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Leximals, the Lexicore and the Average Lexicographic Value

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Abstract

The lexicographic vectors of a balanced game, called here leximals, are used to define a new solution concept, the lexicore, on the cone of balanced games. Properties of the lexicore and its relation with the core on some classes of games are studied. It is shown that on cones of balanced games where the core is additive, the leximals, the lexicore and the Average Lexicographic (AL-)value are additive, too. Further, it turns out that the leximals satisfy a consistency property with respect to a reduced game *à* la Davis and Maschler, which implies an average consistency property of the AL-value. Explicit formulas for the AL-value on the class of k-convex games and on the class of balanced almost convex games are provided.

JEL classification codes: C71.
Keywords: cooperative games, the core, the AL-value, the Shapley value

1 Introduction

In cooperative game theory efficient payoff vectors, referred to as pre-imputations, are basic ingredients for defining different solution concepts. In particular, (individual rational) payoff vectors associated to different orderings of players play an important role. First, we mention the marginal worth vectors whose average was used to define the Shapley value (cf. Shapley, 1953) and whose convex hull was used to define the Weber set (cf. Weber, 1988). Furthermore, the marginal vectors of a convex game (cf. Shapley, 1971) are the extreme points of the core (cf. Ichiishi, 1981). Second, for balanced games, i.e. games whose core is non-empty, payoff vectors that are lexicographic vectors (cf. Tijs, 2005) have turned out to play a similar role. In particular, the average of the lexicographic vectors was used to define the average lexicographic value, in short the AL-value, on the class of balanced games.

In this paper, we also consider the convex hull of these lexicographic vectors to define the lexicore as a new solution concept on the class of balanced games. In what follows, we call leximals the operators corresponding to orderings of the players which assign to each game the lexicographic (payoff) vector of the core according to the given ordering. Leximals, the lexicore and the AL-value are the study object in this paper. We prove that on cones of balanced games where the core is additive, the leximals, the lexicore and the AL-value are additive, too. We also show that leximals satisfy a consistency property with respect to a reduced game *à* la Davis-Maschler (Davis and Maschler, 1965), implying that the AL-value possesses an average consistency property. Some classes of cooperative games with some convexity flavour, like the class of k-convex games and the class of almost convex games, are considered for answering the question whether or not the lexicore and the core coincide as well as for computational aspects of the leximals and the AL-value on these classes of games.

The outline of this paper is as follows. Section 2 discusses some preliminaries on cooperative games with transferable utility and solution concepts. In Section 3, we provide the definition of the lexicore and some sufficient conditions for the coincidence of the lexicore and the core. In Section 4, we provide a condition for the additivity of leximals, the lexicore and the AL-value. In Section 5, we provide a consistency property of leximals and an average consistency property of the AL-value. In Section 6, we give explicit formulas for the leximals and the AL-value of k-convex games and balanced almost convex games. In Section 7, we give concluding remarks.

2 Preliminaries

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of players with $n = |N| \geq 2$, and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is called the *worth* of coalition S. This is what the members of coalition S can obtain by agreeing to cooperate. We denote the class of all TU-games by $\mathcal{G}$. Then the set of games with player set N is denoted by $\mathcal{G}^N$. We also denote $b_i^v = v(N) - v(N \setminus \{i\})$ as the *marginal contribution* of i to N in v, and $I^*(v) = \{x \in \mathbb{R}^N | x(N) = v(N)\}$ as the set of *pre-imputations* of v.

The very basic solution concept of this paper is the *core* given (cf. Gillies, 1953) by

$$C(v) = \{x \in \mathbb{R}^N | x(N) = v(N), x(S) \geq v(S) \text{ for all } S \subset N\},$$

for each $v \in G^N$, where we denote $x(S) = \sum_{i \in S} x_i$. A game with a non-empty core is called a *balanced game* and the set of all balanced games is denoted by $\mathcal{G}_C$. Then, we denote the set of balanced games with player set N by $\mathcal{G}_C^N$.

Let $(N, v) \in \mathcal{G}_C^N$ and let $\Pi(N)$ be the set of all orders on N, that is, one to one onto mappings from $\{1, 2, ..., n\}$ to N. We also denote $\sigma = \{\sigma(1), \sigma(2), ..., \sigma(n)\}$. For each $\sigma \in \Pi(N)$, the lexicographic vector $L^\sigma(v)$ is inductively defined by, for $i \in N$,

$$L^\sigma_{\sigma(i)}(v) = \max\{x_{\sigma(i)} | x \in C(v), L^\sigma_{\sigma(j)}(v) = x_{\sigma(j)} \text{ for each } j \in N \text{ with } j < i\}.$$

We note that $L^\sigma(v)$, $\sigma \in \Pi(N)$, is an extreme point of the core $C(v)$; each $L^\sigma(v)$ is the lexicographic maximum of the core of v with respect to the ordering σ. In the sequel, we refer to each $L^\sigma(v)$, $\sigma \in \Pi(N)$, as the *leximal* of v with respect to σ, whereas we refer to L^σ as the leximal (operator) with respect to σ.

The *AL-value*, defined by Tijs (2005) as the average of the leximals, is a solution concept on the domain $\mathcal{G}_C^N$, which is uniquely determined by the core. The AL-value is the function $AL : \mathcal{G}_C^N \to \mathbb{R}^N$ defined by

$$AL(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^\sigma(v)$$

for each $(N, v) \in \mathcal{G}_C^N$.

On special classes of balanced games the AL-value coincides with specific solutions on those classes (see Tijs, 2005 for details, and also Lohmann, 2006). In this paper we refer to its relations with the Shapley value, the Center of the Imputation Set (CIS) value and the Equal Split of Non-separable Rewards (ESNR) value.

The coincidence of the Shapley value and the AL-value on some classes of balanced games seems not unexpected because the both values share the same principle of averaging. The Shapley value is the average of the marginals $m^\sigma : G^N \to \mathbb{R}^N$, $\sigma \in \Pi(N)$, defined by

$$m^\sigma_{\sigma(i)}(v) := v(\{\sigma(j) | j \in N, j \leq i\}) - v(\{\sigma(j) | j \in N, j < i\}) \quad \text{for each } i \in N.$$

Then the Shapley value $\phi(v)$ of $(N, v) \in \mathcal{G}$ is given (cf. Shapley, 1953) by

$$\phi(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v).$$

The class of convex games is the most important class of balanced games on which the Shapley value and the AL-value coincide (see Tijs, 2005). Recall that a game $(N, v) \in \mathcal{G}$ is *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subset N.$$

The CIS value is defined on the class $\mathcal{I}^N$ of games with (non-empty) imputation set $I(v) := \{x \in I^*(v) | x_i \geq v(\{i\}) \text{ for each } i \in N\}$. The CIS value, $CIS : \mathcal{I}^N \to \mathbb{R}^N$, of $(N, v) \in \mathcal{I}^N$ is defined by

$$CIS(v) = \frac{1}{n} \sum_{k \in N} f^k(v),$$

where, for $k \in N$, and $i \in N \setminus \{k\}$,

$$(f^k(v))_i = v(\{k\}) \text{ for } i \neq k, \text{ and } (f^k(v))_k = v(N) - \sum_{j \in N \setminus \{k\}} v(\{j\}).$$

It also holds that $I(v) = conv(\{(f^k(v))_{k \in N}\})$.

The ESNR value is defined on the the class $\mathcal{I}_d^N$ of games with (non-empty) dual imputation set $I_d(v) := \{x \in I^*(v) \mid x_i \leq b_i^v \text{ for each } i \in N\}$. The ESNR value, $ESNR : \mathcal{I}_d^N \to \mathbb{R}^N$, is defined by

$$ESNR(v) = \frac{1}{n} \sum_{k \in N} g^k(v),$$

for each $(N, v) \in \mathcal{I}_d^N$, where, for $k \in N$, and $i \in N \setminus \{k\}$,

$$(g^k(v))_i = b_k^v \text{ for } i \neq k, \text{ and } (g^k(v))_k = v(N) - \sum_{j \in N \setminus \{k\}} b_j^v.$$

Clearly, this is equivalent to $ESNR_i(v) = b_i^v + \frac{1}{n}(v(N) - \sum_{j \in N} b_j^v)$ for each $i \in N$. It also holds that $I_d(v) = conv(\{(g^k(v))_{k \in N}\})$.

On the class of dual simplex games (also called 1-convex games in Driessen, 1985) the AL-value and the ESNR value coincide (see Tijs, 2005). A game (N, v) is a dual simplex game (cf. Branzei and Tijs, 2001; Tijs and Branzei, 2002) if its core $C(v)$ is equal to its (non-empty) dual imputation set $I_d(v)$.

3 The lexicore and the extreme points of the core

Instead of concentrating on the average of the leximals of a balanced cooperative game, we consider here the convex hull of the leximals of a balanced game, i.e. its lexicore. The *lexicore*, LEC, is defined by

$$LEC(v) = conv(\{L^\sigma(v) | \sigma \in \Pi(N)\}),$$

for each $(N, v) \in \mathcal{G}_C^N$. Clearly, the AL-value $AL(v)$ is the center of gravity of $LEC(v)$, and $LEC(v) \subseteq C(v)$ for each $(N, v) \in \mathcal{G}_C$. The next example shows that this inclusion relation may be strict.

Example 3.1 (Derks and Kuipers (2002)) Consider the following game (N, v):

$N = \{1, 2, 3, 4\}, v(N) = 22, v(S) = 12$ for $|S| = 3, v(S) = 7$ for $|S| = 2, v(S) = 0$ for $|S| = 1$.

This is not a convex game. The set $ext(C(v))$ of extreme points of the core of v has 24 elements, where 12 extreme points are orders of $(10, 5, 5, 2)$, and 12 extreme points are orders of $(7, 7, 8, 0)$. Since each leximal $L^\sigma(v)$ is equal to an order of the vector $(10, 5, 5, 2)$, the lexicore $LEC(v)$ is the convex hull of the set of all vectors obtained by all the orders of $(10, 5, 5, 2)$, which is different from $C(v)$. This means that $LEC(v) \neq C(v)$.

On the class of convex games the lexicore and the core coincide (see Tijs(2005)). However, the lexicore and the core of a game may coincide even if the game is not convex, as the following example illustrates.

Example 3.2. Consider the following game (N, v):

$$N = \{1, 2, 3, 4\}, v(N) = 10, v(1, 2, 3) = 1, v(1, 2, 4) = 2, v(1, 3, 4) = a, v(2, 3, 4) = b,$$

$$v(3, 4) = 5, v(S) = 0 \text{ otherwise, where } 0 \leq a \leq 5, 0 \leq b \leq 5.$$

This is not a convex game except for the case $a = b = 5$. The extreme points of the core are given by

$$A = (5, 0, 0, 5), B = (5, 0, 5, 0), C = (0, 5, 5, 0), D = (0, 5, 0, 5),$$

$$E = (1, 0, 0, 9), F = (0, 1, 0, 9), G = (0, 0, 1, 9), H = (2, 0, 8, 0), I = (0, 2, 8, 0), J = (0, 0, 8, 2).$$

For all 24 orders σ, A, B, C, D correspond each to three different orderings and each of E, F, G, H, I, J appear twice as L^σ. Each lexicore element is given by a convex combination of the all above 10 extreme points. Thus $LEC(v) = C(v)$.

In the sequel, we describe classes of balanced games for which the lexicore and the core coincide. First, we give the following obvious but useful lemma.

Lemma 3.1. Let $(N, v) \in \mathcal{G}_C$. If each extreme point of the core $C(v)$ is a leximal $L^\sigma(v)$ for some $\sigma \in \Pi(N)$, then $LEC(v) = C(v)$.

The above Example 3.2 satisfies the condition of this lemma. Convex games also satisfy the condition in Lemma 3.1 since the leximals of (the core of) a convex game are all marginal vectors (see Tijs, 2005), which are the extreme points of the core of a convex game. To be more precise, for each convex game (N, v), we have $L^\sigma(v) = m^{\bar{\sigma}}(v)$ for each $\sigma \in \Pi(N)$, where $\bar{\sigma} = (\sigma(n), \sigma(n-1), ..., \sigma(2), \sigma(1))$. Thus, we obtain by Lemma 3.1 that $LEC(v) = C(v)$ for each convex game (N, v).

Let (N, v) be an arbitrary balanced game. We consider the game $(N, \tilde{v})$ defined by

$$\tilde{v}(S) = \max\{v(S), v(N) - \sum_{j \in N \setminus S} b_j^v\}, \quad \text{for each } S \subseteq N.$$

The games (N, v) and $(N, \tilde{v})$ have the same core (see the last line of page 197 in Potters and Tijs, 1995). If (N, v) is a balanced game and $(N, \tilde{v})$ is convex then the equality $LEC(v) = C(v)$ holds true. There are two interesting classes of balanced games satisfying the condition that $(N, \tilde{v})$ is convex, namely the class of k-convex games (Driessen, 1988) and the class of clan games (Potters et al., 1989). For a proof, see Potters and Tijs (1995).

Let k be a natural number $1 \leq k \leq n$. If a game (N, v) satisfies the following two conditions, it is called a *k-convex game* (see Driessen (1988) and p.198 in Potters and Tijs (1995)):

(1) $v(S) \leq v(N) - \sum_{j \in N \setminus S} b_j^v$ for S with $|S| \geq k$;

(2) The game $\tilde{v}^k$ is convex, where $\tilde{v}^k$ is defined by $\tilde{v}^k(S) = \tilde{v}(S)$ for $|S| \geq k$ and $\tilde{v}^k(S) = v(S)$ for $|S| < k$.

A necessary and sufficient condition for a game (N, v) be a k-convex game (cf. Driessen, 1988) is given in Theorem 6.1. Consequently, on the class of k-convex games the lexicore and the core coincide. We notice here that n-convex games are convex games. So, we get as a particular case the known result that for each convex game its lexicore and core coincide.

Let C be a non-empty subset of N. A game (N, v) is a *clan game* (cf. Potters et al., 1989) with clan $C \subset N$ if $v \geq 0, b_i^v \geq 0 \ \forall \ i \in N, v(S) = 0$ for $S \not\supseteq C$ and $v(S) \leq v(N) - \sum_{j \in N \setminus S} b_j^v$ for $S \supseteq C$.

Each clan game is a balanced game, and it is proved in Potters and Tijs (1995) that the related game $(N, \tilde{v})$ is convex. Consequently, on the class of clan games the lexicore and the core coincide. We notice that for big boss games, which are clan games whose clan consists of one player, a description of the leximals of such a game has been given in Tijs (2005).

So, we proved the following:

Proposition 3.1. Let $(N, v) \in \mathcal{G}$. If there exists a natural number $1 \leq k \leq n$ such that (N, v) is a k-convex game or if there exists a non-empty set $C \subset N$, such that the game (N, v) is a clan game with clan C, then the equality $LEC(v) = C(v)$ holds true.

Note that the game in Example 3.2 is neither a k-convex game nor a clan game, but the equality $LEC(v) = C(v)$ holds true. In Proposition 3.2 we provide a more general sufficient condition for the coincidence of the lexicore and the core of a game. This sufficient condition is based on the exactification of a balanced game.

It is known that for each balanced game (N, v) there is a unique exact game (N, v^E) with the same core as the original game (see Schmeidler, 1972). The *exact game* (N, v^E) is defined by $v^E(\emptyset) = 0$ and for each $S \subseteq N, S \neq \emptyset$,

$$v^E(S) = \min\{x(S) | x \in C(v)\}.$$

Clearly, if the exact game (N, v^E) is convex, then the equality $LEC(v) = C(v)$ holds true. So, we proved the following

Proposition 3.2. Let $(N, v) \in \mathcal{G}_C$ and let (N, v^E) be the unique exact game corresponding to (N, v). If the game (N, v^E) is convex, then the equality $LEC(v) = C(v)$ holds true.

Remark 3.1. Since each convex game is also exact, for k-convex games and clan games (N, v), by the uniqueness of the game (N, v^E), we have that the games $(N, \tilde{v})$ and (N, v^E) coincide. So, we obtain the result in Proposition 3.1 as a particular case of Proposition 3.2.

Next, we prove that on the class of balanced almost convex games the lexicore and the core coincide as well. Let $(N, v) \in \mathcal{G}_C$. The game (N, v) is an *almost convex game* if all its proper subgames are convex. This means that all the convexity conditions hold true except those involving the grand coalition. In particular,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subset N \text{ with } S \cup T \neq N.$$

A characterization of the extreme points of the core of a balanced almost convex game was provided in Theorem 15 in Núñez and Rafels (1998) based on the reduced marginal worth vectors of the game (cf. Definition 11 in Núñez and Rafels, 1998; see also Section 5). The proof of the following proposition of the coincidence of the lexicore and the core of such a game is given in Section 6 (see Theorem 6.4 (iii)).

Proposition 3.3. Let (N, v) be a balanced almost convex game. Then, $LEC(v) = C(v)$.

In the following we investigate some properties of the lexicore on the domain of arbitrary balanced games.

From the definition of the exactification (N, v^E) of (N, v), we conclude that $C(v^E) = C(v)$ for each balanced game (N, v), and $v^E = v$ iff (N, v) is an exact game. Clearly, if for $(N, v), (N, w)$, we have $C(v) = C(w)$, then $LEC(v) = LEC(w)$. We conclude that $LEC(v) = LEC(v^E)$ for each balanced game (N, v). This property of the lexicore is known as the invariance with respect to the exactification (cf.Tijs, 2005).

4 On additivity of leximals, the lexicore and the AL-value

Let $A \subset \mathcal{G}_C^N$ be a cone, that is, $v, w \in A$ implies $pv + qw \in A$, for each $p, q \in \mathbb{R}_+$, where $\mathbb{R}_+$ stands for the set of non-negative real numbers.

Let $\sigma \in \Pi(N)$. We say that L^σ, LEC, AL are *additive* on A if for all $v, w \in A$:

$$L^\sigma(v + w) = L^\sigma(v) + L^\sigma(w),$$

$$LEC(v + w) = LEC(v) + LEC(w),$$

$$AL(v + w) = AL(v) + AL(w).$$

The next example illustrates that the leximals, the lexicore and the AL-value are not necessarily additive on the class of balanced games.

Example 4.1. Let (N, v) and (N, w) be the following games:

$$N = \{1, 2, 3\},\ v(1, 3) = v(2, 3) = v(N) = 1,\ v(S) = 0 \text{ otherwise},$$

$$w(1, 2) = w(1, 2, 3) = 1,\ w(S) = 0 \text{ otherwise}.$$

Then,

$$C(v) = \{(0, 0, 1)\},\ C(w) = conv(\{(0, 0, 0), (0, 1, 0)\}),$$

and

$$C(v+w) = conv(\{(1,0,1),(0,1,1),(1,1,0)\}).$$

So, $C(v+w) \supset C(v) + C(w)$.

Further,

$$L^{(1,2,3)}(v) + L^{(1,2,3)}(w) = (0,0,1) + (1,0,0) \neq L^{(1,2,3)}(v+w) = (1,1,0),$$

$$LEC(v+w) = C(v+w) \supset LEC(v) + LEC(w) = C(v) + C(w).$$

Since all these solutions are selections of the core, and many subcones of $\mathcal{G}_C$ are known where the core is additive (see, for example, Tijs and Branzei, 2002), an interesting question is whether or not L^σ, $\sigma \in \Pi(N)$, LEC, and AL are additive on such cones. To tackle this question we use a result of Kohlberg (1972) and Tijs (2006) saying that lexicographic optimization on polytopes is linear programming. To be more precise, given a polytope P in $\mathbb{R}^N$ and an order $\sigma \in \Pi(N)$, for all $\epsilon > 0$ sufficiently small, there is a unique $\hat{x} \in P$ such that

$$a^{\sigma,\epsilon} \cdot \hat{x} = \max\{a^{\sigma,\epsilon} \cdot x | x \in P\}, \text{ and } \hat{x} \text{ is the lexicographic maximum of P w.r.t. } \sigma,$$

where $(a^{\sigma,\epsilon})_{\sigma(k)} = \epsilon^{k-1}$ for each $k \in N$, and $a^{\sigma,\epsilon} \cdot x = \sum_{i \in N} a_i^{\sigma,\epsilon} x_i$.

Theorem 4.2. Let K be a cone of n-person balanced games on which the core correspondence is additive. Then,
(i) L^σ is additive for each $\sigma \in \Pi(N)$;
(ii) AL is additive;
(iii) LEC is additive.

Proof. (i) Take $v, w \in K$ and $\sigma \in \Pi(N)$. We have to prove that $L^\sigma(v+w) = L^\sigma(v) + L^\sigma(w)$. Since $C(v), C(w)$ and $C(v+w)$ are polytopes we can find an $\epsilon > 0$, such that

$$\arg\max\{a^{\sigma,\epsilon} \cdot x | x \in C(u)\} = L^\sigma(u)$$

for $u \in \{v, w, v+w\}$. Further, since $C(v+w) = C(v) + C(w)$, we have $L^\sigma(v) + L^\sigma(w) \in C(v+w)$, and for each $z \in C(v+w)$, there exist $x \in C(v)$ and $y \in C(w)$ such that $z = x + y$. From

$$a^{\sigma,\epsilon} \cdot x \leq a^{\sigma,\epsilon} \cdot L^\sigma(v), \; a^{\sigma,\epsilon} \cdot y \leq a^{\sigma,\epsilon} \cdot L^\sigma(w),$$

follows

$$a^{\sigma,\epsilon} \cdot z \leq a^{\sigma,\epsilon} \cdot (L^\sigma(v) + L^\sigma(w)).$$

So, $L^\sigma(v+w) = L^\sigma(v) + L^\sigma(w)$.
(ii) From (i) follows straightforwardly that $AL(v+w) = AL(v) + AL(w)$ for all $v, w \in K$.
(iii) Let $v, w \in K$. Since $L^\sigma(v+w) = L^\sigma(v) + L^\sigma(w)$ for all $\sigma \in \Pi(N)$, by (ii) we obtain

$$LEC(v+w) = conv(\{L^\sigma(v+w) | \sigma \in \Pi(N)\}) = conv(\{L^\sigma(v) + L^\sigma(w) | \sigma \in \Pi(N)\})$$

$$\subseteq conv(\{L^\sigma(v)|\sigma \in \Pi(N)\}) + conv(\{L^\sigma(w)|\sigma \in \Pi(N)\}) = LEC(v) + LEC(w).$$

Now, we prove the reverse inclusion, i.e.

$$LEC(v+w) \supseteq LEC(v) + LEC(w).$$

Notice that the compact and convex set $LEC(v+w)$ is the intersection of all hyper-half spaces H that contain $LEC(v+w)$. It suffices to show that $LEC(v) + LEC(w) \subseteq H$ for each such halfspace H. Take H containing $LEC(v+w)$. Then, H is of the form $\{x \in \mathbb{R}^N | m \cdot x \leq \alpha\}$ with $m \in \mathbb{R}^N \setminus \{0\}$ and $\alpha \in \mathbb{R}$.

Let $\beta = \max\{m \cdot x \mid x \in LEC(v+w)\}$ and let $H' = \{x \in \mathbb{R}^N | m \cdot x \leq \beta\}$. Then, $LEC(v+w) \subseteq H' \subseteq H$ and there is a $\sigma \in \Pi(N)$ with $L^\sigma(v+w) \in H'$ and $m \cdot L^\sigma(v+w) = \beta$.

Define K' and L' by $K' = \{x \in \mathbb{R}^N | m \cdot x \leq m \cdot L^\sigma(v)\}$ and $L' = \{x \in \mathbb{R}^N | m \cdot x \leq m \cdot L^\sigma(w)\}$. Because $L^\sigma(v+w) = L^\sigma(v) + L^\sigma(w)$, it is not difficult to check that $H' = K' + L', LEC(v) \subseteq K', LEC(w) \subseteq L'$. Then, we have

$$LEC(v) + LEC(w) \subseteq K' + L' = H' \subseteq H.$$

■

5 Consistency of leximals and average consistency of the AL-value

5.1 Consistency of leximals

In this section, we use notations $I^*(N,v), C(N,v)$, $L^\sigma(N,v)$, $AL(N,v)$ instead of $I^*(v)$, $C(v)$, $L^\sigma(v)$, $AL(v)$. We also denote $N \setminus j$ instead of $N \setminus \{j\}$ in this section and the next section.

Consistency with respect to a reduced game is one of the very important properties of solutions of a game, which requires the coincidence of the payoffs in the original game and its reduced game. Peleg (1986) shows that the core satisfies this consistency with respect to the reduced game *à* la Davis and Maschler. Núñez and Rafels (1998) show that each extreme point of the core satisfies the same consistency property.

Since each leximal is one of the extreme points of the core, the payoffs of the leximal of the original game coincide with the payoffs of an extreme point of the core of the reduced game. However, it might not be the payoffs of the leximal of the reduced game. More precisely, we have to show the coincidence of the payoffs of the leximal with respect to an order σ of the original game and the payoffs of the leximal with respect to an induced order from order σ of the reduced game *à* la Davis and Maschler.

The *reduced game à la Davis and Maschler*, in short *DM-reduced game* $(N \setminus k, v^x)$ is, for $x \in I^*(N,v)$ and $j \in N$, defined (cf. Davis and Maschler, 1965) by

$$v^x(N \setminus j) = v(N) - x_j,$$

$$v^x(S) = \max\{v(S \cup \{j\}) - x_j, v(S)\} \text{ for all } S \subset N \setminus j,$$

$$v^x(\emptyset) = 0.$$

Next, we will show a consistency of the leximals with respect to the reduced game. To prove this, following Caprari et al. (2006), we define a function $\mathcal{L}^\sigma$ with respect to $\sigma \in \Pi(N)$, $\mathcal{L}^\sigma : \mathcal{K} \to \mathbb{R}^N$, by

$$\mathcal{L}^\sigma_{\sigma(i)}(K) = \max\{x_{\sigma(i)} | x \in K, \mathcal{L}^\sigma_{\sigma(j)}(K) = x_{\sigma(j)}, \text{ for each } j < i\}$$

for each $K \in \mathcal{K}$, where $\mathcal{K} = \{K | K \subset \mathbb{R}^N, K \text{ is convex and compact}\}$. It holds $L^\sigma(N, v) = \mathcal{L}^\sigma(C(N, v))$.[1]

For any subset $S \subset N$ and any order $\sigma \in \Pi(N)$, take $T \subset \{1, 2, ..., n\}$ such that $\sigma(T) = S$; then, we can define a function σ_S on T, $\sigma_S : T \to S$, by $\sigma_S(i) = \sigma(i)$ for $i \in T$. We also denote a set of such functions by $\tilde{\Pi}(S)$. Then σ_S is not in $\Pi(S)$, but it induces a natural order on S. Let $\mathcal{K}_S = \{K_S | \exists K \in \mathcal{K} \text{ s.t. } K_S = K \cap \mathbb{R}^S\}$. We define $(\mathcal{L}^{\sigma_S})|_{\mathcal{K}_S}$ by

$$(\mathcal{L}^{\sigma_S}|_{\mathcal{K}_S})_{\sigma_S(i)}(K_S) = \max\{x_{\sigma_S(i)} \mid x|_S \in K_S, (\mathcal{L}^{\sigma_S}|_{\mathcal{K}_S})_{\sigma_S(j)}(K_S) = x_{\sigma_S(j)}, \forall j < i \text{ with } j \in \sigma^{-1}(S)\}$$

for $i \in \sigma^{-1}(S)$ and for any compact convex set $K_S \in \mathcal{K}_S$. We also denote $\mathcal{L}^{\sigma_S}|_{\mathcal{K}_S}$ by $\mathcal{L}^\sigma$, and σ_S by σ if there is no confusion. For leximals, we also use a similar notation, that is, for any $\sigma \in \Pi(N)$, any $S \subset N$, and any game (S, w), we denote $L^{\sigma_S}(S, w) = \mathcal{L}^{\sigma_S}(C(S, w))$ by $L^\sigma(S, w)$. Then we can present our Theorem.

Theorem 5.1. For any $\sigma \in \Pi(N)$, the leximal L^σ satisfies the DM- consistency, that is, for any $(N, v) \in \Gamma^C$ and $j \in N$, the DM-reduced game $(N \setminus j, v^{L^\sigma(N,v)})$ belongs to $\mathcal{G}_C$, and

$$L^\sigma_i(N, v) = L^\sigma_i(N \setminus j, v^{L^\sigma(N,v)}) \quad \text{for each } i \in N \setminus j.$$

Proof. Without loss of generality, we assume that $N = \{1, 2, ..., n\}$. Take any $j \in N$, $\sigma \in \Pi(N)$, and let $y = L^\sigma(N, v)$. Consider the reduced game $(N \setminus j, v^y)$ for $j \in N$. Let l be such that $\sigma(l) = j$. We distinguish two cases.
First, we consider the case when $\sigma(i) < \sigma(l)$. Let $i = 1$.

$$\begin{aligned} y_{\sigma(1)} &= \mathcal{L}^\sigma_{\sigma(1)}(C(N, v)) \\ &= \max\{x_{\sigma(1)} \in \mathbb{R} | x \in C(N, v)\} \\ &= \max\{x_{\sigma(1)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N\} \\ &= \max\{x_{\sigma(1)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(l)} = y_{\sigma(l)}\}, \end{aligned}$$

where the last equality holds because y is an element of $\arg\max\{x_{\sigma(1)} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N\}$.

On the other hand,

$$\begin{aligned} & \{x \in \mathbb{R}^{N \setminus j} | x(N) = v(N), x(S) \geq v(S),\ \forall S \subset N, x_j = y_j\} \\ = & \{x \in \mathbb{R}^{N \setminus j} | x(N \setminus j) = v(N) - y_j, x(S) \geq v(S) \text{ and } x(S) \geq v(S \cup \{j\}) - y_j\ \forall S \subseteq N \setminus j\} \\ = & \{x \in \mathbb{R}^{N \setminus j} | x(N \setminus j) = v^y(N \setminus j), x(S) \geq \max\{v(S), v(S \cup \{j\}) - y_j\}\ \forall S \subseteq N \setminus j \text{ with } S \neq \emptyset, \\ & \qquad 0 \geq v(j) - y_j\} \\ = & \{x \in \mathbb{R}^{N \setminus j} | x(N \setminus j) = v^y(N \setminus j), x(S) \geq v^y(S),\ \forall S \subseteq N \setminus j \text{ with } S \neq \emptyset, y_j \geq v(j)\} \\ = & \{x \in \mathbb{R}^{N \setminus j} | x(N \setminus j) = v^y(N \setminus j), x(S) \geq v^y(S),\ \forall S \subseteq N \setminus j\} = C(N \setminus j, v^y). \end{aligned}$$

[1]In Caprari et al. (2006), K is a share set, not a general compact convex set.

So, $\{x \in \mathbb{R}^{N\setminus j} | x(N) = v(N), x(S) \geq v(S),\ \forall S \subset N, x_j = y_j\} = C(N \setminus j, v^y)$. (5.1)

Then, we have

$$y_{\sigma(1)} = \max\{x_{\sigma(1)} \in \mathbb{R} | x \in C(N \setminus j, v^y)\} = \mathcal{L}^{\sigma}_{\sigma(1)}(C(N \setminus j, v^y)) = L^{\sigma}_{\sigma(1)}(N \setminus j, v^y).$$

Next consider $i = 2$. Based on $y_{\sigma(1)} = \mathcal{L}^{\sigma}_{\sigma(1)}(C(N \setminus j, v^y))$, we have

$$\begin{aligned} y_{\sigma(2)} &= \mathcal{L}^{\sigma}_{\sigma(2)}(C(N, v)) \\ &= \max\{x_{\sigma(2)} \in \mathbb{R} | x \in C(N, v), x_{\sigma(1)} = y_{\sigma(1)}\} \\ &= \max\{x_{\sigma(2)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(1)} = y_{\sigma(1)}\} \\ &= \max\{x_{\sigma(2)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(l)} = y_{\sigma(l)}, x_{\sigma(1)} = y_{\sigma(1)}\}. \end{aligned}$$

Further, by (5.1) we obtain,

$$\begin{aligned} &\{x \in \mathbb{R}^{N\setminus j} | x(N) = v(N), x(S) \geq v(S),\ \forall S \subset N, x_j = y_j, x_{\sigma(1)} = y_{\sigma(1)}\} \\ &\qquad = \{x \in \mathbb{R}^{N\setminus j} | x \in C(N \setminus j, v^y), x_{\sigma(1)} = y_{\sigma(1)}\}. \end{aligned}$$

This implies that

$$\begin{aligned} y_{\sigma(2)} &= \max\{x_{\sigma(2)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(l)} = y_{\sigma(l)}, x_{\sigma(1)} = y_{\sigma(1)}\} \\ &= \max\{x_{\sigma(2)} \in \mathbb{R} | x \in C(N \setminus j, v^y), x_{\sigma(1)} = y_{\sigma(1)}\} \\ &= \mathcal{L}^{\sigma}_{\sigma(2)}(C(N \setminus j, v^y)) = L^{\sigma}_{\sigma(2)}(N \setminus j, v^y). \end{aligned}$$

Based on $y_{\sigma(1)} = \mathcal{L}^{\sigma}_{\sigma(1)}(C(N\setminus j, v^y))$, $y_{\sigma(2)} = \mathcal{L}^{\sigma}_{\sigma(2)}(C(N\setminus j, v^y))$,..., $y_{\sigma(i-1)} = \mathcal{L}^{\sigma}_{\sigma(i-1)}(C(N\setminus j, v^y))$, by (5.1) we obtain $y_{\sigma(i)} = L^{\sigma}_{\sigma(i)}(N \setminus j, v^y)$ for the case $\sigma(i) < \sigma(l)$.

Now, consider the case when $\sigma(i) > \sigma(l)$. Based on the fact that $y_{\sigma(s)} = \mathcal{L}^{\sigma}_{\sigma(s)}(C(N \setminus j, v^y))$ for $\sigma(s) < \sigma(i)$, by (5.1) we obtain

$$\begin{aligned} y_{\sigma(i)} &= \mathcal{L}^{\sigma}_{\sigma(i)}(C(N, v)) \\ &= \max\{x_{\sigma(i)} \in \mathbb{R} | x \in C(N, v), x_{\sigma(t)} = y_{\sigma(t)} \text{ for } \sigma(t) < \sigma(i)\} \\ &= \max\{x_{\sigma(i)} \in \mathbb{R} | x(N) = v(N),\ x(S) \geq v(S)\ \forall S \subset N, x_{\sigma(t)} = y_{\sigma(t)} \text{ for } \sigma(t) < \sigma(i)\} \\ &= \max\{x_{\sigma(i)} \in \mathbb{R} | x \in C(N \setminus j, v^y), x_{\sigma(t)} = y_{\sigma(t)} \text{ for } \sigma(t) < \sigma(i)\} \\ &= \mathcal{L}^{\sigma}_{\sigma(i)}(C(N \setminus j, v^y)) = L^{\sigma}_{\sigma(i)}(N \setminus j, v^y). \end{aligned}$$

■

Let σ be a one to one and onto mapping from $\mathbb{N}$ to $\mathbb{N}$. For any different $i, j \in \mathbb{N}$, we consider the two-person game $(\{i, j\}, v)$ where $v(i, j) \geq v(i) + v(j)$. Then, we can define a value p^{σ} with respect to σ on the class of such games by

$$p^{\sigma}_i(v) = v(i, j) - v(j),\ p^{\sigma}_j(v) = v(j)$$

if $\sigma(i) < \sigma(j)$.

We notice that this value p^{σ} coincides with L^{σ} on the class of two-person games v as defined above. An interesting open question is whether L^{σ}, $\sigma \in \Pi(N)$, is the unique value which coincides with p^{σ} on the set of two-person balanced games and satisfies the DM-consistency.

5.2 Average Consistency of the AL-value

Next, we consider a special type of reduced games. Let $k \in N$ and let $\sigma^k \in \Pi(N)$ be an order σ which satisfies $\sigma(1) = k$. We denote the set of such orders by $\Pi^k(N)$. We also denote $z^k = \max\{x_k \in \mathbb{R} | x \in C(N,v)\} = L^{\sigma^k}_{\sigma^k(1)}(N,v)$.

We can consider the reduced game *à* la Davis and Maschler with respect to z^k, $(N \setminus k, v^{-k})$, given by

$$v^{-k}(N \setminus k) = v(N) - z^k,$$

$$v^{-k}(S) = \max\{v(S \cup \{k\}) - z^k, v(S)\} \text{ for all } S \subset N \setminus k,$$

$$v^{-k}(\emptyset) = 0.$$

Theorem 5.1 implies that for any $k \in N$,

$$L^{\sigma^k}_i(N,v) = L^{\sigma^k}_i(N \setminus k, v^{-k}) \text{ for any } j \in N \setminus \{k\}.$$

Definition 5.1. *(Average Consistency)* Let $(N,v) \in \mathcal{G}^N$. For any $k \in N$, let $z^k = \max\{x_k \in \mathbb{R} | x \in C(N,v)\}$. Then, a value $\phi : \mathcal{G}^N \to \mathbb{R}^n$ satisfies the *average consistency* if and only if for any $i \in N$,

$$\phi_i(N,v) = \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} \phi_i(N \setminus k, v^{-k}).$$

Theorem 5.1 easily implies the following theorem.

Theorem 5.2. The AL-value satisfies the average consistency, that is, for any $i \in N$,

$$AL_i(N,v) = \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} AL_i(N \setminus k, v^{-k}).$$

Proof. For any $k \in N$ and $i \in N \setminus k$,

$$\begin{aligned} AL_i(N,v) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^{\sigma}_i(N,v) \\ &= \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} \frac{1}{(n-1)!} \sum_{\sigma^k \in \Pi^k(N)} L^{\sigma^k}_i(N,v) \\ &= \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} \frac{1}{(n-1)!} \sum_{(\sigma^k)_{N \setminus k} \in \Pi(N \setminus k)} L^{(\sigma^k)_{N \setminus k}}_i(N \setminus k, v^{-k}) \\ &= \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} \frac{1}{(n-1)!} \sum_{\sigma \in \Pi(N \setminus k)} L^{\sigma}_j(N \setminus k, v^{-k}) \\ &= \frac{1}{n} z^i + \frac{1}{n} \sum_{k \neq i} AL_i(N \setminus k, v^{-k}). \end{aligned}$$

■

Corollary 5.1. Let $(N,v) \in \mathcal{G}^N$. For any $k \in N$, let $z^k = \max\{x_k \in \mathbb{R} | x \in C(N,v)\}$. Then, for any $i \in N$,

$$AL_i(N,v) = \frac{1}{n}\sum_{j\neq i}\left(\frac{1}{n-1}L_i^{\sigma^i}(N\setminus\{j\}, v^{L^{\sigma^i}(N,v)}) + AL_i(N\setminus j, v^{L^{\sigma^j}(N,v)})\right).$$

Proof. By the consistency of the leximals, we have

$$z^i = L^{\sigma^i}_{\sigma^i(1)}(N,v) = L_i^{\sigma^i}(N,v) = L_i^{\sigma^i}(N\setminus j, v^{L^{\sigma^i}(N,v)}) \quad \text{for each } j \in N \setminus i.$$

Then, we have

$$z^i = \frac{1}{n-1}\sum_{j\neq i} z^i = \frac{1}{n-1}\sum_{j\neq i} L_i^{\sigma^i}(N\setminus j, v^{L^{\sigma^i}(N,v)}).$$

■

6 The AL-value for some classes of games

In this section, we give explicit formulas for the leximals and the AL-value of games belonging to some classes of games with some convexity flavour.

6.1 The AL-value of k-convex games

Let $1 \le k \le n$ and let (N,v) be a k-convex game. Driessen (1988) gives a necessary and sufficient condition of k-convexity of a game.

Theorem 6.1.(Driessen (1988)) Let $1 \le k \le n$. A game (N,v) is k-convex if and only if it satisfies the following four conditions:

(i) $v(S\cup\{i\}) - v(S) \le v(T\cup\{i\}) - v(T)$ for all $i \in N, S \subset T \subset N \setminus i$ with $|T| \le k-2$,

(ii) $v(N) - v(S) \ge \sum_{j\in N\setminus S} b_j^v$ for all $S \subset N$ with $|S| \ge k$,

(iii) $v(N) - v(S) \le \sum_{j\in N\setminus S} b_j^v$ for all $S \subset N$ with $|S| = k-1$,

(iv) $v(N) - v(S) \ge \sum_{j\in (N\setminus S)\setminus i} b_j^v + \max_{j\in S}\{v((S\setminus j)\cup\{i\}) - v(S\setminus j)\}$ for all $i \in N$ and all $S \subset N \setminus i$ with $S \neq \emptyset$ and $|S| = k-1$.

Note that n- and $(n-1)$-convex games are convex games. Driessen (1988) completely characterizes the extreme points of the core of k-convex games.

Theorem 6.2. (Driessen(1988)) Let (N,v) be a k-convex game. Then, the set of extreme points of the core, $ext(C(v))$, coincides with $\{x^\tau | \tau \in \Pi(N)\}$, where x^τ is given by

$$x^\tau_{\tau(i)} = \begin{cases} v(\{\tau(j)\in N | j \le i\}) - v(\{\tau(j)\in N | j < i\}) & \text{if } i < k \\ v(N) - v(\{\tau(j)\in N | j < i\}) - \sum_{j>i} b^v_{\tau(j)} & \text{if } i = k \\ b^v_{\tau(i)} & \text{if } i > k. \end{cases}$$

Theorem 6.3. Let (N, v) be a k-convex game. Then for each $\sigma \in \Pi(N)$, the leximal $L^\sigma(v)$ is given by

$$L^\sigma_{\sigma(i)}(v) = \begin{cases} b^v_{\sigma(i)} & \text{if } i < n-k+1 \\ v(N) - v(\{\sigma(j) \in N | j > i\}) - \sum_{j<i} b^v_{\sigma(j)} & \text{if } i = n-k+1 \\ v(\{\sigma(j) \in N | j \geq i\}) - v(\{\sigma(j) \in N | j > i\}) & \text{if } i > n-k+1. \end{cases} \tag{6.1}$$

Moreover, the AL-value $AL(v)$ is given by

$$AL_i(v) = (1 - \frac{k}{n})b^v_i + \sum_{s=0}^{k-2} \sum_{S \subset N \setminus i, |S|=s} \frac{s!(n-1-s)!}{n!}(v(S \cup \{i\}) - v(S))$$

$$+\frac{1}{n}v(N) - \frac{(k-1)!(n-k)!}{n!} \sum_{T \subset N \setminus i, |T|=k-1} (v(T) + \sum_{j \in (N \setminus T) \setminus i} b^v_j).$$

Proof. For any order $\tau \in \Pi(N)$, take any player $t \in N$ such that $\tau(i) = t, i \in N$. Then, player t receives one of the payoffs b^v_t, $v(S \cup \{t\}) - v(S)$ and $v(N) - v(T) - \sum_{j \in N \setminus (T \cup \{i\})} b^v_{\tau(j)}$, where $S = \{\tau(j) \in N | \tau(j) < \tau(s)\}$ for $s < i, s \in N$ and $T = \{\tau(j) \in N | \tau(j) < \tau(i)\}$. Therefore, we have to compare b^v_t, $v(S \cup \{t\}) - v(S)$, $v(S' \cup \{t\}) - v(S')$ and $v(N) - v(T) - \sum_{u \in N \setminus (T \cup \{t\})} b^v_u$, where $|T| = k-1, T \subseteq N \setminus t$ and $S' \subset S \subseteq T$.

Then, the condition (i) of Theorem 6.1 implies

$$v(S \cup \{t\}) - v(S) \geq v(S' \cup \{t\}) - v(S')$$

for all S' and S such that $S' \subset S \subseteq N \setminus t$ with $|S| \leq k-1$. The condition (ii) of Theorem 6.1 implies

$$\begin{aligned} &\left(v(N) - v(T) - \sum_{u \in N \setminus (T \cup \{t\})} b^v_u \right) - (v(T \cup \{t\}) - v(T)) \\ = \;& v(N) - v(T \cup \{t\}) - \sum_{u \in N \setminus (T \cup \{t\})} b^v_u \geq 0 \end{aligned}$$

The condition (iii) of Theorem 6.1 implies

$$b^v_t - \left(v(N) - v(T) - \sum_{u \in N \setminus (T \cup \{t\})} b^v_u \right) = -v(N) + v(T) + \sum_{u \in N \setminus T} b^v_u \geq 0$$

These inequalities imply that for any $t \in N$ and $T \subseteq N \setminus \{t\}$ with $|T| = k-1$ and for $S' \subset S \subseteq T$,

$$b^v_t \geq v(N) - v(T) - \sum_{u \in N \setminus (T \cup \{t\})} b^v_u$$

$$\geq v(T\cup\{t\})-v(T) \geq v(S\cup\{t\})-v(S) \geq v(S'\cup\{t\})-v(S'). \qquad (6.2)$$

Then, for each order $\sigma \in \Pi(N)$, the first $n-k$ players j get their marginal contribution b_j^v, which is, by (6.2), their maximal payoff in the core $C(v)$. The $(n-k+1)$-th player t gets $v(N)-v(T)-\sum_{u\in N\setminus(T\cup\{t\})} b_u^v$, where $T=\{k\in N|\sigma^{-1}(k)>t\}$, because after the first $n-k$ players j got b_j^v, this is, by (6.2), the maximal payoff of t in the core . After the $(n-k+1)$-th player t got his payoff the restricted game $v|_T = v$ to T is a convex game. Then, the $(n-k+2)$-th player j_1 gets $v(T)-v(T\setminus\{j_1\})$ because of the convexity of $v|_T$, the $(n-k+3)$-th player j_2 gets $v(T\setminus\{j_1\})-v(T\setminus\{j_1,j_2\})$, and so on. This implies that the leximal $L^\sigma(v)$ is given by (6.1).

Now, since the leximals are given by (6.1), the AL-value of a k-convex game (N,v) is given by

$$\begin{aligned}
AL_i(v) &= \frac{n-k}{n}b_i^v + \frac{1}{n}\binom{n-1}{0}^{-1} v(i) + \frac{1}{n}\binom{n-1}{1}^{-1}\sum_{j\in N\setminus i}(v(i,j)-v(j)) \\
&+\frac{1}{n}\binom{n-1}{2}^{-1}\sum_{j,k\in N\setminus i, j\neq k}(v(i,j,k)-v(j,k))+\cdots \\
&+\frac{1}{n}\binom{n-1}{k-3}^{-1}\sum_{\substack{i_1,i_2,\ldots i_{k-3}\in N\setminus i\\ \text{mutually different}}}(v(\{i_1,i_2,,\ldots,i_{k-3},i\})-v(\{i_1,i_2,,\ldots,i_{k-3}\})) \\
&+\frac{1}{n}\binom{n-1}{k-2}^{-1}\sum_{\substack{i_1,i_2,\ldots i_{k-2}\in N\setminus i\\ \text{mutually different}}}(v(\{i_1,i_2,,\ldots,i_{k-2},i\})-v(\{i_1,i_2,,\ldots,i_{k-2}\})) \\
&+\frac{1}{n}\binom{n-1}{k-1}^{-1}\sum_{\substack{i_1,i_2,\ldots i_{k-1}\in N\setminus i\\ \text{mutually different}}}(v(N)-v(\{i_1,i_2,,\ldots,i_{k-1}\})-\sum_{j\in N\setminus\{i_1,i_2,\ldots,i_{k-1},i\}} b_j^v) \\
&= (1-\frac{k}{n})b_i^v + \frac{1}{n}\sum_{s=0}^{k-2}\sum_{S\subset N\setminus i,|S|=s}\binom{n-1}{s}^{-1}(v(S\cup\{i\})-v(S)) \\
&+\frac{1}{n}\binom{n-1}{k-1}^{-1}\sum_{T\subset N\setminus i,|T|=k-1}(v(N)-v(T)-\sum_{j\in(N\setminus T)\setminus i} b_j^v)
\end{aligned}$$

$$= (1-\frac{k}{n})b_i^v + \frac{1}{n}\sum_{s=0}^{k-2}\sum_{S\subset N\setminus i,|S|=s}\frac{s!(n-1-s)!}{(n-1)!}(v(S\cup\{i\})-v(S))$$

$$+\frac{(k-1)!(n-k)!}{n!}\sum_{T\subset N\setminus i,|T|=k-1}(v(N)-v(T)-\sum_{j\in(N\setminus T)\setminus i}b_j^v)$$

$$= (1-\frac{k}{n})b_i^v + \sum_{s=0}^{k-2}\sum_{S\subset N\setminus i,|S|=s}\frac{s!(n-1-s)!}{n!}(v(S\cup\{i\})-v(S))$$

$$+\frac{1}{n}v(N) - \frac{(k-1)!(n-k)!}{n!}\sum_{T\subset N\setminus i,|T|=k-1}(v(T)+\sum_{j\in(N\setminus T)\setminus i}b_j^v)$$

$$= \sum_{s=0}^{k-2}\sum_{S\subset N\setminus i,|S|=s}\frac{s!(n-1-s)!}{n!}(v(S\cup\{i\})-v(S))$$

$$+b_i^v + \frac{1}{n}v(N) - \frac{k}{n}b_i^v - \frac{(k-1)!(n-k)!}{n!}\sum_{\substack{T\subset N\setminus i\\|T|=k-1}}\sum_{j\in(N\setminus T)\setminus i}b_j^v - \frac{(k-1)!(n-k)!}{n!}\sum_{\substack{T\subset N\setminus i\\|T|=k-1}}v(T).$$

■

Remark 6.1. We can look at the expression of $AL_i(v), i\in N$, as consisting of three parts:
• $\sum_{s=0}^{k-2}\sum_{S\subset N\setminus i,|S|=s}\frac{s!(n-1-s)!}{n!}(v(S\cup\{i\})-v(S))$, which is similar to the Shapley value formula up to $k-2$;
• $b_i^v+\frac{1}{n}v(N)-\frac{k}{n}b_i^v-\frac{(k-1)!(n-k)!}{n!}\sum_{T\subset N\setminus i,|T|=k-1}\sum_{j\in(N\setminus T)\setminus i}b_j^v$, which is similar to the $ESNR$-value;
• $-\frac{(k-1)!(n-k)!}{n!}\sum_{T\subset N\setminus i,|T|=k-1}v(T)$, which can be seen as an adjustment term.

Remark 6.2. The AL-value coincides with the ESNR-value on the class of 1-convex games (see also Tijs (2005)).

Remark 6.3. If (N,v) is a 2-convex game,

$$AL_i(v) = \left[\frac{1}{n-1}v(i)+\frac{n-2}{n-1}b_i^v\right]+\frac{1}{n}(v(N)-\sum_{j\in N}\left[\frac{v(j)}{n-1}+\frac{n-2}{n-1}b_j^v\right])$$

$$= \frac{1}{n-1}CIS_i(v)+\frac{n-2}{n-1}ESNR_i(v).$$

So, on the class of 2-convex games the AL-value is a convex combination of the CIS-value and the ESNR-value.

Remark 6.4. By the proof of Theorem 6.1, we can find that each extreme point of the core $C(v)$ appears as a leximal $L^\sigma(v)$ for some $\sigma\in\Pi(N)$, in case v is a k-convex game. This implies $LEC(v)=C(v)$ by Lemma 3.1.

6.2 The AL-value of balanced almost convex games

The notion of almost convex game is introduced by Núñez and Rafels (1998). As stated in Section 3, a game (N, v) is an almost convex game if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for all } S, T \subset N \text{ with } S \cup T \neq N.$$

For any $\sigma \in \Pi(N)$, define $\sigma(1) = i_n, \sigma(2) = i_{n-1}, ..., \sigma(n) = i_1$ The extreme points of the core of a balanced almost convex game are, by Theorem 15 of Núñez and Rafels (1998), the reduced marginal vectors $\{rm^\sigma\}_{\sigma\in\Pi(N)}$ given by:

$$\begin{aligned}
rm^\sigma_{\sigma(1)} &= rm^\sigma_{i_n} = v(i_1, i_2, ..., i_n) - v(i_1, i_2, ..., i_{n-1}) = b^v_{i_n} = b^v_{\sigma(1)}; \\
rm^\sigma_{\sigma(2)} &= rm^\sigma_{i_{n-1}} = v_{i_n}(i_1, i_2, ..., i_{n-1}) - v_{i_n}(i_1, i_2, ..., i_{n-2}) = b^{v_{i_n}}_{i_{n-1}} = b^{v_{\sigma(1)}}_{\sigma(2)}; \\
& \quad ... \\
rm^\sigma_{\sigma(n-1)} &= rm^\sigma_{i_2} = v_{i_n i_{n-1}...i_3}(i_1, i_2) - v_{i_n i_{n-1}...i_3}(i_1) = b^{v_{i_n i_{n-1}\cdots i_3}}_{i_2} = b^{v_{\sigma(1)\sigma(2)...\sigma(n-2)}}_{\sigma(n-1)}; \\
rm^\sigma_{\sigma(n)} &= rm^\sigma_{i_1} = v_{i_n i_{n-1}...i_2}(i_1) = b^{v_{i_n i_{n-1}\cdots i_2}}_{i_1} = b^{v_{\sigma(1)\sigma(2)...\sigma(n-1)}}_{\sigma(n)}.
\end{aligned}$$

Here, the i-th marginal game $(N \setminus \{i\}, v_i)$ for $i \in N$ is given by

$$v_i(S) = \max\{v(S \cup \{i\}) - b^v_i, v(S)\} \text{ for all } S \subseteq N \setminus \{i\}, S \neq \emptyset, \text{ and } v_i(\emptyset) = 0,$$

which is the reduced game *à* la Davis and Maschler when $x_i = b^v_i$; v_{i_n} denotes the i_n-th marginal game of v; $v_{i_n i_{n-1}} = (v_{i_n})_{i_{n-1}}$, the i_{n-1}-th marginal game of the game v_{i_n}; $v_{i_n i_{n-1}...i_{k+1} i_k} = (v_{i_n i_{n-1}...i_{k+1}})_{i_k}$ is the i_k-th marginal game of the game $v_{i_n i_{n-1}...i_{k+1}}$, and so on.

Theorem 6.4. Let (N, v) be a balanced almost convex game. Take any $\sigma \in \Pi(N)$, and let rm^σ be the reduced marginal vectors with respect to σ as defined above. Then, for $i \in \{1, 2, ..., n\}$, we have

(*i*) $L^\sigma_{\sigma(i)}(v) = rm^\sigma_{\sigma(i)}$,

(*ii*) $AL_{\sigma(i)}(v) = \dfrac{1}{n!} \sum_{\sigma\in\Pi(N)} rm^\sigma_{\sigma(i)}$,

(*iii*) $LEC(v) = C(v)$.

Proof. (i) Take any $i \in N$. For $k = 1, 2, .., n$, let us consider an order $\tau^k \in \Pi(N)$ such that $\tau^k(k) = i$. Then we show that

$$rm^{\tau^1}_i \geq rm^{\tau^2}_i \geq ... \geq rm^{\tau^{n-1}}_i \geq rm^{\tau^n}_i. \tag{6.3}$$

This is obtained from the fact that for any $k < n$, any mutually different $i_n, i_{n-1}, ..., i_{k+1}, i_k$ and any $i \in N \setminus \{i_n, i_{n-1}, ..., i_{k+1}, i_k\}$,

$$v_{i_n i_{n-1}...i_{k+1}}(N \setminus \{i_n, i_{n-1}, ..., i_{k+1}\}) - v_{i_n i_{n-1}...i_{k+1}}(N \setminus \{i_n, i_{n-1}, ..., i_{k+1}, i\})$$

$$\geq v_{i_n i_{n-1}...i_{k+1} i_k}(N \backslash \{i_n, i_{n-1}, ..., i_{k+1}, i_k\}) - v_{i_n i_{n-1}...i_{k+1} i_k}(N \backslash \{i_n, i_{n-1}, ..., i_{k+1}, i_k, i\}). \tag{6.4}$$

Now, we show inequality (6.4). We put $w = v_{i_n i_{n-1}...i_{k+1}}$ and $T = \{i_n, i_{n-1}, ..., i_{k+1}\}$. Then, we have to show that for any $i \in N \setminus \{i_n, i_{n-1}, ..., i_{k+1}, i_k\}$,

$$w(N \setminus T) - w(N \setminus (T \cup \{i\})) \geq w_{i_k}(N \setminus (T \cup \{i_k\})) - w_{i_k}(N \setminus (T \cup \{i_k, i\})).$$

This is because

$$\begin{aligned} & w_{i_k}(N \setminus (T \cup \{i_k\})) - w_{i_k}(N \setminus (T \cup \{i_k, i\})) \\ = \; & \max\{w(N \setminus T) - b^w_{i_k}, w(N \setminus (T \cup \{k\})\} - \max\{w(N \setminus (T \cup \{i\})) - b^w_{i_k}, w(N \setminus (T \cup \{k, i\}))\} \\ = \; & w(N \setminus (T \cup \{i_k\})) - \max\{w(N \setminus (T \cup \{i\})) - b^w_{i_k}, w(N \setminus (T \cup \{k, i\}))\} \\ \leq \; & w(N \setminus T) - w(N \setminus (T \cup \{i\})). \end{aligned}$$

The last inequality is implied by the fact that w is also a almost convex game (See Proposition 13 of Núñez and Rafels (1998)). So, we proved (6.4), and, consequently (6.3). The important fact is that, for $i \in N$, $rm_i^{\tau^1}$ is the maximal payoff in the core $C(v)$.

Take any order $\sigma \in \Pi(N)$. We will compute $L^\sigma(v)$. The first player i in the order σ gets $rm_i^\sigma = rm_{\sigma(1)}^\sigma$ because of (6.3). After i got the payoff rm_i^σ, we consider the game $(N \setminus \{i\}, v_i)$. The important facts for this game are

$$x \in C(N, v) \iff x|_{N \setminus \{i\}} \in C(N \setminus \{i\}, v_i),$$

and the game $(N \setminus \{i\}, v_i)$ is also balanced and almost convex. The first fact follows from the consistency of the core by Peleg(1986), and the second fact is from Núñez and Rafels (1998).

Hence, the second player j according to the order σ should get $rm_j^\sigma = rm_{\sigma(2)}^\sigma$ by the above argument. The same reasoning works out for all $k \in \{1, 2, ..., n\}$. Then, this consideration implies that

$$L^\sigma_{\sigma(k)}(v) = rm^\sigma_{\sigma(k)} \text{ for all } k \in \{1, 2, ..., n\}.$$

(ii) From (i) straightforwardly follows

$$AL_{\sigma(k)}(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} rm^\sigma_{\sigma(k)},$$

for all $k \in \{1, 2, ..., n\}$.

(iii) Since, by (i), all the extreme points of the core $C(v)$ are leximals $L^\sigma(v)$ for some $\sigma \in \Pi(N)$, we obtain by Lemma 3.1 that $LEC(v) = C(v)$. ■

Remark 6.5. The AL-value and the Shapley value coincide on the class of convex games as stated in Section 2. They also coincide on the class of symmetric games, which are games where the worth of any coalition only depends on the number of members of the coalition, because both of the values satisfy the efficiency and the symmetry properties (see Tijs (2005) and Shapley(1953)).

However, the two values do not necessarily coincide for non-convex and asymmetric games as the following example illustrates.

Example 6.1. Consider the game (N, v) in Example 3.2. The AL-value $AL(v)$ of (N, v) equals $(\frac{3}{2}, \frac{3}{2}, \frac{10}{3}, \frac{11}{3})$. The Shapley value $\phi(v)$ of (N, v) is

$$(\frac{21 + a - 3b}{12}, \frac{21 + b - 3a}{12}, \frac{30 + a + b}{12}, \frac{34 + a + b}{12}).$$

Notice that only if $a = b = 5$, $AL(v)$ coincides with $\phi(v)$, and in this case, the game is (convex and, consequently,) exact. Otherwise, $AL(v) \neq \phi(v)$.

Remark 6.6. It is proved by Tijs (2005) that if the exactification v^E of v is convex, then $AL(v^E) = \phi(v^E)$. Since k-convex games (N, v) satisfy this condition, we have $AL(v) = AL(v^E) = \phi(v^E)$ for any k-convex game (N, v).

7 Concluding Remarks

In this paper, we investigate several properties of the leximals, the lexicore and the AL-value. We provide some sufficient conditions for the coincidence of the lexicore and the core, a condition for the additivity of leximals, the lexicore and the AL-value, a consistency property of leximals and an average consistency property of the AL-value. We also give explicit formulas for the leximals and the AL-value of k-convex games and balanced almost convex games.

Though the lexicore is a refinement of the core, there is another interesting set called *Weber set* $W(v)$, which is the convex hull of the marginals and includes the core $C(v)$. It is known that $C(v) = W(v)$ if and only if the game v is convex (cf. Weber, 1988, and Ichiishi, 1981). Moreover we have $LEC(v) = C(v) = W(v)$ if v is convex in Section 3. As we showed in the paper, the convexity is not necessary for $LEC(v) = C(v)$. To find a necessary and sufficient condition for $LEC(v) = C(v)$ is an interesting topic to study.

The leximals satisfy the DM-consistency and coincide with p^σ values in two-person games. It is left for further research to characterize the leximals based on these two properties.

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