

A Model Distinguishing Production and Consumption Bundles

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August 1998

Abstract

In contrast to the neo-classical theory of Arrow and Debreu, a model of a private ownership economy is presented in which production and consumption bundles are treated separately. Each of the two types of bundles is assumed to establish a convex cone. The main part in the modelling is the introduction of production technologies which can be thought of as replacing the notion of production sets in Arrow and Debreu's model. It is a point of further investigation whether the notion of production technology is also generating the notion of production set.

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- The author would like to thank Stef van Eijndhoven and Dolf Talman for their valuable comments.
- This research is part of the VF-program "Competition and Cooperation".
- The author is financially supported by the Cooperation Centre Tilburg and Eindhoven Universities, The Netherlands.

Introduction

The introduction of a new mathematical model of a private ownership economy, a corresponding Walrasian equilibrium theorem and the mathematics surrounding them are the main topics of this paper. Thus, we hope to achieve a more realistic model of a private ownership economy.

As far as the model is concerned, it differs from the neo-classical models, described in the standard works of [Debreu] and [Arrow/Debreu], in the following two features.

- The model recognises commodity bundles instead of separate commodities.
- The model treats production and consumption on a different level.

Our model of a private ownership economy is only in terms of convex cones and their properties, and not in terms of vector spaces, whereas the neo-classical models are set in terms of the Euclidean space \mathbb{R}^n . We emphasize this use of convex cones by the axiomatic introduction of the concept of salient half-space. We define a salient half-space to be a set in which addition and scalar multiplication over the positive reals are defined such that the set is an addition semi-group and such that the distributive axioms are satisfied. The main difference between a salient half-space and a vector space is that for a salient half-space multiplication is allowed over the non-negative real numbers, only. Each pointed convex cone in which addition and scalar multiplication are defined in the natural way by its surrounding vector space, is a salient half-space. Furthermore, each salient half-space induces an ordered vector space for which the salient half-space is the positive cone. A great deal of effort in this paper is put in the presentation of this mathematical concept and related topics.

The use of salient half-spaces allows us to not distinguish separate commodities. In fact, we do not need to consider the concept of commodity at all, and will consider the concept of "economy bundle" instead. In a worldlike example, our model can describe the non-neo-classical situation in which fixed links between different commodities may be assumed present, for instance an economy in which only fixed, prescribed combinations of commodities can be traded.

In the presented model, an economy bundle is a unique concatenation of a production (economy) bundle and a consumption (economy) bundle. Here, only production bundles can be used as input for a production process whereas the output of this process is always a consumption bundle. The set C of economy bundles is taken to be the product set $C_{\text{prod}} \times C_{\text{cons}}$ where the salient half-spaces C_{prod} and C_{cons} contain the production and consumption bundles, respectively.

If it is possible to produce consumption bundle x^{cons} from production bundle x^{prod} , we call economy bundle $(x^{\text{prod}}, x^{\text{cons}}) \in C$ a production process. A collection T of production processes is called a production technology if it satisfies certain conditions, to be specified later. As far as we know, in the neo-classical models, consumption (economy) bundles and production (economy) bundles are not distinguished explicitly: instead of introducing a production technology T as a subset of $C_{\text{prod}} \times C_{\text{cons}}$, the neo-classical models recognise a

production technology (production set) as a subset Y of the Euclidean vector space \mathbb{R}^n . Globally speaking, the vector lattice \mathbb{R}^n with corresponding production set Y is replaced by the salient half-space $C_{\text{prod}} \times C_{\text{cons}}$ with production technology T . Indeed, \mathbb{R}^n can be regarded as the product of the positive cone $(\mathbb{R}^n)^+$ and the negative cone $(\mathbb{R}^n)^-$ by corresponding to each input-output vector $x \in \mathbb{R}^n$, the pair (x^-, x^+) with output vector x^+ and input vector x^- defined by $x^+ := 0 \vee x$ and $x^- := (-x) \vee 0$. So, to each $x \in Y$ there is associated a unique pair $(x^+, x^-) \in (\mathbb{R}^n)^+ \times (\mathbb{R}^n)^+$, and thus Y can be seen as a subset \tilde{Y} of $(\mathbb{R}^n)^+ \times (\mathbb{R}^n)^+$. We emphasize that the natural lattice structure of \mathbb{R}^n with positive cone $(\mathbb{R}^n)^+$ enables to regard Y this way. However, \tilde{Y} does not satisfy the conditions we impose on T , in general. In fact, in our model, lattice structures are not involved at all. In this paper, we shall not discuss whether the neo-classical notion of production technology (Y) is generalised by our notion of production technology (T). This will be part of further research.

Disregarding the concept of commodity, we cannot speak of the price of a commodity, and so, we use the notion of "pricing function" which gives a value to every economy bundle. Furthermore, the introduction of the concept of production and consumption bundles gives rise to a slightly altered definition of Walrasian equilibrium. Although the model is presented in the general terms of salient half-spaces, existence of these Walrasian equilibria can be guaranteed only if some assumptions are made, of which the assumption that the vector space for which the salient half-space is the positive cone, is finite dimensional, is the strongest. Despite this, we feel that the essential idea of this model is the use of the concept of salient half-space and concepts related to it. Forcing ourselves to cope with this general model structure, we have to apply an analysis and techniques which may be of use when tackling models for private ownership economies where the finite dimensionality restriction is not satisfied.

We conclude this introduction by describing the contents of the different sections. Section 1 contains the introduction of the mathematical concepts and theorems which are used to construct the model and to prove the Equilibrium Existence Theorem. Its main item is the introduction of the concept of salient half-space and its relationship with vector spaces. The presentation in this section is almost self containing. In Section 2 we describe the mathematical model introducing the features of the economic agents, and of the production technologies. The Equilibrium Existence Theorem is stated and the mathematical assumptions, needed in its proof, are introduced. Furthermore, a sketch of the proof is presented.

1 Mathematical concepts

The purpose of this section is the description of the mathematical concepts involved in the model of a private ownership economy presented in Section 2.

1.1 Salient half-space

We start with the concept of salient half-space, since we shall use this notion to model the set of economy bundles. Thereafter, we describe some similarities and differences between salient half-spaces, vector spaces, and convex cones.

Definition 1.1.1 *A salient half-space is a set C with the following properties:*

- *An addition is defined on C , which is commutative, associative and satisfies*
 - 1.1.1.a)** *there exists an element $v \in C$, called the vertex of C , such that $x + y = v \iff x = y = v$, for all $x, y \in C$,*
 - 1.1.1.b)** *for every $x \in C$ the mapping $\text{add}_x : C \rightarrow C$, defined by $\text{add}_x(y) := y + x$, is injective.*
- *To every pair $x \in C$ and $\alpha \geq 0$, there corresponds an element $\alpha x \in C$, called the (scalar) product of α and x . Scalar multiplication over \mathbb{R}^+ thus defined, is associative and satisfies the distributive laws. Furthermore, $1x = x$ for every $x \in C$.*

Note that Condition 1.1.1.a implies that the mapping add_x is surjective if and only if $x = v$. Given $x, y, z \in C$, with $x = y + z$, it is meaningful to write $z = x - y$. To avoid confusion, we shall not use this notation.

Example

Let C be a pointed convex cone in a vector space V , then C is a salient half-space with the zero-element of V as vertex, and addition and multiplication defined in the natural way. Recall that a subset C of a vector space V is called a cone if $\alpha x \in C$ for all $x \in C$ and $\alpha \geq 0$. A cone is called pointed if the zero-element of V is the only extreme point of C . A subset D of a vector space is called convex if $\tau x + (1 - \tau)y \in D$ for all $x, y \in D$ and $\tau \in [0, 1]$. Thus, a cone in a vector space is convex if and only if it is closed under addition.

We shall see that the converse also holds: For every salient half-space C , there is a vector space $V[C]$ such that C is a pointed convex cone in $V[C]$.

It is not difficult to prove that the vertex of a salient half-space is unique and satisfies

- a) $\forall \alpha > 0 : \alpha v = v$,
- b) $\forall x \in C : x + v = x$,
- c) $\forall x \in C : 0x = v$.

From the second property together with Conditions 1.1.1.a and 1.1.1.b, we conclude that $(C, +)$ is an addition semi-group with zero-element v . Since in a salient half-space, scalar

multiplication is defined only over \mathbb{R}^+ and due to Condition 1.1.1.a, $(C, +)$ is not a group. However, we can extend $(C, +)$ to a group in a similar way as $\mathbb{N} \cup \{0\}$ extends to \mathbb{Z} . We shall present this extension in short. Define the equivalence relation \sim on the product set $C \times C$ by:

$$(x_1, x_2) \sim (y_1, y_2) : \iff x_1 + y_2 = y_1 + x_2.$$

Let $V[C]$ be the collection of all equivalent classes $[(y_1, y_2)] := \{(z_1, z_2) \in C \times C \mid (z_1, z_2) \sim (y_1, y_2)\}$, so $V[C] := (C \times C)/\sim$. Unambiguously, we can define the following addition and scalar multiplication on $V[C]$:

$$\begin{aligned} [(y_1, y_2)] + [(z_1, z_2)] &:= [(y_1 + z_1, y_2 + z_2)] \\ \alpha[(y_1, y_2)] &:= \begin{cases} [(\alpha y_1, \alpha y_2)] & \text{if } \alpha \geq 0 \\ [(-\alpha y_2, -\alpha y_1)] & \text{if } \alpha < 0. \end{cases} \end{aligned}$$

We shall make plausible that with these definitions, the set $V[C]$ becomes a real vector space. We call $V[C]$ the vector space generated by the salient half-space C .

In general, if $(A, +)$ is a semi-group with a zero-element, then the above construction can be applied to construct a group. So the proof that $V[C]$ is indeed a vector space can concentrate on the introduction of the scalar product over negative α . The construction yields that $[(v, v)]$ is the origin of $V[C]$ and $-[(y_1, y_2)] = [(y_2, y_1)]$. Note that multiplication by negative scalars is defined properly. Let $\alpha > 0$ then

$$(-\alpha)[(y_1, y_2)] = \alpha(-1)[(y_1, y_2)] = \alpha[(y_2, y_1)] = \alpha(-[(y_1, y_2)]).$$

Furthermore, the salient half-space C is a total subset of the vector space $V[C]$, i.e., the linear span of C equals $V[C]$. The vertex v of C coincides with the origin of the vector space $V[C]$, and henceforward we shall denote the vertex of a salient half-space by 0.

Definition 1.1.2 *On a salient half-space C the partial ordering \leq_C is given by*

$$\begin{aligned} x \leq_C y & \text{ if and only if } \exists z \in C : x + z = y, \\ x <_C y & \text{ if and only if } \exists z \in C \setminus \{0\} : x + z = y. \end{aligned}$$

The salient half-space C , when identified with $\{[(y_1, y_2)] \in V[C] \mid \exists x \in C : [(y_1, y_2)] \sim [(x, 0)]\}$, can be regarded as a subset of $V[C]$. The partial ordering \leq_C , defined on C , can be extended to a partial ordering on $V[C]$ by defining for all $[(y_1, y_2)], [(z_1, z_2)] \in V[C]$:

$$[(y_1, y_2)] \leq_C [(z_1, z_2)] \text{ if } \exists [(x_1, x_2)] \in C : [(y_1, y_2)] + [(x_1, x_2)] = [(z_1, z_2)].$$

Note that this is equivalent with $y_1 + x_1 + z_2 = y_2 + x_2 + z_1$, or

$$y_1 + z_2 \leq_C y_2 + z_1.$$

Also, note that $C := \{[(y_1, y_2)] \in V[C] \mid [(0, 0)] \leq_C [(y_1, y_2)]\}$.

It is customary to introduce a pointed convex cone in a vector space, therewith introducing a partial ordering on this vector space. Since we consider the salient half-space, rather than the vector space, to be the essential element of the model, we introduce these notions the other way around,

Definition 1.1.3 An element u of C is called an order unit for C if

$$\forall x \in C \exists \lambda \geq 0 : x \leq_C \lambda u.$$

Lemma 1.1.4 Let u be an order unit for C , and let $[(y_1, y_2)] \in V[C]$. Then

$$\exists \lambda \geq 0 : -\lambda[(u, 0)] \leq_C [(y_1, y_2)] \leq_C \lambda[(u, 0)].$$

Proof

Since u is an order unit for C , we find $\begin{cases} \exists \lambda_1 \geq 0 : y_1 \leq_C \lambda_1 u \\ \exists \lambda_2 \geq 0 : y_2 \leq_C \lambda_2 u. \end{cases}$

Define $\lambda := \max\{\lambda_1, \lambda_2\}$, then $\begin{cases} y_1 \leq_C y_2 + \lambda u \\ y_2 \leq_C y_1 + \lambda u. \end{cases}$ □

1.2 Salient half-dual space

Let C be salient half-space.

Definition 1.2.1 A functional $p : C \rightarrow \mathbb{R}^+$ is said to be half-linear if p satisfies

$$\begin{cases} p(x + y) = p(x) + p(y) & \forall x, y \in C \\ p(\alpha x) = \alpha p(x) & \forall x \in C \forall \alpha \geq 0. \end{cases}$$

The set of all half-linear functionals defined on C will be denoted by C^* . From the definition it follows that the set C^* is a salient half-space also, where the zero-functional is its vertex and addition and positive scalar multiplication are defined pointwise; for $p, q \in C^*$ and $\alpha \geq 0$:

$$\begin{cases} (p + q)(x) := p(x) + q(x) & \forall x \in C \\ (\alpha p)(x) := \alpha p(x) & \forall x \in C. \end{cases}$$

We call C^* the salient half-dual space of C or, in short, the half-dual of C .

It turns out (cf. [Conway]) that existence of an order unit in C is sufficient to guarantee that C^* is non-trivial, i.e., $C^* \neq \{0\}$.

Proposition 1.2.2 If C has an order unit, then $C^* \neq \{0\}$.

Proof

Let u be an order unit for C . Define the set $U \subset V[C]$ by $U := \{\lambda[(u, 0)] \mid \lambda \in \mathbb{R}\}$, then U is a subspace of $V[C]$. By Lemma 1.1.4, we find

$$\forall [(y_1, y_2)] \in V[C] \exists \lambda \geq 0 : -\lambda[(u, 0)] \leq_C [(y_1, y_2)] \leq_C \lambda[(u, 0)].$$

Thus, we can define the sublinear functional $q : V[C] \rightarrow \mathbb{R}$ by

$$q([(y_1, y_2)]) := \inf\{\lambda \mid [(y_1, y_2)] \leq_C \lambda[(u, 0)]\}.$$

Define $f(\lambda[(u, 0)]) := \lambda$, for every $\lambda \in \mathbb{R}$. With this definition, $f : U \rightarrow \mathbb{R}$ is a positive linear functional on U satisfying $\forall \lambda \in \mathbb{R} : f(\lambda[(u, 0)]) = q(\lambda[(u, 0)])$. By the Hahn-Banach Theorem, there exists a linear functional $\tilde{f} : V[C] \rightarrow \mathbb{R}$ such that on the set U , \tilde{f} is equal to f , and $\forall [(y_1, y_2)] \in V[C] : \tilde{f}([(y_1, y_2)]) \leq q([(y_1, y_2)])$. For every $[(x_1, x_2)] \in C$ it holds that $q([(x_1, x_2)]) \geq 0$. We conclude that the functional \tilde{f} acts positively on C since for all $[(x_1, x_2)] \in C : \tilde{f}(-[(x_1, x_2)]) \leq q(-[(x_1, x_2)]) \leq 0$. \square

Applying Definition 1.1.2 on the salient half-dual space, we find the partial ordering \leq_{C^*} on C^* , which is given by

$$\begin{aligned} p \leq_{C^*} q & \text{ if and only if } \exists r \in C^* : p + r = q. \\ p <_{C^*} q & \text{ if and only if } \exists r \in C^* \setminus \{0\} : p + r = q. \end{aligned}$$

Note that this partial ordering is equivalent with the standard partial ordering on functionals in $(V[C])^*$:

$$\begin{aligned} p \leq_{C^*} q & \iff \forall x \in C : p(x) \leq q(x). \\ p <_{C^*} q & \iff (\forall x \in C : p(x) \leq q(x)) \wedge (\exists x \in C : p(x) < q(x)). \end{aligned}$$

First we examine the relationship between the vector space $V[C^*]$, generated by the half-dual C^* of C , and the dual space $(V[C])^*$ of $V[C]$.

Proposition 1.2.3 $V[C^*]$ is canonically injected in $(V[C])^*$ and therefore can be considered a subspace of $(V[C])^*$. Furthermore, $C^* = \{p \in (V[C])^* \mid \forall x \in C : p(x) \geq 0\}$.

Proof

Let $[(p_1, p_2)] \in V[C^*]$ and define for every $[(y_1, y_2)] \in V[C]$:

$$[(p_1, p_2)]([(y_1, y_2)]) := p_1(y_1) - p_1(y_2) - p_2(y_1) + p_2(y_2).$$

It is easy to check that this definition is independent of the choice of the representatives (y_1, y_2) and (p_1, p_2) , and that with this definition $[(p_1, p_2)]$ acts as a linear functional on $V[C]$. Secondly, it is easy to check that the mapping, described above, which adds a linear functional to every pair $[(p_1, p_2)] \in V[C^*]$ is linear. Furthermore, if $\forall [(x_1, x_2)] \in V[C]$ it holds that $[(p_1, p_2)]([(x_1, x_2)]) = 0$, then $\forall x \in C : [(p_1, p_2)]([(x, 0)]) = p_1(x) - p_2(x) = 0$, and we conclude $p_1 = p_2$, or, in other words, $[(p_1, p_2)] = [(0, 0)]$. \square

In the sequel we shall regard C^* as a subset of $(V[C])^*$.

Let W be a vector space. Then $S \subset W^*$ is said to be separating the elements of a subset $M \subset W$ if $\forall x, y \in M, x \neq y \exists p \in S : p(x) \neq p(y)$. If M is linear, this comes down to $\forall x \in M \setminus \{0\} \exists p \in S : p(x) \neq 0$.

Lemma 1.2.4 A set $S_0 \subset C^*$ separates the elements of C if and only if the collection $S := \{[(p_1, p_2)] \mid p_1, p_2 \in S_0\} \subset V[C^*]$ separates the elements of $V[C]$.

Proof

Let $x, y \in C$. Consider the following sequence of equivalent statements

$$\begin{aligned} \forall p \in S_0 : p(x) &= p(y), \\ \forall p_1, p_2 \in S_0 : p_1(x) + p_2(y) &= p_1(y) + p_2(x), \\ \forall [(p_1, p_2)] \in S : p_1(x) + p_2(y) - p_1(y) - p_2(x) &= 0, \\ \forall [(p_1, p_2)] \in S : [(p_1, p_2)]((x, y)) &= 0. \end{aligned}$$

Note that $x \neq y$ is equivalent with $[(x, y)] \neq [(0, 0)]$. □

From now on, we assume that $V[C]$ is finite-dimensional. As usual in this situation, we identify $V[C]$ and its bidual $(V[C])^{**}$, i.e., we identify each $x \in V[C]$ with its action $p \mapsto p(x)$ on $(V[C])^*$. To show this duality to full advantage, instead of $p(x)$, we write $[x, p]$ for every $p \in (V[C])^*$ and $x \in V[C]$. Note that with this identification, we have $C \subseteq C^{**}$. Since in this paper, we are particularly interested in salient half-spaces, and since we regard the vector space generated by a salient half-space merely as a mathematical tool, we shall often adopt the notation $[x, p]_C$ to denote $p(x)$ where $x \in C$ and $p \in C^*$.

Because $C \subseteq C^{**}$, we can consider the partial ordering $\leq_{C^{**}}$ on C as follows. Let $x, y \in C$, then

$$\begin{aligned} x \leq_{C^{**}} y &\iff \exists z \in C^{**} : x + z = y \\ &\iff \forall p \in C^* : [p, x]_{C^*} \leq [p, y]_{C^*} \\ &\iff \forall p \in C^* : [x, p]_C \leq [y, p]_C. \end{aligned}$$

So, if $C^{**} = C$, then $x \leq_C y$ is equivalent with $\forall p \in C^* : [x, p]_C \leq [y, p]_C$.

Proposition 1.2.5 *Let $C^{**} = C$. Then C^* separates the elements of C .*

Proof

Let $x, y \in C$, and suppose $\forall p \in C^* : [x, p]_C = [y, p]_C$. Of course, since $C^{**} = C$, this means $x \leq_C y$ and $y \leq_C x$. The partial ordering \leq_C being anti-symmetric, this implies $x = y$. □

Assuming $C^{**} = C$, Lemma 1.2.4 yields that $V[C^*]$ is a subspace of $(V[C])^*$, separating the elements of the finite dimensional vector space $V[C]$. This yields

$$C^{**} = C \implies V[C^*] = (V[C])^*.$$

It is in general not true, that $V[C^*] = (V[C])^*$ implies $C^{**} = C$, since the latter equality is related to a non-algebraic condition on C .

Finally, we mention the consequences of the condition $C^{**} = C$ for the partial ordering on C :

$$\begin{aligned} x \leq_C y &:\iff \exists z \in C : x + z = y \\ &\iff \forall p \in C^* : [x, p]_C \leq [y, p]_C, \\ x <_C y &:\iff \exists z \in C \setminus \{0\} : x + z = y \\ &\iff (\forall p \in C^* : [x, p]_C \leq [y, p]_C) \wedge (\exists p \in C^* : [x, p]_C < [y, p]_C). \end{aligned}$$

1.3 Topology and order units

We start by introducing the topology $\mathcal{T}(C, C^*)$ for a salient half-space C .

Definition 1.3.1 Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in C , then we say that $(x_n)_{n \in \mathbb{N}}$ converges to x (notation: $x_n \rightarrow x$), if $\forall f \in C^* : \lim_{n \rightarrow \infty} f(x_n) = f(x)$.

Definition 1.3.2 A set $S \subset C$ is $\mathcal{T}(C, C^*)$ -closed in C , if for all sequences $(x_n)_{n \in \mathbb{N}}$ in S , satisfying $x_n \rightarrow x \in C$, it holds that $x \in S$.

Thus, a topology on C is defined, where $O \subset C$ is an open set if and only if $C \setminus O$ is $\mathcal{T}(C, C^*)$ -closed. The proof that the collection of all such open sets satisfies the conditions of a topology for C is straightforward. We shall denote this topology by $\mathcal{T}(C, C^*)$.

In the following, we shall assume C to be a salient half-space satisfying the conditions presented at the end of Subsection 1.2, i.e. $C \neq \{0\}$, $\dim(V[C]) < \infty$, and $C^{**} = C$. Note that if a salient half-space C satisfies these conditions, so does its dual C^* , since $(V[C^*])^* = (V[C^*])^{***}$. Therefore, every result derived for C has a dual result for C^* . Furthermore, note that the construction of $V[C]$ from C implies that C is solid in $V[C]$.

On $V[C]$ we introduce the unique linear topology \mathcal{T} . We note that this topology is induced by the choice of any norm on $V[C]$. Since C^* contains a basis for $V[C^*]$, we find the following lemma which yields that the relative topology on C equals $\mathcal{T}(C, C^*)$.

Lemma 1.3.3 Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $V[C]$. Then $(y_n)_{n \in \mathbb{N}}$ is convergent if and only if $\exists y \in V[C] \forall f \in C^* : \lim_{n \rightarrow \infty} f(y_n) = f(y)$.

Henceforward, we shall refer to topology $\mathcal{T}(C, C^*)$ as the relative topology on C . We shall denote the \mathcal{T} -interior of a set $A \subset V[C]$ by $\text{int}(A)$ and the boundary of A by ∂A . In particular, we shall use the notation $\text{int}(C)$ to denote the \mathcal{T} -interior of C , where C is regarded as a subset of $V[C]$. With the notation ∂C , we denote $C \setminus \text{int}(C)$.

Lemma 1.3.4

$$C^{**} = C \iff C \text{ is closed in } V[C].$$

Proof

Suppose $C^{**} = C$, and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in C which is convergent in $V[C]$, with respect to topology \mathcal{T} , with limit $x \in V[C]$. Since by Lemma 1.3.3

$$\forall f \in C^* \forall n \in \mathbb{N} : f(x_n) \geq 0,$$

we conclude that $x \in C^{**} = C$.

For the converse, suppose that C is \mathcal{T} -closed. We shall prove that $C^{**} \subseteq C$. Let $x \in C^{**}$ and suppose $x \notin C$. Then by the Strong Separation Theorem of Minkowski ([Panik, p.59])

$$\exists f \in V[C^*] \exists \alpha \in \mathbb{R} : \begin{cases} f(x) < \alpha \\ \forall y \in C : f(y) > \alpha. \end{cases}$$

Suppose there exists $y \in C$ with $f(y) < 0$, then we come to a contradiction since $\lambda y \in C$ for all $\lambda > 0$. Hence, $f \in C^*$, which is in contradiction with $x \in C^{**}$. \square

Since C is solid in $V[C]$, $\text{int}(C) \neq \emptyset$. Since, in this paper, we regard the salient half-space C , rather than the vector space $V[C]$, to be the essential concept, we would like to have a salient half-space related characterisation of $\text{int}(C)$.

Lemma 1.3.5 *Let $x_0 \in C$. Then $x_0 \in \text{int}(C)$ if and only if $\forall p \in C^* \setminus \{0\} : [x_0, p]_C > 0$.*

Proof

Let $x_0 \in \text{int}(C)$. Suppose there exists $p \in C^*$ such that $[x_0, p]_C = 0$. Since $x_0 \in \text{int}(C)$ there is an open set $O \in \mathcal{T}$ satisfying $\{x_0\} + O \subset C$. For all $y \in O$, $[y, p]_C = [x_0 + y, p]_C \geq 0$, from which we conclude that $p = 0$.

For the converse, suppose $x_0 \in \partial C \setminus \{0\}$. Since C is a convex cone, $\text{int}(C)$ is a convex cone. By the Weak Separation Theorem of Minkowski ([Panik, p.60])

$$\exists p_0 \in (V[C])^* \setminus \{0\} \exists \alpha \in \mathbb{R} : \begin{cases} \forall \lambda \geq 0 : & [\lambda x_0, p_0] \leq \alpha \\ \forall x \in \text{int}(C) : & [x, p_0] \geq \alpha. \end{cases}$$

Choosing λ equal to 0, and choosing a sequence in $\text{int}(C)$ converging to 0, we find $\alpha = 0$. As a consequence $p_0 \in C^* \setminus \{0\}$. By subsequently choosing λ equal to 1, we find $[x_0, p_0]_C \leq 0$. \square

Note that as a consequence of this lemma, every element $x \in \partial C$ satisfies $\exists p \in C^* \setminus \{0\} : [x, p]_C = 0$.

Proposition 1.3.6 *Let $p_0 \in \text{int}(C)^*$. Then there is a unique norm $\| \cdot \|_{p_0}$ on $V[C]$, where $\forall x \in C : \| x \|_{p_0} = [x, p_0]_C$.*

Proof

For every $y \in V[C]$ define $\| y \|_{p_0} := \inf\{[x_1 + x_2, p_0]_C \mid x_1, x_2 \in C \text{ with } y + x_2 = x_1\}$. It is not difficult to check that $\| \cdot \|_{p_0}$ indeed is a norm on $V[C]$. To prove that $\forall x \in C : \| x \|_{p_0} = [x, p_0]_C$, we remark that $\forall x \in C : [x, p_0]_C \leq \| x \|_{p_0}$, since for all $x, x_1, x_2 \in C$ satisfying $x + x_2 = x_1$ it holds that $x \leq_C x + 2x_2 = x_1 + x_2$. Furthermore, we can choose $x_1 = x$ and $x_2 = 0$ to obtain that $\| x \|_{p_0} \leq [x, p_0]_C$. \square

Since $C^{**} = C$, interchanging the role of C and C^* in the above proposition yields that each $x_0 \in \text{int}(C)$ induces the unique norm $\| \cdot \|_{x_0}$ on C^* .

Corollary 1.3.7 *Let $p_0 \in \text{int}(C^*)$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in C . Then $(x_n)_{n \in \mathbb{N}}$ converges to 0 with respect to the relative topology if and only if $\lim_{n \rightarrow \infty} [x_n, p_0]_C = 0$.*

Corollary 1.3.8 *Let S be a subset of C and let $p_0 \in \text{int}(C^*)$. Then S is bounded if and only if the set $\{[x, p_0]_C \mid x \in S\}$ is bounded.*

Corollary 1.3.9 For all $p_0 \in \text{int}(C^*)$, the sets $K_1(p_0) := \{x \in C \mid [x, p_0]_C \leq 1\}$ and $L_1(p_0) := \{x \in C \mid [x, p_0]_C = 1\}$ are compact.

Proof

Let $p_0 \in \text{int}(C^*)$ be given. The sets $K_1(p_0)$ and $L_1(p_0)$ are closed subsets of $V[C]$. \square

Proposition 1.3.10 Let $x_0 \in \text{int}(C)$. Then x_0 is an order unit for C for the partial ordering \leq_C . So, there is a function $\mathcal{U}_{x_0} : C \rightarrow \mathbb{R}^+$ satisfying $\forall x \in C : x \leq_C \mathcal{U}_{x_0}(x)x_0$. Moreover, there is a function $\mathcal{L}_{x_0} : C \rightarrow \mathbb{R}^+$ satisfying $\forall x \in C : \mathcal{L}_{x_0}(x)x_0 \leq_C x$ and $\forall x \in \text{int}(C) : \mathcal{L}_{x_0}(x) > 0$.

Proof

The statement

$$\forall x \in C \exists \psi, \varphi \geq 0 : \psi x_0 \leq_C x \leq_C \varphi x_0 \quad (1)$$

is equivalent with

$$\forall x \in C \exists \psi, \varphi \geq 0 \forall p \in C^* : \psi[x_0, p]_C \leq [x, p]_C \leq \varphi[x_0, p]_C.$$

Consider the compact set $L_1(x_0) := \{p \in C^* \mid [x_0, p]_C = 1\}$. Then $C^* = \{\alpha p \mid p \in L_1(x_0), \alpha \geq 0\}$. So, statement (1) is equivalent with

$$\forall x \in C \exists \psi, \varphi, \geq 0 \forall p \in L_1(x_0) : \psi \leq [x, p]_C \leq \varphi.$$

If we define $\mathcal{U}_{x_0} : C \rightarrow \mathbb{R}^+$ and $\mathcal{L}_{x_0} : C \rightarrow \mathbb{R}^+$ by

$$\begin{aligned} \mathcal{U}_{x_0}(x) &:= \max\{[x, p]_C \mid p \in L_1(x_0)\} \\ \mathcal{L}_{x_0}(x) &:= \min\{[x, p]_C \mid p \in L_1(x_0)\}. \end{aligned}$$

Then $\mathcal{L}_{x_0}(x) \leq [x, p]_C \leq \mathcal{U}_{x_0}(x)$ for all $p \in L_1(x_0)$. Clearly, $\mathcal{L}_{x_0}(x) > 0$ if $x \in \text{int}(C)$. \square

From the definition of \mathcal{U}_{x_0} and \mathcal{L}_{x_0} in the above proof, it is not difficult to prove that these functions are continuous on C .

Brouwer's Fixed Point Theorem [Conway, p.149]

Let K be a non-empty compact convex subset of a finite-dimensional normed vector space X and let $\mathcal{F} : K \rightarrow K$ be a continuous function, then there exists $x \in K$ such that $\mathcal{F}(x) = x$, i.e., \mathcal{F} has a fixed point in K .

Since we assumed the salient half-space C to satisfy $V[C]$ is finite-dimensional and $C^{**} = C$, Brouwer's Fixed Point Theorem yields the following consequence for continuous functions on C .

Proposition 1.3.11 Let $\mathcal{G} : C \setminus \{0\} \rightarrow C$ be a continuous function. Then there exists an $x \in C \setminus \{0\}$ such that $\mathcal{G}(x) = \alpha x$ for some $\alpha \geq 0$. In fact, for all $p_0 \in \text{int}(C^*)$ there is $x \in C$ such that $\mathcal{G}(x) = [\mathcal{G}(x), p_0]x$.

Proof

Let $p_0 \in \text{int}(C^*)$. The set $K_1(p_0) := \{x \in C \mid [x, p_0]_C = 1\}$ is non-empty, convex and compact by Corollary 1.3.9. Define the function $\mathcal{F} : K_1(p_0) \rightarrow K_1(p_0)$ by $\mathcal{F}(x) := \frac{x + \mathcal{G}(x)}{1 + [\mathcal{G}(x), p_0]_C}$. Then \mathcal{F} is a continuous function. By the preceding theorem the function \mathcal{F} has a fixed point x in $K_1(p_0)$, so

$$x = \mathcal{F}(x) = \frac{x + \mathcal{G}(x)}{1 + [\mathcal{G}(x), p_0]_C}.$$

□

We finish this subsection with the introduction of a Lebesgue measure. Let $x_0 \in \text{int}(C)$ and consider the hyperplane $H_1(x_0) := \{p \in (V[C])^* \mid [x_0, p] = 1\}$ of the dual space $(V[C])^*$. Let $\Phi : \mathbb{R}^{n-1} \rightarrow H_1(x_0)$ be an affine parametrisation of $H_1(x_0)$, where $n = \dim V[C]$ and endow $H_1(x_0)$ with the topology such that Φ is a homeomorphism. Take the standard Lebesgue measure λ on \mathbb{R}^{n-1} and define μ to be the measure on $H_1(x_0)$ induced by Φ and λ . Hence, for every subset A of $H_1(x_0)$ we have $\mu(A) = \lambda(\Phi^{-1}(A))$ and for a real-valued function f on (a subset of) $H_1(x_0)$, for which $f \circ \Phi$ is continuous, f is integrable with respect to μ , and

$$\int_A f d\mu = \int_{\Phi^{-1}(A)} (f \circ \Phi) d\lambda.$$

This measure μ is a regular Borel measure. Therefore, if f is continuous on a subset A of $H_1(x_0)$ with a dense interior, and if the set $L := \{x \in A \mid f(x) < 0\}$ satisfies $\mu(L) = 0$, then $L = \emptyset$, i.e. $\forall x \in A : f(x) \geq 0$.

Let W denote a finite-dimensional real vector space with $\{g_1, \dots, g_m\}$ a basis in the dual space W^* , and let $f : H_1(x_0) \rightarrow W$ be continuous. Then $\forall i \in \{1, \dots, m\} : g_i \circ f$ is continuous from $H_1(x_0)$ into \mathbb{R} . Furthermore, for a subset A of $H_1(x_0)$, we denote the unique element w in W which satisfies

$$\forall i \in \{1, \dots, m\} : \int_A (g_i \circ f) d\mu = g_i \circ w,$$

by $\int_A f d\mu$. For a norm $\| \cdot \|$ on the vector space W , we have

$$\left\| \int_A f d\mu \right\| \leq \int_A \| f \| d\mu.$$

1.4 Direct sums

In our model (cf. Section 2) we shall define a production technology set which will be a subset of a direct sum of two salient half-spaces.

Definition 1.4.1 *Let C_a and C_b be two salient half-spaces. Their direct sum is the salient half-space $C_a \oplus C_b$, consisting of all ordered pairs $x = (x^a, x^b)$ with $x^a \in C_a$ and $x^b \in C_b$. The salient half-space operations are for all $x, y \in C_a \oplus C_b$ and for all $\alpha \geq 0$ given by:*

$$\left\{ \begin{array}{l} (x + y)^a := x^a + y^a \\ (\alpha x)^a := \alpha x^a \end{array} \right. \text{ and } \left\{ \begin{array}{l} (x + y)^b := x^b + y^b \\ (\alpha x)^b := \alpha x^b. \end{array} \right.$$

For every $x \in C_a \oplus C_b$, there are unique $x^a \in C_a$ and $x^b \in C_b$ such that $x = (x^a, x^b)$. Since $C_a \oplus C_b$ is a salient half-space, every property for salient half-spaces derived thusfar, is also applicable to $C_a \oplus C_b$.

On the direct sum $C_a \oplus C_b$ the partial ordering $\leq_{(C_a \oplus C_b)}$ is given by:

$$x \leq_{(C_a \oplus C_b)} y \iff \begin{cases} x^a \leq_{C_a} y^a \\ x^b \leq_{C_b} y^b. \end{cases}$$

We continue this subsection on direct sums by remarking that

$$V[C_a \oplus C_b] = V[C_a] \oplus V[C_b],$$

where the second \oplus denotes the usual direct sum defined for two vector spaces (cf. [Halmos]), and that

$$(C_a \oplus C_b)^* = C_a^* \oplus C_b^*,$$

where the action of $p \in C_a^* \oplus C_b^*$ on an element $x \in C_a \oplus C_b$ is defined by

$$[x, p]_{(C_a \oplus C_b)} = [x^a, p^a]_{C_a} + [x^b, p^b]_{C_b}.$$

To simplify notation we shall use C to denote $C_a \oplus C_b$. Furthermore, we shall write $[\cdot, \cdot]_a$ and $[\cdot, \cdot]_b$ instead of $[\cdot, \cdot]_{C_a}$ and $[\cdot, \cdot]_{C_b}$, respectively. Hence, for every $x \in C, p \in C^*$ we write $[x, p]_C = [x^a, p^a]_a + [x^b, p^b]_b$. Also, we shall write \leq_a and \leq_b instead of \leq_{C_a} and \leq_{C_b} .

Definition 1.4.2 For all $x \in C$ we define the set F_x by

$$F_x := \{z \in C \mid x^a \leq_a z^a \text{ and } z^b \leq_b x^b\}.$$

Let $U \subset C$. For all $x \in U$ we define the set $R_x(U)$ by

$$R_x(U) := \{z \in U \mid x \in F_z \text{ and } F_z \subset U\}.$$

Furthermore, the set $E(U)$ is defined by

$$E(U) := \{e \in U \mid R_e(U) = \{e\}\}.$$

Without proof we state the following two properties.

Lemma 1.4.3 Let $x \in C$. Then

- $\forall y \in F_x : F_y \subset F_x$.
- If $y \in F_x$ and $x \neq y$, then $x \notin F_y$.

Lemma 1.4.4 Let $U \subset C$ satisfy

- $U = \bigcup_{e \in E(U)} F_e$

- $\forall e, f \in E(U) \forall \tau \in [0, 1] : \tau e + (1 - \tau)f \in U$.

Then the set U is convex.

Proof

Let $x, y \in U$ and $\tau \in [0, 1]$. By the first statement of U , there exist $e, f \in E(U)$ such that $x \in F_e$ and $y \in F_f$. Thus,

$$\left\{ \begin{array}{l} \exists \tilde{x}^a \in C_a : x^a = e^a + \tilde{x}^a \\ \exists \tilde{x}^b \in C_b : e^b = x^b + \tilde{x}^b \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \exists \tilde{y}^a \in C_a : y^a = f^a + \tilde{y}^a \\ \exists \tilde{y}^b \in C_b : f^b = y^b + \tilde{y}^b. \end{array} \right.$$

To prove convexity of U we shall show that $\tau x + (1 - \tau)y \in F_{(\tau e + (1 - \tau)f)}$. Indeed, this proves the assertion since both properties of U , combined with the first statement of Lemma 1.4.3, yield $F_{(\tau e + (1 - \tau)f)} \subset U$.

Firstly, note that

$$\begin{aligned} \tau x^a + (1 - \tau)y^a &= \tau(e^a + \tilde{x}^a) + (1 - \tau)(f^a + \tilde{y}^a) \\ &= (\tau e^a + (1 - \tau)f^a) + (\tau \tilde{x}^a + (1 - \tau)\tilde{y}^a), \end{aligned}$$

and secondly,

$$(\tau x^b + (1 - \tau)y^b) + (\tau \tilde{x}^b + (1 - \tau)\tilde{y}^b) = \tau e^b + (1 - \tau)f^b.$$

Since $\tau \tilde{x}^a + (1 - \tau)\tilde{y}^a \in C_a$ and $\tau \tilde{x}^b + (1 - \tau)\tilde{y}^b \in C_b$, we conclude that $\tau x + (1 - \tau)y \in F_{(\tau e + (1 - \tau)f)}$. \square

2 The private ownership model

2.1 Economy bundles and pricing functions

As mentioned in the introduction, the main goal of this paper is the introduction of a model of a private ownership economy, which differs from the neo-classical models in the following two aspects.

- Commodities are not assumed to occur separately. Instead of introducing the commodity space $(\mathbb{R}^n)^+$ describing n different commodities, we shall only assume appearance of so called economy bundles. Here, we use the term “economy bundle” to describe exchangeable objects in the economy. Thus, economy bundles can represent a single commodity, a bundle of commodities or a fixed combination of commodities, of which one of the elements can only be obtained by buying this specific fixed combination, i.e., of which one element is not sold separately. The latter case describes a situation in which our model allows for links between commodities.
- Production and consumption are not treated on the same level. In the model, two different types of economy bundles occur: production bundles which can be used as input to production processes, and consumption bundles which can be output of these processes. Bundles of both types can be consumed by economic agents and bundles of both types will be present in the initial endowment. However, the production processes can convert only production bundles into consumption bundles and not the other way around.

In our model, we incorporate the above described situation as follows.

Firstly, considering economy bundles instead of separate commodities, we model the set of all economy bundles in the economy by a salient half-space C , reflecting that the only possible manipulations with economy bundles are adding and scaling over \mathbb{R}^+ . If $x, y \in C$ represent two economy bundles then we can speak of the sum $x + y$ of x and y , and if $\alpha \geq 0$ we can speak of the scaled version αx of x . Both $x + y$ and αx are economy bundles in C . Requiring the economy bundle set C to be salient (Condition 1.1.1.a) describes the fact that it is impossible for two economy bundles to cancel each other out after addition.

Secondly, considering two types of economy bundles, we assume that C is the direct sum of two salient half-spaces C_{prod} and C_{cons} , where C_{prod} and C_{cons} consists of all production bundles and all consumption bundles, respectively. Both C_{prod} and C_{cons} are assumed to be non-trivial, i.e., assumed to be different from $\{0^{\text{prod}}\}$ and $\{0^{\text{cons}}\}$, respectively. So, C is also non-trivial. In every economy bundle $x \in C$, each of the two types is uniquely represented: $x = (x^{\text{prod}}, x^{\text{cons}})$ with $x^{\text{prod}} \in C_{\text{prod}}$ and $x^{\text{cons}} \in C_{\text{cons}}$.

Since in our model commodities are not assumed to occur separately, the price of a single commodity is not a meaningful concept. Instead, we speak of the value of an economy bundle, which will be determined on the basis of “pricing functions”. These pricing functions are described by subadditive positive functionals on C . The set of all such

functionals has been introduced in Section 1 as the salient half-dual space C^* and we have seen that $C^* = (C_{\text{prod}})^* \oplus (C_{\text{cons}})^*$. Let $x \in C$ and $p \in C^*$, then the value of economy bundle x with respect to the pricing function p equals

$$[x, p]_C := [x^{\text{prod}}, p^{\text{prod}}]_{\text{prod}} + [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}}.$$

Instead of the notation $[x, p]_C$ we shall mostly write $\mathcal{V}(x, p)$ for the value of the pair (x, p) with $x \in C$ and $p \in C^*$.

2.2 Economic agents

The features of an economic agent are an economy bundle $w = (w^{\text{prod}}, w^{\text{cons}}) \in C$, called initial endowment, and a preference relation \succeq defined on C , on the basis of which the agent is supposed to make choices. By $x \succeq y$ we denote that the agent considers economy bundle x to be at least as preferable as bundle y . By $x \succ y$ we mean $x \succeq y$ and $\neg(y \succeq x)$. This preference relation \succeq on C satisfies reflexivity, transitivity and completeness.

For a given value $\kappa \geq 0$ and a pricing function $p \in C^*$, the budget set $B(p, \kappa) := \{x \in C \mid \mathcal{V}(x, p) \leq \kappa\}$ consists of all economy bundles that can be afforded given value κ and pricing function p . The set $D(p, \kappa) := \{x \in B(p, \kappa) \mid \forall y \in B(p, \kappa) : x \succeq y\}$ of all best (most preferable) elements of the budget set $B(p, \kappa)$, is called the demand set. In the final model, κ will be specified as being the value $\mathcal{V}(w, p)$ of the initial endowment plus the values of the shares in the profit of production.

2.3 Production processes and technologies

Since we deal with an exchange economy with production, we have to model so called production processes, i.e., processes that incorporate the possibility of converting production bundles into consumption bundles. For our model this means that we say that an economy bundle $x \in C$ represents the production process which converts production bundle $x^{\text{prod}} \in C_{\text{prod}}$ into consumption bundle $x^{\text{cons}} \in C_{\text{cons}}$. A collection of production processes being technologically feasible is said to be a production technology. Hence, a production technology is modelled by a subset T of C . Each production technology T will satisfy the following natural assumptions from a feasible point of view:

- a) The production process "no production" belongs to T ;
- b) A production process in T with zero input has zero output;
- c1) Free disposal of input;
- c2) Free disposal of output.

Free disposal of input states that if $x = (x^{\text{prod}}, x^{\text{cons}})$ is an feasible production process and $\tilde{x}^{\text{prod}} = x^{\text{prod}} + y^{\text{prod}}$ for some $y^{\text{prod}} \in C_{\text{prod}}$, then $(\tilde{x}^{\text{prod}}, x^{\text{cons}})$ is also a feasible production process since after disposal of y^{prod} , production process x can be executed. Put differently, if $x \in T$ and $\tilde{x}^{\text{prod}} \in C_{\text{prod}}$ with $x^{\text{prod}} \leq_{\text{prod}} \tilde{x}^{\text{prod}}$ then $(\tilde{x}^{\text{prod}}, x^{\text{cons}}) \in T$. Similarly, free disposal of output states that if $x = (x^{\text{prod}}, x^{\text{cons}})$ is a feasible production process and $x^{\text{cons}} = y^{\text{cons}} + \tilde{x}^{\text{cons}}$

for some $y^{\text{cons}}, \tilde{x}^{\text{cons}} \in C_{\text{cons}}$, then $(x^{\text{prod}}, \tilde{x}^{\text{cons}})$ is also a feasible production process since after production of x^{cons} out of $x^{\text{prod}}, y^{\text{cons}}$ can be disposed of, leaving \tilde{x}^{cons} as output. So, if $x \in T$ and $\tilde{x}^{\text{cons}} \in C_{\text{cons}}$ with $\tilde{x}^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}}$ then $(x^{\text{prod}}, \tilde{x}^{\text{cons}}) \in T$.

In fact, for every $x \in T$, the set F_x (as defined in Definition 1.4.2) is a subset of T , since F_x consists of precisely all the production processes in C which are executable due to the fact that x is executable and the two free disposal properties c1 and c2.

So, we come to the following definition of the concept of production technology.

A set $T \subset C$ is a production technology if the set T has the following properties:

- a) $(0^{\text{prod}}, 0^{\text{cons}}) \in T$,
- b) If $(0^{\text{prod}}, x^{\text{cons}}) \in T$ then $x^{\text{cons}} = 0^{\text{cons}}$,
- c) $T = \bigcup_{x \in T} F_x$.

We call a production process $(x^{\text{prod}}, x^{\text{cons}})$ of a technology T efficient, if at least x^{prod} is needed to produce x^{cons} , and if it is not possible to produce more than x^{cons} out of x^{prod} . Mathematically speaking, this boils down to the following definition.

Definition 2.3.1 For a production technology T , a production process $e \in T$ is efficient if $\forall x \in C$:

- $((x^{\text{prod}}, e^{\text{cons}}) \in T \text{ and } x^{\text{prod}} \leq_{\text{prod}} e^{\text{prod}}) \implies x^{\text{prod}} = e^{\text{prod}}$;
- $((e^{\text{prod}}, x^{\text{cons}}) \in T \text{ and } e^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}}) \implies e^{\text{cons}} = x^{\text{cons}}$.

Put differently, e is efficient if and only if $e \in E(T)$ (cf. Definition 1.4.2). Note that $(0^{\text{prod}}, 0^{\text{cons}}) \in E(T)$.

Given a pricing function $p \in C^*$ and a production process $x \in T$, the gain $\mathcal{G}(x, p)$ of the pair (x, p) equals the value of the produced economy bundle x^{cons} minus the value of the production bundle x^{prod} , used as input. So,

$$\mathcal{G}(x, p) := [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} - [x^{\text{prod}}, p^{\text{prod}}]_{\text{prod}}. \quad (2)$$

Note that the following two properties are a direct consequence of the definition of \mathcal{G} and F_x .

- Let $x \in C, p \in C^*$ and $y \in F_x$, then $\mathcal{G}(x, p) \geq \mathcal{G}(y, p)$.
- Let $x \in C, p \in \text{int}(C^*)$ and let $y \in F_x$ satisfy $y \neq x$, then $\mathcal{G}(x, p) > \mathcal{G}(y, p)$.

Since for every pair $(x, p) \in C \times C^*$ we can speak of both its value, where x is considered as an economy bundle, and its gain, where x is considered as a production process, we have introduced the distinguishing notation $\mathcal{V}(x, p)$ and $\mathcal{G}(x, p)$. Note that \mathcal{V} is a mapping from $C \times C^*$ into \mathbb{R}^+ , while \mathcal{G} is a mapping into \mathbb{R} .

Given $p \in C^*$, the (possibly empty) set of all gain maximizing production processes in T is called the supply set $S(p)$ of T , i.e.,

$$S(p) = \{x \in T \mid \forall y \in T : \mathcal{G}(x, p) \geq \mathcal{G}(y, p)\}. \quad (3)$$

The conditions on T and the definition of $E(T)$ imply that $\forall p \in C^* : S(p) \subseteq E(T)$.

2.4 Agents, production and equilibrium

Let I denote the number of economic agents and J the number of production technologies present in the private ownership economy. The set of agents and the set of production technologies is labelled by $i \in \{1, \dots, I\}$ and $j \in \{1, \dots, J\}$, respectively. For each $i \in \{1, \dots, I\}, j \in \{1, \dots, J\}$, agent i has initial endowment $w_i \in C$, and share θ_{ij} , $0 \leq \theta_{ij} \leq 1$, in the gain of production technology T_j , i.e., if production process $x_j \in T_j$ is executed at pricing function p , the gain $\mathcal{G}(x_j, p)$ of this production process is divided amongst the agents, such that agent i receives $\theta_{ij}\mathcal{G}(x_j, p)$. So, for all $j \in \{1, \dots, J\}$ these shares satisfy $\sum_{i=1}^I \theta_{ij} = 1$.

At pricing function $p \in C^*$ and executed production processes $x_j \in T_j, j \in \{1, \dots, J\}$, the income $\kappa_i(p; x_1, \dots, x_J)$ of agent i is defined by

$$\kappa_i(p; x_1, \dots, x_J) := \mathcal{V}(w_i, p) + \sum_{j=1}^J \theta_{ij} \mathcal{G}(x_j, p),$$

where the first term denotes the value of the initial endowment of agent i and the second term denotes the total value received from shares in the gain of the production technologies. In this setting, an equilibrium concept analogous to that of the neo-classical Walrasian equilibrium can be introduced.

Definition 2.4.1 A Walrasian equilibrium is an $(I + J + 1)$ -tuple $((s_j)_{j=1}^J, (d_i)_{i=1}^I, p_{eq})$ consisting of

- $p_{eq} \in C^* \setminus \{0\}$,
- $s_j \in S_j(p_{eq})$ for all $j \in \{1, \dots, J\}$;
- $d_i \in D_i(p_{eq}, \kappa_i(p_{eq}; s_1, \dots, s_J))$ for all $i \in \{1, \dots, I\}$;
- $\sum_{i=1}^I d_i + \sum_{j=1}^J (s_j^{\text{prod}}, 0^{\text{cons}}) = \sum_{i=1}^I w_i + \sum_{j=1}^J (0^{\text{prod}}, s_j^{\text{cons}})$.

We call p_{eq} a (Walrasian) equilibrium pricing function.

Finally, we present additional assumptions for this model, such that existence of such equilibria is guaranteed.

Equilibrium Existence Theorem

The model of a private ownership economy, described above, admits a Walrasian equilibrium, under the following assumptions:

A1 $V[C]$ is finite-dimensional.

A2 $C^{**} = C$.

A3 For every $j \in \{1, \dots, J\}$, production technology T_j satisfies

- a) $T_j = \bigcup_{e \in E(T_j)} F_e$.
- b) T_j is closed with respect to topology $\mathcal{T}(C, C^*)$,
- c) if $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0, 1)$ then $\tau e_1 + (1 - \tau)e_2 \in T_j$ and $\tau e_1 + (1 - \tau)e_2 \notin E(T_j)$.

A4 For every $i \in \{1, \dots, I\}$, preference relation \succeq_i is

- a) *monotone*: $\forall x, y \in C : x \leq_C y$ implies $y \succeq_i x$,
- b) *strictly convex*: $\forall x, y \in C, \tau \in (0, 1) : x \succeq_i y$ and $x \neq y$ imply $\tau x + (1 - \tau)y \succ_i y$,
- c) *continuous*: $\forall y \in C$ the sets $\{x \in C \mid x \succeq_i y\}$ and $\{x \in C \mid y \succeq_i x\}$ are closed in C .

A5 Furthermore,

- a) $\exists p \in \text{int}(C^*) \forall j \in \{1, \dots, J\} : S_j(p) \neq \emptyset$,
- b) for all $p^{\text{cons}} \in C_{\text{cons}}^* \setminus \{0^{\text{cons}}\}$ satisfying $\forall i \in \{1, \dots, I\} : [w_i^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = 0$, there is $j_0 \in \{1, \dots, J\}$ and $x \in T_{j_0}$ such that $[x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} > 0$,
- c) $\forall p^{\text{prod}} \in C_{\text{prod}} \setminus \{0^{\text{prod}}\} : [\sum_{i=1}^I w_i^{\text{prod}}, p^{\text{prod}}]_{C_{\text{prod}}} > 0$.

Let us shortly discuss these extra assumptions, and give a short sketch of the proof of this theorem. The complete proof can be found in [Schalk].

Assumptions 1 and 2 guarantee that C is a closed subset of $V[C]$ with respect to the natural topology \mathcal{T} . Furthermore, they guarantee that every bounded set is pre-compact and so the budget sets are compact for interior pricing functions. The interpretation of Assumption 3.a is that for every production process $x \in T_j$, there is an efficient production process $e \in E(T_j)$ such that $x \in F_e$, i.e., x is the result of e and the free disposal properties. On the basis of Assumptions 3.b and 3.c it can be proved that instead of dealing with supply sets, we deal with supply functions \mathcal{S}_j with values in $E(T_j)$. So, in order to guarantee that we can use supply functions, we introduce Assumption 3.b, which resembles “decreasing returns to scale” or “strict convexity conditions”. Assumption 3.c guarantees the continuity of the supply functions. Similarly, Assumption 4 implies that we can deal with continuous demand functions. Assumption 5.a yields that the total supply function has a non-trivial domain. Existence of a Walrasian equilibrium, in the sense of Definition 2.4.1, follows from a generalisation of Brouwers’ Fixed Point Theorem, for continuous functions on salient half-spaces (cf. Proposition 1.3.11). In this, Assumptions 5.b and 5.c will be used. Assumption 5.b states that if p^{cons} is such that $[w_i^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = 0$ for every $i \in \{1, \dots, I\}$, there is a production technology j_0 which can produce something with positive value at p^{cons} . If this were not the case, every consumer would have zero income at pricing function $(0^{\text{prod}}, p^{\text{cons}})$. Since Assumption A5.c only requires that the production part $\sum_{i=1}^I w_i^{\text{prod}}$ of the total initial endowment is strictly positive, it is an assumption more natural than the one which is usually made (cf. [Debreu]), stating that $\sum_{i=1}^I w_i$ is strictly positive. Hence, in

this model, the existence of a Walrasian equilibrium is guaranteed even if $\sum_{i=1}^I w_i^{\text{cons}} = 0^{\text{cons}}$. Moreover, as can be seen in the proof below, Assumption A5.b and 5.c can be replaced by the weaker condition

A5.b') for every sequence $(p_n)_{n \in \mathbb{N}}$ in the domain of the total supply function with non-zero limit, there is $i_0 \in \{1, \dots, I\}$ such that

$$\liminf_{n \rightarrow \infty} \{\kappa_{i_0}(p_n; \mathcal{S}_1(p_n), \dots, \mathcal{S}_J(p_n)) \mid n \in \mathbb{N}\} > 0.$$

Proof

Let $(p_n)_{n \in \mathbb{N}}$ be a sequence in the domain of the total supply function, with limit $p \in C^* \setminus \{0\}$. We have to prove

$$\exists i_0 \in \{1, \dots, I\} : \liminf_{n \rightarrow \infty} \left(\underbrace{[w_i^{\text{prod}}, p_n^{\text{prod}}]_{\text{prod}}}_{\geq 0} + \underbrace{[w_i^{\text{cons}}, p_n^{\text{cons}}]_{\text{cons}}}_{\geq 0} + \sum_{j=1}^J \theta_{i_0 j} \underbrace{\mathcal{G}(\mathcal{S}_j(p_n), p_n)}_{\geq 0} \right) > 0.$$

Since, by Assumption 5.c, $\sum_{i=1}^I w_i^{\text{prod}} \in \text{int}(C_{\text{prod}})$, we may as well assume $p^{\text{prod}} = 0^{\text{prod}}$.

Furthermore, we may as well assume that $\forall i \in \{1, \dots, I\} : [w_i^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = 0$. By Assumption 5.c, $\exists j_0 \in \{1, \dots, J\} \exists x \in T_{j_0} : [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} > 0$. The continuity of the function \mathcal{G} yields $\exists N \in \mathbb{N} \forall n > N : \mathcal{G}(\mathcal{S}_{j_0}(p_n), p_n) \geq \mathcal{G}(x, p_n) > \frac{1}{2} \mathcal{G}(x, p) > 0$. Take $i_0 \in \{1, \dots, I\}$ such that $\theta_{i_0 j_0} \neq 0$ and the proof is done. \square

Sketch of the proof of the Equilibrium Existence Theorem: first we establish continuity of the total supply function \mathcal{S} and the total demand function \mathcal{D} on a suitable domain in C^* . Then, we introduce the function \mathcal{Z} by

$$\mathcal{Z}(p, q) := \mathcal{V}(\mathcal{D}(p), q) - \mathcal{G}(\mathcal{S}(p), q) - \mathcal{V}\left(\sum_{i=1}^I w_i, q\right).$$

Now, p^{eq} is an equilibrium pricing function if and only if for all $q \in C^* : \mathcal{Z}(p^{\text{eq}}, q) \leq 0$. To find p^{eq} we introduce the function \mathcal{F} by

$$\mathcal{F}(p) := \int_{L_1(x_0)} \max\{0, \mathcal{Z}(p, q)\} q d\mu(q),$$

where $L_1(x_0) := \{q \in C^* \mid \mathcal{V}(x_0, q) = 1\}$, where μ is the standard Lebesgue measure on $L_1(x_0)$ and where $x_0 \in \text{int}(C)$ can be taken arbitrarily. Precisely those p for which there is $\alpha \geq 0$ such that $\mathcal{F}(p) = \alpha p$, are equilibrium pricing functions. To prove existence, we use Proposition 1.3.11, by extending \mathcal{F} in an appropriate way.

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