gives rise to an NTU-game (N, V) by defining $V(S) = \{x \in \mathbb{R}^S \,|\, \sum_{i \in S} x_i \leq v(S)\}$ for all $S \subset N, S \neq \emptyset$.

The set of *Pareto efficient allocations* for coalition $S \subset N, S \neq \emptyset$, denoted by $Par(S)$, is defined by

$$Par(S) = \{x \in V(S) \,|\, \nexists_{y \in V(S)} : y \geq x, y \neq x\},$$

its set of *weak Pareto efficient allocations* $WPar(S)$ is defined by

$$WPar(S) = \{x \in V(S) \,|\, \nexists_{y \in V(S)} : y > x\}$$

and its set of *individually rational allocations* is defined by

$$IR(S) = \{x \in V(S) \,|\, \forall_{i \in S} : x_i \geq 0\}.$$

The *imputation set* of an NTU-game (N, V), denoted by $I(V)$, is defined by

$$I(V) = IR(N) \cap WPar(N).$$

The *core* of an NTU-game (N, V) consists of those elements of $V(N)$ for which it holds that no coalition $S \subset N, S \neq \emptyset$ has an incentive to split off:

$$C(V) = \{x \in V(N) \,|\, \forall_{S \subset N, S \neq \emptyset} \nexists_{y \in V(S)} : y > x_S\}.$$

An NTU-game (N, V) is called *superadditive* if for all coalitions $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset, S \cap T = \emptyset$ we have

$$V(S) \times V(T) \subset V(S \cup T).$$

This definition of superadditivity is a straightforward generalisation of the concept of superadditivity for TU-games. In addition, we define a weaker property concerning only the merger between individual players and coalitions rather than between two arbitrary coalitions. An NTU-game (N, V) is called *individually superadditive* if for all $i \in N$ and for all $S \subset N \backslash \{i\}, S \neq \emptyset$ we have

$$V(S) \times V(\{i\}) \subset V(S \cup \{i\}).$$

Note that individual superadditivity is stronger than monotonicity. We define the *marginal vector* m^σ corresponding to the order $\sigma \in \Pi(N)$ by

$$\begin{aligned} M^\sigma_{\sigma(k)}(V) = \quad \max\{x_{\sigma(k)} \,|\quad & x \in V(\{\sigma(1), \dots, \sigma(k)\}), \\ & \forall_{i \in \{1, \dots, k-1\}} : x_{\sigma(i)} = M^\sigma_{\sigma(i)}(V)\}. \end{aligned}$$

for all $k = 1, \dots, n$. Note that we use the assumption of monotonicity to ensure that the sets over which the maximums are taken are nonempty. By construction, $M^\sigma(V) \in WPar(N)$. If a game is individually superadditive, then all marginal vectors belong to $IR(N)$.

3 Convexity

A TU-game (N, v) is called *convex* if it satisfies the following four equivalent conditions (cf. Shapley (1971) and Ichiishi (1981)):

$$\forall_{S,T \subset N} : v(S) + v(T) \leq v(S \cap T) + v(S \cup T), \tag{3.1}$$

$$\forall_{U \subset N} \forall_{S \subset T \subset N \backslash U} : v(S \cup U) - v(S) \leq v(T \cup U) - v(T), \tag{3.2}$$

$$\forall_{i \in N} \forall_{S \subset T \subset N \backslash \{i\}} : v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \tag{3.3}$$

$$\forall_{\sigma \in \Pi(N)} : m^\sigma(v) \in C(v). \tag{3.4}$$

Condition (3.1), which is called the *supermodularity property*, was originally stated in Shapley (1971) as the definition of convexity for TU-games. Subsequently, Vilkov

(1977) and Sharkey (1981) generalised this property to ordinal and cardinal convexity for NTU-games, respectively. An NTU-game (N, V) is called *ordinally convex* if for all coalitions $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset$ and for all $x \in \mathbb{R}^N$ such that $x_S \in V(S)$ and $x_T \in V(T)$ we have

$$x_{S\cap T} \in V(S \cap T) \text{ or } x_{S\cup T} \in V(S \cup T). \tag{3.5}$$

A game is called *cardinally convex* if for all coalitions $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset$ we have

$$V^{\circ}(S) + V^{\circ}(T) \subset V^{\circ}(S \cap T) + V^{\circ}(S \cup T). \tag{3.6}$$

In contrast with these *supermodular* definitions of convexity by Vilkov (1977) and Sharkey (1981), we define three new types of convexity for NTU-games, based on the *marginalistic* properties (3.2)-(3.4). First of all, we define *coalition-merge convexity*[1], which generalises property (3.2). For $U = \emptyset$ and $S = T$, (3.2) is trivial and these cases can therefore be ignored when defining an analogous property for NTU-games. If $S = \emptyset$, (3.2) is equivalent to superadditivity. Because we do not define $V(\emptyset)$ for NTU-games, we require superadditivity as a separate condition. For $S \neq \emptyset$, (3.2) states that for any coalition U, the marginal contribution to the larger coalition T is larger than the marginal contribution to the smaller coalition S. In terms of allocations, this can be interpreted as follows: given the situation in which coalitions S and T have agreed upon a weak Pareto efficient and individually rational allocation of $v(S)$ and $v(T)$ (say, p and q, resp.), if coalition U joins the smaller coalition S, then for any allocation r of $v(S \cup U)$ such that the players in S get at least their previous amount ($r_S \geq p$), it is possible for U to join the larger coalition T using allocation s of $v(T \cup U)$, which gives the players in T at least their previous amount ($s_T \geq q$) and makes all players in U better off than in case they join S ($s_U \geq r_U$). Using this interpretation of (3.2), we can now define an analogous property for NTU-games.

An NTU-game (N, V) is called *coalition-merge convex*, if it is superadditive and it satisfies the coalition-merge property, i.e., for all $U \subset N$ such that $U \neq \emptyset$ and all $S \subsetneq T \subset N \backslash U$ such that $S \neq \emptyset$ the following statement is true: for all $p \in WPar(S) \cap IR(S)$, all $q \in V(T)$ and all $r \in V(S \cup U)$ such that $r_S \geq p$, there exists an $s \in V(T \cup U)$ such that

[1] This notion is introduced for stochastic cooperative games in Suijs and Borm (1999). The name *coalition-merge convexity* and the subsequent names *individual-merge* and *marginal convexity* are from Timmer et al. (2000)

$$\begin{cases} \forall_{i\in T} : s_i \geq q_i \\ \forall_{i\in U} : s_i \geq r_i. \end{cases} \tag{3.7}$$

Note that it makes no differences whether we require the coalition-merge property for all $q \in V(T)$ or only for $q \in WPar(T) \cap IR(T)$. The extension of (3.3) towards NTU-games goes in a similar manner: an NTU-game (N, V) is called *individual-merge convex* if it is individually superadditive and it satisfies the individual-merge property, i.e., for all $k \in N$ and all $S \subsetneq T \subset N\backslash\{k\}$ such that $S \neq \emptyset$, the following statement is true: for all $p \in WPar(S) \cap IR(S)$, all $q \in V(T)$ and all $r \in V(S \cup \{k\})$ such that $r_S \geq p$ there exists an $s \in V(T \cup \{k\})$ such that

$$\begin{cases} \forall_{i\in T} : s_i \geq q_i \\ s_k \geq r_k. \end{cases} \tag{3.8}$$

And finally, an NTU-game (N, V) is called *marginal convex* if for all $\sigma \in \Pi(N)$ we have

$$M^\sigma(V) \in C(V). \tag{3.9}$$

One important aspect of the five convexity properties defined in this section is that within the class of NTU-games that correspond to TU-games, they are all equivalent and coincide with TU-convexity.

Another property of these concepts is the following: if an NTU-game (N, V) satisfies some form of convexity, then all its subgames do, where the subgame of (N, V) with respect to coalition $S \subset N, S \neq \emptyset$ is defined as the NTU-game (S, V^S) with $V^S(T) = V(T)$ for all $T \subset S, T \neq \emptyset$.

4 Relations between the Five Types of Convexity

In this section we investigate the relations between the five types of convexity for NTU-games that were presented in the previous section. For 2-player NTU-games, all five types are equivalent to (individual) superadditivity, as is shown in Proposition 4.1.

Proposition 4.1 Let $(N, V) \in NTU^N$ such that $|N| = 2$. Then ordinal, cardinal, coalition-merge, individual-merge and marginal convexity are equivalent to (individual) superadditivity.

Proof: For the first four convexity notions, the statement follows immediately from the definitions. For marginal convexity, first note that (N, V) is superadditive if and only if $V(N) \supset \mathbb{R}^N_-$. If (N, V) is marginal convex, then both marginal vectors belong to $C(V) \subset IR(N)$ and using comprehensiveness, this implies that (N, V) is superadditive. Conversely, if (N, V) is superadditive, then both marginal vectors are individually rational and hence belong to the core. □

For general n-player NTU-games, equivalence between the five types of convexity does not hold. The remainder of this section shows which relations do exist between these properties.

The following proposition states an implication, which follows immediately from the definitions of coalition-merge and individual-merge convexity.

Proposition 4.2 If an NTU-game (N, V) is coalition-merge convex, then it is individual-merge convex.

The following Example shows that the reverse need not be the case.

Example 4.3 Consider the following NTU-game with player set $N = \{1, 2, 3, 4\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 1\} \text{ if } S = \{1, 2\} \text{ or } S = \{3, 4\},$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 0\} \text{ for other } S \subset N, |S| = 2,$$

$$V(\{1, 2, 3\}) = \{x \in \mathbb{R}^{\{1,2,3\}} \mid x_1 \leq 1, x_2 \leq 1, x_3 \leq 0\},$$

$$V(\{1, 2, 4\}) = \{x \in \mathbb{R}^{\{1,2,4\}} \mid x_1 \leq 1, x_2 \leq 1, x_4 \leq 0\},$$

$$V(\{1, 3, 4\}) = \{x \in \mathbb{R}^{\{1,3,4\}} \mid x_1 \leq 0, x_3 \leq 1, x_4 \leq 1\},$$

$$V(\{2, 3, 4\}) = \{x \in \mathbb{R}^{\{2,3,4\}} \mid x_2 \leq 0, x_3 \leq 1, x_4 \leq 1\},$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 3\}.$$

This game is not superadditive and therefore not coalition-merge convex[2]: take $S = \{1, 2\}, T = \{3, 4\}$, then $(1, 1) \in V(S)$ and $(1, 1) \in V(T)$, but $(1, 1, 1, 1) \notin$

[2] One can even construct an individual-merge convex game that is superadditive, but which does not satisfy the coalition-merge property.

$V(S \cup T)$. This game does, however, satisfy individual-merge convexity. First, individual superadditivity can easily be checked to be satisfied. Next, let $k \in N$, let $S \subsetneq T \subset N \backslash \{k\}$ such that $S \neq \emptyset$ and let $p \in WPar(S) \cap IR(S), q \in V(T)$ and $r \in V(S \cup \{k\})$ such that $r_S \geq p$. Define $s = (q, r_k) \in \mathbb{R}^{T \cup \{k\}}$. If $|T| = 3$, we have $T \cup \{k\} = N$. Because $\sum_{i \in T} q_i \leq 2$ and $r_k \leq 1$ (which follows from $|S| \leq 2$), we have $\sum_{i \in N} s_i \leq 3$ and hence, $s \in V(N)$. If $T = \{1, 2\}$ or $T = \{3, 4\}$, we have $|S| = 1$ and $r_k \leq 0$ and because of individual superadditivity, $s \in V(T \cup \{k\})$. Finally, for other coalitions T with $|T| = 2$, we have $\max_{i \in T} q_i \leq 0$, $r_k \leq 1$ and therefore $s \in V(T \cup \{k\})$. Hence, this game satisfies the individual-merge property. $\triangleleft$

In the following lemma, we show that individual-merge convexity implies marginal convexity.

Proposition 4.4 Let $(N, V) \in NTU^N$. If (N, V) is individual-merge convex, then it is marginal convex.

Proof: Assume (N, V) is individual-merge convex and let $\sigma \in \Pi(N)$. To simplify notation, assume without loss of generality that $\sigma(i) = i$ for all $i \in N$. We prove that $M^\sigma(V) \in C(V)$ by induction on the player set. For this, we define for $k = 1, \ldots, n$ the subgame (N^k, V^k) where $N^k = \{1, \ldots, k\}$ and $V^k(S) = V(S)$ for all $S \subset N^k, S \neq \emptyset$. For $k = 1$, $M^\sigma(V^k) \in C(V^k)$ by construction. Next, let $k \in \{2, \ldots, n\}$ and assume $M^\sigma(V^{k-1}) \in C(V^{k-1})$. We show that $M^\sigma(V^k) \in C(V^k)$, i.e., no coalition has an incentive to leave the "grand" coalition N^k. Define $T = \{1, \ldots, k-1\}$ and let $S \subsetneq T, S \neq \emptyset$. Then it is sufficient to show that coalitions S, T, $\{k\}$, $T \cup \{k\}$ and $S \cup \{k\}$ have no incentive to split off:

- Because $M^\sigma(V^{k-1}) \in C(V^{k-1})$, by definition there does not exist an $y \in V(S)$ such that $y > M^\sigma_S(V^{k-1})$. By construction, $M^\sigma_S(V^k) = M^\sigma_S(V^{k-1})$, so there does not exist an $y \in V(S)$ such that $y > M^\sigma_S(V^k)$. Hence, coalition S has no incentive to leave N^k when the payoff is $M^\sigma(V^k)$. The same argument holds for coalition T.

- Player k will not deviate on his own, because individual-merge convexity implies individual superadditivity and hence, $M^\sigma(V^k) \in IR(V^k)$.

- Because $M^{\sigma,k}(V^k) \in WPar(N^k)$, there exists no $y \in V^k(N^k)$ such that $y > M^{\sigma,k}(V^k)$ and hence, the "grand" coalition $T \cup \{k\}$ has no incentive to deviate.

- Finally, we show that coalition $S \cup \{k\}$ has no incentive to split off. Define $R = \{r \in V(S \cup \{k\}) \,|\, r_S \geq M^\sigma_S(V^k)\}$ to be the set of allocations in $V(S \cup \{k\})$ according to which the players in S get at least the amount they get according to the marginal vector $M^\sigma(V^k)$. If $R = \emptyset$, then $S \cup \{k\}$ will be satisfied with the allocation $M^\sigma(V^k)$. Because $M^\sigma(V^k) \in IR(N^k)$, it follows from the basic assumptions of an NTU-game that R is closed and bounded, so if $R \neq \emptyset$, we can compute $\max\{r_k \,|\, r = (r_S, r_k) \in R\}$. Let $r \in R$ be a point in which this maximum is reached. Because $M^\sigma(V^{k-1}) \in C(V^{k-1})$, we must have $M^\sigma_S(V^k) \notin V(S)$ or $M^\sigma_S(V^k) \in WPar(S)$. Let p be the intersection point of the line segment between 0 and $M^\sigma_S(V^k)$ and the set $WPar(S) \cap IR(S)$. By construction, $r \in V(S \cup \{k\})$ is such that $r_S \geq p$.
  Next, take $q = M^\sigma(V^{k-1}) \in V(T)$. As a result of individual-merge convexity and comprehensiveness, there exists an $s \in V(T \cup \{k\})$ such that $s_T = q$ and $s_k \geq r_k$. Because $s_T = M^\sigma(V^{k-1})$, it follows from the construction of $M^\sigma(V^k)$ that $M^\sigma_k(V^k) \geq s_k$. But then, $M^\sigma_k(V^k) \geq r_k$. We constructed r_k as the maximum amount player k can obtain by cooperating with coalition S, while giving each player $i \in S$ at least $M^\sigma_i(V^k)$. We conclude that there does not exist a $y \in V(S \cup \{k\})$ such that $y_i > M^\sigma_i(V^k)$ for all $i \in S \cup \{k\}$.

From these four cases we conclude $M^\sigma(V^k) \in C(V^k)$ and by induction on k, we obtain $M^\sigma(V) \in C(V)$. □

In Example 4.5 we show that the reverse implication of Proposition 4.4 need not hold.

Example 4.5 The following game with player set $N = \{1, 2, 3\}$ is the NTU-analogon of Example 4.6 in Timmer et al. (2000), which is a cooperative game with stochastic payoffs:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \,|\, x_1 + x_2 \leq 3\},$$

$$V(\{1,3\}) = \{x \in \mathbb{R}^{\{1,3\}} \,|\, x_1 + x_3 \leq 2\},$$

$$V(\{2,3\}) = \{x \in \mathbb{R}^{\{2,3\}} \,|\, x_2 + x_3 \leq 6\},$$

$$V(N) = \{x \in \mathbb{R}^N \,|\, \frac{x_1}{6} + \frac{x_2}{10} + \frac{x_3}{14} \leq 1\}.$$

The marginal vectors of this games are stated in the following table, where $\sigma = (a, b, c)$ is shorthand notation for $\sigma(1) = a$, $\sigma(2) = b$ and $\sigma(3) = c$.

σ	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$
$M^\sigma(V)$	$(0,3,\frac{49}{5})$	$(0,\frac{60}{7},2)$	$(3,0,7)$	$(\frac{24}{7},0,6)$	$(2,\frac{20}{3},0)$	$(\frac{12}{5},6,0)$

and the core is given by

$$C(V) = \{x \in \mathbb{R}^N_+ \mid \frac{x_1}{6} + \frac{x_2}{10} + \frac{x_3}{14} = 1, x_1 + x_3 \geq 3, x_1 + x_3 \geq 2, x_2 + x_3 \geq 6\}.$$

It is easy to check that $M^\sigma(V) \in C(V)$ for all $\sigma \in \Pi(N)$ and hence, (N, V) is marginal convex. Next, we show that this game is not individual-merge convex. Take $k = 1, S = \{2\}, T = \{2, 3\}$ and take $p = 0 \in WPar(S) \cap IR(S), q = (6, 0) \in V(T)$ and $r = (3, 0) \in V(S \cup \{k\})$. Note that $r_S \geq p$. Suppose (N, V) is individual-merge convex. Then there exists an $s \in V(T \cup \{k\})$ such that (3.8) holds, i.e., $s_2 \geq 6, s_3 \geq 0$ and $s_1 \geq 3$. But $s \in V(T \cup \{k\})$ implies $\frac{s_1}{6} + \frac{s_2}{10} + \frac{s_3}{14} \leq 1$, which gives a contradiction. Hence, (N, V) is not individual-merge convex. ◁

In the following example we show that ordinal convexity is not implied by any of the other four types of convexity.

Example 4.6 Consider the following NTU-game with player set $N = \{1, 2, 3, 4\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 1\} \text{ for all } S \subset N, |S| = 2,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq 4\} \text{ for all } S \subset N, |S| = 3,$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 7\}.$$

First, we show that this game is not ordinally convex. Consider $S = \{1, 2, 3\}, T = \{2, 3, 4\}$ and $x = (4, -3, 3, 4) \in \mathbb{R}^N$. Then we have both $x_S \in V(S)$ and $x_T \in V(T)$, but neither $x_{S \cap T} \in V(S \cap T)$ nor $x_{S \cup T} \in V(S \cup T)$. Hence, (3.5) is not satisfied and (N, V) is not ordinally convex.

Next, we show that (N, V) is coalition-merge convex. Let $U \subset N, U \neq \emptyset$ and let $S \subsetneq T \subset N \backslash U$ such that $S \neq \emptyset$. Let $p \in WPar(S) \cap IR(S)$, let $q \in V(T)$ and let $r \in V(S \cup U)$ such that $r_S \geq p$. Define $s = (q, r_U)$. If $|T| = 3$, then $\sum_{i \in T} q_i \leq 4$ and

$r_U \leq 3$. If $|T| = 2$ and $|U| = 2$, then $\sum_{i \in T} q_i \leq 2$ and $\sum_{i \in U} r_i \leq 4$. In both cases, we have $\sum_{i \in T \cup U} s_i \leq 7$ and hence, $s \in V(T \cup U) = V(N)$. In case $|T| = 2$ and $|U| = 1$, we have $\sum_{i \in T} q_i \leq 2$ and $r_U \leq 1$ and hence, $\sum_{i \in T \cup U} s_i \leq 3$, implying $s \in V(T \cup U)$. Noting that (N, V) is superadditive, we conclude that this game is coalition-merge convex, and because of Propositions 4.2 and 4.4, also individual-merge and marginal convex.

Finally, we show that (N, V) is cardinally convex. Let $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset$ and let $x^S \in V^\circ(S), x^T \in V^\circ(T)$. If $S \subset T$ or $T \subset S$, then (3.6) is trivially satisfied. If $S \cap T = \emptyset$, (3.6) follows from superadditivity. We distinguish between three further cases. First, if $|S| = |T| = 3$, then $|S \cap T| = 2$ and $S \cup T = N$. Take $x^{S \cap T} = (1^{S \cap T}, 0^{N \setminus (S \cup T)}) \in V^\circ(S \cap T)$ and define $x = x^S + x^T - x^{S \cap T}$. Then $\sum_{i \in S \cup T} x_i = \sum_{i \in S} x_i^S + \sum_{i \in T} x_i^T - 2 \leq 4 + 4 - 2 = 6$. Hence, $x \in V^\circ(S \cup T)$. Second, if $|S| = 2, T = |3|$, then $|S \cap T| = 1$ and $S \cup T = N$. Take $x^{S \cap T} = (0, 0, 0, 0) \in V^\circ(S \cap T)$ and define x as before. Then $\sum_{i \in S \cup T} x_i \leq 2 + 4 - 0 = 6$ and hence, $x \in V^\circ(S \cup U)$. Third, if $|S| = |T| = 2$, then $|S \cap T| = 1$ and $|S \cup T| = 3$. Take $x^{S \cap T} = (0, 0, 0, 0) \in V^\circ(S \cap T)$ and define x as before. Then $\sum_{i \in S \cup T} x_i \leq 2 + 2 - 0 = 4$ and hence, $x \in V^\circ(S \cup U)$. From these three cases we conclude that (N, V) is cardinally convex. ◁

Next, we prove that ordinal convexity does not imply any of the other four types of convexity.

Example 4.7 Consider the following NTU-game with player set $N = \{1, 2, 3\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(\{1, 2\}) = \{x \in \mathbb{R}^{\{1,2\}} \,|\, x_1 \leq 0, x_2 \leq 2\},$$

$$V(\{1, 3\}) = \{x \in \mathbb{R}^{\{1,3\}} \,|\, x_1 + x_3 \leq 1\},$$

$$V(\{2, 3\}) = \{x \in \mathbb{R}^{\{2,3\}} \,|\, x_2 \leq 0, x_3 \leq 0\},$$

$$V(N) = \{x \in \mathbb{R}^N \,|\, \sum_{i \in N} x_i \leq 2\}.$$

This game (N, V) is ordinally convex: let $S, T \subset N$ such that $S \neq \emptyset, T \neq \emptyset$ and let $x \in \mathbb{R}^N$ such that $x_S \in V(S)$ and $x_T \in V(T)$. We distinguish between four cases: if $S \subset T$ or $T \subset S$, (3.5) is trivially satisfied. If $S \cap T = \emptyset$, (3.5) is equivalent to superadditivity, which is satisfied by this game. If $S = \{1, 2\}$ and $T = \{1, 3\}$, then $x_1 \leq 0$ and hence, $x_{S \cap T} \in V(S \cap T)$. Otherwise, $\sum_{i \in N} x_i \leq 2$ and hence, $x_{S \cup T} \in V(S \cup T)$. From these four cases we conclude that (3.5) is satisfied and (N, V) is ordinally convex. However, this game is not marginal convex, because the marginal vector corresponding to $\sigma = (1, 2, 3)$, $M^\sigma(V) = (0, 2, 0)$, does not belong to the core, because player 1 and 3 have an incentive to leave the grand coalition. Using Propositions 4.2 and 4.4, we conclude that (N, V) is neither coalition-merge nor individual-merge convex. Furthermore, this game is not cardinally convex: $(0, 2, 0) \in V^\circ(\{1, 2\})$ and $(0, 0, 1) \in V^\circ(\{1, 3\})$, but $(0, 2, 0) + (0, 0, 1) = (0, 2, 1) \notin V^\circ(\{1\}) + V^\circ(N)$. $\triangleleft$

The example below shows that cardinal convexity does not imply any of the marginalistic types of convexity.

Example 4.8 Consider the following NTU-game with player set $N = \{1, 2, 3, 4\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(\{1, 2\}) = \{x \in \mathbb{R}^{\{1,2\}} \,|\, x_1 + x_2 \leq 2, x_2 \leq 1\},$$

$$V(S) = \{x \in \mathbb{R}^S \,|\, \max_{i \in S} x_i \leq 0\} \text{ for other } S \subset N, |S| = 2,$$

$$V(\{1, 2, 3\}) = \{x \in \mathbb{R}^{\{1,2,3\}} \,|\, x_1 + x_2 + x_3 \leq 2, x_3 \leq 2\},$$

$$V(\{1, 2, 4\}) = \{x \in \mathbb{R}^{\{1,2,4\}} \,|\, x_1 + x_2 + x_4 \leq 2, x_4 \leq 1\},$$

$$V(S) = \{x \in \mathbb{R}^S \,|\, \max_{i \in S} x_i \leq 0\} \text{ for other } S \subset N, |S| = 3,$$

$$V(N) = \{x \in \mathbb{R}^N \,|\, \sum_{i \in N} x_i \leq 2, x_3 \leq 2, x_4 \leq 1\}.$$

For the cardinal property (3.6), only the case with $S = \{1, 2, 3\}$ and $T = \{1, 2, 4\}$ is nontrivial. Let $x^S \in V^\circ(S), x^T \in V^\circ(T)$. Because $(1, 1, 0, 0) \in V^\circ(S \cap T)$, it is sufficient to show that $x = x^S + x^T - (1, 1, 0, 0) \in V^\circ(S \cup T) = V(N)$. Now,

$$\sum_{i \in N} x_i = \sum_{i \in S} x_i^S + \sum_{i \in T} x_i^T - 2 \leq 2 + 2 - 2 = 2,$$

$$x_3 = x_3^S + x_3^T = x_3^S + 0 \leq 2,$$

$$x_4 = x_4^S + x_4^T = 0 + x_4^T \leq 1.$$

Hence, $x \in V(N)$ and (N, V) is cardinally convex. For $\sigma = (1,2,3,4)$ we have $M^\sigma = (0,1,1,0)$. The players of coalition $\{1,2,4\}$ have an incentive to deviate from this vector, because the allocation $(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}) \in V(\{1,2,4\})$ gives them a strictly higher payoff. Hence, $M^\sigma(V) \notin C(V)$ and (N, V) is not marginal convex. Using Propositions 4.2 and 4.4, we conclude that (N, V) is neither coalition-merge nor individual-merge convex. ◁

Finally, we show that the three marginalistic convexity properties do not imply cardinal convexity.

Example 4.9 Consider the following NTU-game with player set $N = \{1,2,3\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 1\} \text{ for } S \subset N, |S| > 1.$$

This game is a 1-corner game (see Section 5.2) and it follows from Proposition 5.4 that (N, V) is coalition-merge convex (and hence, individual-merge and marginal convex as well). This game is, however, not cardinally convex: take $S = \{1,2\}, T = \{2,3\}$ and take $(1,1,0) \in V^\circ(S), (0,1,1) \in V^\circ(T)$. Then $(1,1,0) + (0,1,1) = (1,2,1) \notin V^\circ(S \cap T) + V^\circ(S \cup T)$. ◁

Summarising Propositions 4.2 and 4.4 and Examples 4.3 to 4.9, the five types of convexity for NTU-games are related as is depicted in Diagram 1. An arrow from one type of convexity to another indicates that the first one implies the second one. Where an arrow is absent, such an implication does not hold in general.

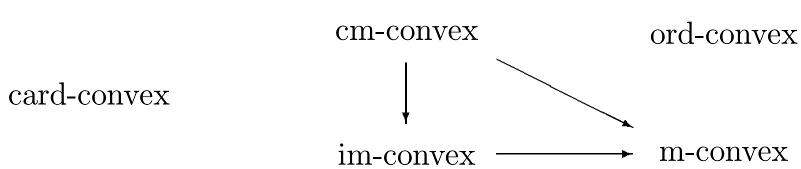

Diagram 1

The results in Diagram 1 hold for general n-player NTU-games. Some of the counterexamples that we used to show that certain implications do not hold, however, are games with four players. So, to round off this section, we state the relations between the five types of convexity for 3-player NTU-games in Diagram 2. The corresponding proofs and examples can be found in the appendix. To keep the picture clear, the arrows from cardinal convexity to ordinal and marginal convexity have been omitted.

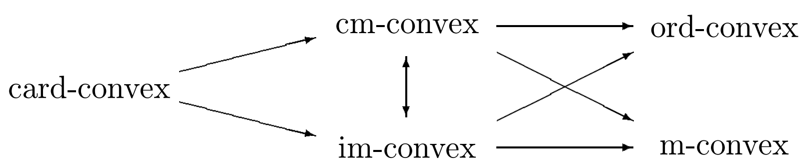

Diagram 2

5 Special Classes of Games

In this section, we look at our convexity notions in some specific classes of NTU-games.

5.1 Hyperplane Games

A *hyperplane game* is an NTU-game (N, V) such that for all coalitions $S \subset N, S \neq \emptyset$ we have

$$V(S) = \{x \in \mathbb{R}^S \,|\, x^\top a^S \leq b^S\}$$

for certain $a^S \in \mathring{\Delta}^S = \{x \in \mathbb{R}^S \,|\, \sum_{i \in S} x_i = 1, x > 0\}$ and $b^S \in \mathbb{R}$. Note that every entry of a^S must be positive to ensure boundedness of $V(S) \cap \mathbb{R}^S_+$. We denote the class of all hyperplane games with player set N by $\mathcal{H}^N$. A property of hyperplane games that we are going to use later on, is that these games possess a convex core.

Lemma 5.1 Let $(N, V) \in \mathcal{H}^N$. Then $C(V)$ is a convex set.

Proof: Let a^S, b^S for all $S \subset N, S \neq \emptyset$ be as in the definition. Then

$$\begin{aligned} C(V) &= \{x \in V(N) \,|\, \forall_{S \subset N, S \neq \emptyset} \nexists_{y \in V(S)} : y > x_S\} \\ &= \bigcap_{S \subset N, S \neq \emptyset} \{x \in \mathbb{R}^N \,|\, \nexists_{y \in V(S)} : y > x_S\} \cap V(N) \\ &= \bigcap_{S \subset N, S \neq \emptyset} \{x \in \mathbb{R}^N \,|\, x_S^\top a^S \geq b^S\} \cap V(N). \end{aligned}$$

$C(V)$ is the intersection of a finite number of halfspaces and a convex set and is hence convex. $\square$

A *parallel hyperplane game* is a hyperplane game (N,V) such that the projection of a^N onto $\mathring{\Delta}^S$ equals a^S for all coalitions $S \subset N, S \neq \emptyset$. We denote the class of parallel hyperplane games with player set N by $\mathcal{P}^N$.

Lemma 5.2 Let $(N,V) \in \mathcal{H}^N$. If (N,V) is individually superadditive, then it belongs to $\mathcal{P}^N$.

Proof: Assume that (N,V) is individually superadditive and let a^S, b^S for all $S \subset N, S \neq \emptyset$ be as in the definition. Let $S \subset N, S \neq \emptyset$. Take $p \in V(S)$ and let $i,j \in S$. Construct for all $\alpha \in \mathbb{R}$ the vector $p_\alpha = p + \alpha(\frac{a_i^S}{a_j^S} e_j - e_i)$, where e_j and e_i are unit vectors in $\mathbb{R}^S$. Then

$$\begin{aligned} p_\alpha^\top a^S &= p^\top a^S + \alpha(\frac{a_i^S}{a_j^S} e_j^\top a^S - e_i^\top a^S) \\ &= p^\top a^S + \alpha(\frac{a_i^S}{a_j^S} a_j^S - a_i^S) \\ &= p^\top a^S \\ &\leq b^S \end{aligned}$$

for all $\alpha \in \mathbb{R}$ and hence, $p_\alpha \in V(S)$. Next, define $q_\alpha = (p_\alpha, 0^{N \backslash S})$ for all $\alpha \in \mathbb{R}$. Applying individual superadditivity $|N \backslash S|$ times yields $q_\alpha \in V(N)$. Hence,

$$q_\alpha^\top a^N = p^\top a_S^N + \alpha(\frac{a_i^S}{a_j^S} e_j^\top a_S^N - e_i^\top a_S^N) \leq b^N$$

for all $\alpha \in \mathbb{R}$. The inequality can only hold for all $\alpha \in \mathbb{R}$ if the expression between parentheses equals zero. Therefore $\frac{a_i^S}{a_j^S} = \frac{a_i^N}{a_j^N}$. Hence, a^S is the projection of a^N onto $\mathring{\Delta}^S$ and $(N,V) \in \mathcal{P}^N$. $\square$

The following lemma relates the five convexity properties within the class of parallel hyperplane games.

Lemma 5.3 Within $\mathcal{P}^N$, coalition-merge, individual-merge, marginal, ordinal and cardinal convexity coincide.

Proof: First of all, note that all five convexity properties are scale invariant: if (N,V) satisfies some form of convexity, then so does (N,V^w) for every vector of scale factors $w \in \mathbb{R}^N_{++}$, where $V^w(S) = \{(w_i x_i)_{i\in S} \,|\, x \in V(S)\}$ for all $S \subset N, S \neq \emptyset$. In a parallel hyperplane game (N,V), we can choose w in such a way that (N,V^w) corresponds to a TU-game. From this the assertion follows. □

The relations between the various forms of convexity for hyperplane games are summarised in Diagram 3. For simplicity, the double arrow between cardinal and ordinal convexity and the arrow from cardinal to marginal convexity have been omitted. It follows from Lemmas 5.2 and 5.3 that within the class $\mathcal{H}^N$, coalition-merge, individual-merge, ordinal and cardinal convexity coincide. Because there are hyperplane games that are marginal convex, but not parallel, marginal convexity is weaker than the other four convexity notions.

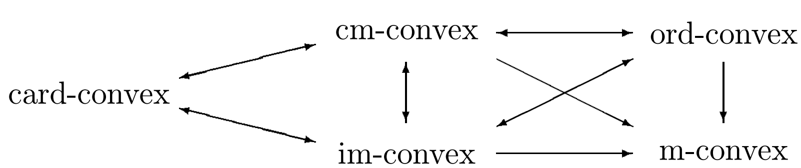

Diagram 3

5.2 1-Corner Games

An NTU-game is called a *1-corner game* if $V(S) = \{x \in \mathbb{R}^S \,|\, x \leq u^S\}$ for some $u^S \in \mathbb{R}^S$ for all $S \subset N, S \neq \emptyset$. We denote the class of 1-corner games with player set N by $\mathcal{C}^N$. Monotonicity implies that for all $S \subset T \subset N, S \neq \emptyset$ we must have $u^T_S \geq u^S$. From this, superadditivity readily follows.

The core of a 1-corner game is given by (cf. Otten (1985)):

$$C(V) = \bigcup_{\sigma\in\Pi(N)} \{x \in V(N) \,|\, x \geq M^\sigma(V)\} \tag{5.10}$$

In the following lemma we show that all 1-corner games are coalition-merge convex.

Proposition 5.4 Let $(N,V) \in \mathcal{C}^N$. Then (N,V) is coalition-merge convex.

Proof: Let $U \subset N$ such that $U \neq \emptyset$, let $S \subsetneq T \subset N \backslash U$ such that $S \neq \emptyset$ and let $p \in WPar(S) \cap IR(S), q \in V(T)$ and $r \in V(S \cup U)$ such that $r_S \geq p$. Then it is sufficient to show that $(q, r_U) \in V(T \cup U)$. First, $q \in V(T)$, so $q \leq u^T$. Similarly, $r \leq u^{S \cup U}$ and hence, $r_U \leq u_U^{S \cup U}$. Because of monotonicity, we have $q \leq u_T^{T \cup U}$ and $r_U \leq u_U^{T \cup U}$. Therefore, $(q, r_U) \leq u^{T \cup U}$ and $(q, r_U) \in V(T \cup U)$. □

It can be shown in a similar fashion that every 1-corner game is ordinally convex. However, a 1-corner game need not be cardinally convex, as is illustrated by Example 4.9.

5.3 Bargaining Games

A *bargaining situation* is a pair (F, d) where $F \subset \mathbb{R}^N$ is a closed, convex and comprehensive set of attainable utility vectors and $d \in F$ is a disagreement point (in F) such that there exists a $y \in F$ with $y > d$.

A bargaining situation with $d = 0$ gives rises to the *bargaining game* (N, V) with $N = \{1, \ldots, n\}, V(S) = \{x \in \mathbb{R}^S \,|\, x \leq 0\}$ for all $S \subsetneqq N, S \neq \emptyset$ and $V(N) = F$. We denote the class of bargaining games with player set N by $\mathcal{B}^N$.

Proposition 5.5 Let $(N, V) \in \mathcal{B}^N$. Then (N, V) satisfies all five convexity properties.

Proof: Define the game (N, W) with $W(S) = \mathbb{R}^S_-$ for all coalitions $S \subset N, S \neq \emptyset$. Then (N, W) trivially satisfies all five convexity properties. Because $V(S) = W(S)$ for all $S \subsetneqq N, S \neq \emptyset$ and $V(N) \supsetneqq W(N)$, it follows from the definitions (3.5)-(3.9) that (N, V) satisfies all five convexity properties as well. □

6 Solution Concepts

In this section we investigate how some solution concepts for NTU-games relate to our convexity notions. A *solution* Ψ on a class $\mathcal{Z}^N \subset NTU^N$ of NTU-games is a correspondence $\Psi : \mathcal{Z}^N \twoheadrightarrow \mathbb{R}^N$, assigning to every $V \in \mathcal{Z}^N$ a set of payoff vectors $\Psi(V) \subset \mathbb{R}^N$. A *value* is a function $\Psi : \mathcal{Z}^N \to \mathbb{R}^N$ assigning to every game $V \in \mathcal{Z}^N$ a single payoff vector $\Psi(V) \in \mathbb{R}^N$.

6.1 The MC-Value

The *marginal based compromise value* or *MC-value* was introduced in Otten et al. (1998) and is defined as

$$MC(V) = \alpha_V \sum_{\sigma \in \Pi(N)} M^\sigma(V),$$

where $\alpha_V = \max\{\alpha \in \mathbb{R}_+ \,|\, \alpha \sum_{\sigma\in\Pi(N)} M^\sigma(V) \in V(N)\}$.

Proposition 6.1 Let $(N,V) \in NTU^N$. If (N,V) is marginal convex and belongs to $\mathcal{H}^N$, $\mathcal{C}^N$ or $\mathcal{B}^N$, then $MC(V) \in C(V)$.

Proof: Assume (N,V) is marginal convex. For $(N,V) \in \mathcal{H}^N$ and $(N,V) \in \mathcal{C}^N$, the statement follows from Lemma 5.1 and equation (5.10), respectively. If $(N,V) \in \mathcal{B}^N$, then it is easily seen that the core includes the set on the right hand side of (5.10), from which $MC(V) \in C(V)$ follows. □

6.2 The Compromise Value

The *compromise value* is introduced in Borm et al. (1992) and is an extension of the τ-value for TU-games (Tijs (1981)). The compromise value is a compromise between two payoff vectors. The first one is the utopia vector $K(V)$, defined by

$$K_i(V) = \sup\{t \in \mathbb{R} \,|\, \exists_{a \in \mathbb{R}_+^{N\setminus\{i\}}} : (a,t) \in V(N), \nexists_{b \in V(N\setminus\{i\})} : b > a\}$$

for all $i \in N$. The second one is the minimal right vector $k(V)$, defined by

$$k_i(V) = \max_{S: i \in S} \rho_i^S(V)$$

for all $i \in N$, where $\rho_i^S(V)$ is the remainder for player i after giving the other members in S their utopia payoff:

$$\rho_i^S(V) = \sup\{t \in \mathbb{R} \,|\, \exists_{a \in \mathbb{R}^{S\setminus\{i\}}} : (t,a) \in V(S), a > K_{S\setminus\{i\}}(V)\}.$$

The following lemma comes from Borm et al. (1992).

Lemma 6.2 Let $(N,V) \in NTU^N$ with $x \in C(V)$. Then $k(V) \leq x \leq K(V)$.

A game (N, V) is called compromise admissible if $k(V) \leq K(V), k(V) \in V(N)$ and there does not exist a $b \in V(N)$ such that $b > K(V)$. In view of Lemma 6.2, every NTU-game with a nonempty core is compromise admissible. For a compromise admissible game, the compromise value $T(V)$ is defined by

$$T(V) = \lambda_V K(V) + (1 - \lambda_V) k(V),$$

where

$$\lambda_V = \max\{\lambda \in [0,1] \mid \lambda K(V) + (1-\lambda) k(V) \in V(N)\}.$$

An NTU-game (N, V) is called *semi-convex* if $k(V) = 0$. This definition extends the definition of semi-convexity for TU-games in Driessen and Tijs (1985)[3]. In TU-games, semi-convexity is implied by convexity and the next lemma states the corresponding result for NTU-games.

Lemma 6.3 Let $(N, V) \in NTU^N$. If (N, V) is marginal convex, then it is semi-convex.

Proof: Assume (N, V) is marginal convex. Let $i \in N$ and let $\sigma \in \Pi(N)$ be such that $\sigma(1) = i$. By construction, $M_i^\sigma(V) = 0$. Because of Lemma 6.2, we have $k_i(V) \leq M_i^\sigma(V) = 0$. On the other hand, $k_i(V) = \max_{S: i \in S} \rho_i^S(V) \geq \rho_i^{\{i\}}(V) = 0$. We conclude that $k_i(V) = 0$ for all $i \in N$ and (N, V) is semi-convex. □

As a corollary, we obtain the following proposition, in which compromise admissibility follows from nonemptiness of the core.

Proposition 6.4 Let $(N, V) \in NTU^N$. If (N, V) is marginal convex, then it is compromise admissible and the compromise value is proportional to the utopia payoff vector.

6.3 The Bargaining Set

The (Maschler) *bargaining set* for an NTU-game (N, V) is defined as (cf. Aumann and Maschler (1964))

[3]Contrary to the TU-game case, we do not require superadditivity in the definition of semi-convexity.

$$\begin{aligned}\mathcal{M}(V) \;\; = \{x \in I(V) \mid \;\; & \forall_{i,j \in N} \forall_{S \subset N, i \in S, j \notin S} \forall_{y \in WPar(S), y > x_S} \\ & \exists_{T \subset N, i \notin T, j \in T} \exists_{z \in WPar(T)} : z \geq (y_{S \cap T}, x_{T \setminus S})\}.\end{aligned}$$

The bargaining set consists of those imputations x such that whenever player i raises an objection against player j by cooperating with coalition S and promising the members of S more than they get according to x, player j can counter this objection by cooperating with coalition T, giving each player in $S \cap T$ at least the amount they are promised by i.

It is a well-known result that in TU-games, this set is always nonempty and contains the core. For convex TU-games, the bargaining set coincides with the core (cf. Solymosi (1999)). In NTU-games, the bargaining set still contains the core, but there are games in which $\mathcal{M}(V)$ is empty. In the next example we show that even a strong form of convexity does not ensure that $\mathcal{M}(V) = C(V)$.

Example 6.5 Consider the same game as in Example 4.9, which is coalition-merge convex. The imputation $x = (\frac{1}{2}, \frac{1}{2}, 1)$ does not belong to the core, but we show that $x \in \mathcal{M}(V)$. By symmetry, we only have to look at objections of player 1 against player 3. Player 1 cannot object on his own, but only through coalition $S = \{1, 2\}$. The maximum payoff vector player 1 can promise is $y = (1, 1)$. But player 3 can counter this objection through coalition $T = \{2, 3\}$ and payoff vector $z = (1, 1)$. Hence, $x \in \mathcal{M}(V)$ although $x \notin C(V)$ and (N, V) is coalition-merge convex. $\triangleleft$

Of course, there might be some subclass of NTU^N for which coalition-merge convexity (or even a weaker form of convexity) implies $\mathcal{M}(V) = C(V)$. The proof in Solymosi (1999) for the corresponding TU-result uses excess games and it might be interesting to investigate how this result can be extended to NTU-games, and in particular, what definition of excess games can be used in this context.

A Appendix

In this appendix we give the proofs and examples that relate the five convexity properties for 3-player NTU-games. First, we prove that in 3-player NTU-games, individual-merge convexity implies coalition-merge convexity.

Proposition A.1 Let $(N,V) \in NTU^N$ such that $|N| = 3$. If (N,V) is individual-merge convex, then it is coalition-merge convex.

Proof: Assume (N,V) is individual-merge convex. Then (N,V) is individually superadditive, and because there are only three players, superadditive. For the coalition-merge property, if $|U| = 1$, then (3.7) is equivalent to (3.8). For $|U| > 1$, we cannot find coalitions S and T such that $S \subsetneq T \subset N \backslash U$ and $S \neq \emptyset$. Hence, the coalition-merge property is satisfied. $\square$

Next, we show that in 3-player games, coalition-merge convexity implies ordinal convexity.

Proposition A.2 Let $(N,V) \in NTU^N$ such that $|N| = 3$. If (N,V) is coalition-merge convex, then it is ordinally convex.

Proof: Assume (N,V) is coalition-merge convex. Let $S_1, S_2 \subset N$ such that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. If $S_1 \subset S_2$ or $S_2 \subset S_1$, then (3.5) is trivially satisfied. If $S_1 \cap S_2 = \emptyset$, (3.5) is satisfied because (N,V) is superadditive. Otherwise, let $x \in \mathbb{R}^N$ such that $x_{S_1} \in V(S_1)$ and $x_{S_2} \in V(S_2)$ and suppose $x_{S_1 \cap S_2} \notin V(S_1 \cap S_2)$. Then $x_{S_1 \cap S_2} > 0$ because $|S_1 \cap S_2| = 1$. Next, define $U = S_2 \backslash S_1$, $S = S_1 \cap S_2$ and $T = S_1$ and take $p = 0 \in WPar(S) \cap IR(S)$, $q = x_{S_1} \in V(T)$ and $r = x_{S_2} \in V(S \cup U)$. Now $r_S = x_{S_1 \cap S_2} > 0 = p$. Because (N,V) is coalition-merge convex, there exists an $s \in V(T \cup U) = V(N)$ such that $s \geq (q, r_U) = (x_T, x_U) = x_{S_1 \cup S_2}$. Hence, $x_{S_1 \cup S_2} \in V(N) = V(S_1 \cup S_2)$ and (N,V) is ordinally convex. $\square$

The following example shows that in 3-player NTU-games, marginal convexity need not imply ordinal convexity.

Example A.3 Consider the following NTU-game with player set $N = \{1,2,3\}$:

$V(\{i\}) = (-\infty, 0]$ for all $i \in N$,

$$V(S) = \{x \in \mathbb{R}^S \mid \max_{i \in S} x_i \leq 1\} \text{ for all } S \subset N, |S| = 2,$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 2\}.$$

The marginal vectors of this game are

σ	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$
M^σ	$(0,1,1)$	$(0,1,1)$	$(1,0,1)$	$(1,0,1)$	$(1,1,0)$	$(1,1,0)$

and the core is

$$C(V) = \{(1,1,0), (1,0,1), (0,1,1)\}.$$

This game is marginal convex. For ordinal convexity, consider $S = \{1,2\}, T = \{2,3\}$ and $x = (1,1,1) \in \mathbb{R}^N$. Then we have both $x_S \in V(S)$ and $x_T \in V(T)$, but neither $x_{S \cap T} \in V(S \cap T)$ nor $x_{S \cup T} \in V(S \cup T)$. Hence, (N,V) is not ordinally convex. $\triangleleft$

Finally, we show that in 3-player games, cardinal convexity implies coalition-merge convexity.

Proposition A.4 Let $(N,V) \in NTU^N$ such that $|N| = 3$. If (N,V) is cardinally convex, then it is coalition-merge convex.

Proof: Assume (N,V) is cardinally convex. Then it is superadditive. For the coalition-merge property, let $U \subset N$ such that $U \neq \emptyset$ and let $S \subsetneq T \subset N \backslash U$ such that $S \neq \emptyset$. Let $p \in WPar(S) \cap IR(S)$, $q \in V^\circ(T)$ and $r \in V^\circ(S \cup U)$ such that $r_S \geq p$. Because $|S| = 1$, we have $p = 0$ and hence, $r_S \geq 0$. Next, define $\hat{S} = S \cup U$. Then $q + r \in V^\circ(\hat{S}) + V^\circ(T)$ and because of cardinal convexity, there exists an $s \in V^\circ(\hat{S} \cap T) + V^\circ(\hat{S} \cup T)$ such that $s \geq q + r$. Because $|\hat{S} \cap T| = |S| = 1$, $V^\circ(\hat{S} \cap T) = \mathbb{R}_-$ and $s \in V^\circ(\hat{S} \cup T) = V(N) = V(T \cup U)$. Furthermore, $s_T = (s_S, s_{T \backslash S}) \geq (r_S + q_S, q_{T \backslash S}) \geq q$ and $s_U = r_U$. So s satisfies (3.7) and (N,V) is coalition-merge convex. $\square$

As a corollary, we obtain that in 3-player NTU-games, cardinal convexity implies individual-merge, marginal and ordinal convexity as well.

References

Abdou, J. and H. Keiding (1991). *Effectivity Functions in Social Choice*. Dordrecht: Kluwer Academic Press.

Aumann, R.J. and M. Maschler (1964). The bargaining set for cooperative games. In: M. Dresher, L.S. Shapley, and A.W. Tucker (Eds.), *Advances in Games Theory*, pp. 443–476. Princeton: Princeton University Press.

Aumann, R.J. and M. Maschler (1985). Game Theoretic Analysis of a Bankruptcy Problem from the Talmud. *Journal of Economic Theory*, **36**, pp. 195–213.

Borm, P., H. Keiding, R.P. McLean, S. Oortwijn, and S. Tijs (1992). The Compromise Value for NTU-Games. *International Journal of Game Theory*, **21**, pp. 175–190.

Curiel, I., G. Pederzoli, and S. Tijs (1989). Sequencing Games. *European Journal of Operational Research*, **54**, pp. 323–334.

Driessen, T.S.H. and S.H. Tijs (1985). The τ-value, the Core and Semiconvex Games. *International Journal of Game Theory*, **14**, pp. 229–247.

Ichiishi, T. (1981). Supermodularity: Applications to Convex Games and the Greedy Algorithm for LP. *Journal of Economic Theory*, **25**, pp. 283–286.

Littlechild, S. and G. Owen (1973). A Simple Expression for the Shapley Value in a Special Case. *Management Science*, **20**, pp. 370–372.

Otten, G.J.M. (1985). *On Decision Making in Cooperative Situations*. Ph. D. thesis, Tilburg University, Tilburg, The Netherlands.

Otten, G.J.M., P.E.M. Borm, B. Peleg, and S.H. Tijs (1998). The MC-value for Monotonic NTU-Games. *International Journal of Game Theory*, **27**, pp. 37–47.

Shapley, L. (1971). Cores of Convex Games. *International Journal of Game Theory*, **1**, pp. 11–26.

Sharkey, W. (1981). Convex Games without Side Payments. *International Journal of Game Theory*, **10**, pp. 101–106.

Solymosi, T. (1999). On the Bargaining Set, Kernel and Core of Superadditive Games. *International Journal of Game Theory*, **28**, pp. 229–240.

Suijs, J. (2000). *Cooperative Decision-Making under Risk*. Theory and Decision Library - Series C: Game Theory, Mathematical Programming and Mathematical

Economics. Boston: Kluwer Academic Publishers.

Suijs, J. and P. Borm (1999). Stochastic Cooperative Games: Superadditivity, Convexity and Certainty Equivalents. *Games and Economic Behavior*, **27**, pp. 331–345.

Tijs, S.H. (1981). Bounds for the Core and the τ-Value. In: O. Moeschlin and D. Pallaschke (Eds.), *Game Theory and Mathematical Economics*, pp. 123–132. Amsterdam: North Holland Publishing Company.

Timmer, J., P. Borm, and S. Tijs (2000). Convexity in Stochastic Cooperative Situations. CentER Discussion Paper 2000-04, Tilburg University, Tilburg, The Netherlands.

Vilkov, V. (1977). Convex Games without Side Payments. *Vestnik Leningradskiva Universitata*, **7**, pp. 21–24. (In Russian).