Evaluation of Value-at-Risk Models Using Historical Data

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Researchers in the field of financial economics have long recognized the importance of measuring the risk of a portfolio of financial assets or securities. Indeed, concerns go back at least four decades, when Markowitz's pioneering work on portfolio selection (1959) explored the appropriate definition and measurement of risk. In recent years, the growth of trading activity and instances of financial market instability have prompted new studies underscoring the need for market participants to develop reliable risk measurement techniques.[1]

One technique advanced in the literature involves the use of "value-at-risk" models. These models measure the market, or price, risk of a portfolio of financial assets—that is, the risk that the market value of the portfolio will decline as a result of changes in interest rates, foreign exchange rates, equity prices, or commodity prices. Value-at-risk models aggregate the several components of price risk into a single quantitative measure of the potential for losses over a specified time horizon. These models are clearly appealing because they convey the market risk of the entire portfolio in one number. Moreover, value-at-risk measures focus directly, and in dollar terms, on a major reason for assessing risk in the first place—a loss of portfolio value.

Recognition of these models by the financial and regulatory communities is evidence of their growing use. For example, in its recent risk-based capital proposal (1996a), the Basle Committee on Banking Supervision endorsed the use of such models, contingent on important qualitative and quantitative standards. In addition, the Bank for International Settlements Fisher report (1994) urged financial intermediaries to disclose measures of value-at-risk publicly. The Derivatives Policy Group, affiliated with six large U.S. securities firms, has also advocated the use of value-at-risk models as an important way to measure market risk. The introduction of the RiskMetrics database compiled by J.P. Morgan for use with third-party value-at-risk software also highlights the growing use of these models by financial as well as nonfinancial firms.

Clearly, the use of value-at-risk models is increas-

ing, but how well do they perform in practice? This article explores this question by applying value-at-risk models to 1,000 randomly chosen foreign exchange portfolios over the period 1983-94. We then use nine criteria to evaluate model performance. We consider, for example, how closely risk measures produced by the models correspond to actual portfolio outcomes.

We begin by explaining the three most common categories of value-at-risk models—equally weighted moving average approaches, exponentially weighted moving average approaches, and historical simulation approaches. Although within these three categories many different approaches exist, for the purposes of this article we select five approaches from the first category, three from the second, and four from the third.

By employing a simulation technique using these twelve value-at-risk approaches, we arrived at measures of price risk for the portfolios at both 95 percent and 99 percent confidence levels over one-day holding periods. The confidence levels specify the probability that losses of a portfolio will be smaller than estimated by the risk measure. Although this article considers value-at-risk models

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only in the context of market risk, the methodology is fairly general and could in theory address any source of risk that leads to a decline in market values. An important limitation of the analysis, however, is that it does *not* consider portfolios containing options or other positions with non-linear price behavior.[2]

We choose several performance criteria to reflect the practices of risk managers who rely on value-at-risk measures for many purposes. Although important differences emerge across value-at-risk approaches with respect to each criterion, the results indicate that none of the

twelve approaches we examine is superior on every count. In addition, as the results make clear, the choice of confidence level—95 percent or 99 percent—can have a substantial effect on the performance of value-at-risk approaches.

INTRODUCTION TO VALUE-AT-RISK MODELS

A value-at-risk model measures market risk by determining how much the value of a portfolio could decline over a given period of time with a given probability as a result of changes in market prices or rates. For example, if the given period of time is one day and the given probability is 1 percent, the value-at-risk measure would be an estimate of the decline in the portfolio value that could occur with a 1 percent probability over the next trading day. In other words, if the value-at-risk measure is accurate, losses greater than the value-at-risk measure should occur less than 1 percent of the time.

The two most important components of value-at-risk models are the length of time over which market risk is to be measured and the confidence level at which market risk is measured. The choice of these components by risk managers greatly affects the nature of the value-at-risk model.

The time period used in the definition of value-at-risk, often referred to as the "holding period," is discretionary. Value-at-risk models assume that the portfolio's composition does not change over the holding period. This assumption argues for the use of short holding periods because the composition of active trading portfolios is apt to change frequently. Thus, this article focuses on the widely used one-day holding period.[3]

Value-at-risk measures are most often expressed as percentiles corresponding to the desired confidence level. For example, an estimate of risk at the 99 percent confidence level is the amount of loss that a portfolio is expected to exceed only 1 percent of the time. It is also known as a 99th percentile value-at-risk measure because the amount is the 99th percentile of the distribution of potential losses on the portfolio.[4] In practice, value-at-risk estimates are calculated from the 90th to 99.9th percentiles, but the most commonly used range is the 95th to 99th percentile range. Accordingly, the text charts and the

tables in the appendix report simulation results for each of these percentiles.

THREE CATEGORIES OF VALUE-AT-RISK APPROACHES

Although risk managers apply many approaches when calculating portfolio value-at-risk models, almost all use past data to estimate potential changes in the value of the portfolio in the future. Such approaches assume that the future will be like the past, but they often define the past quite differently and make different assumptions about how markets will behave in the future.

The first two categories we examine, "variance-covariance" value-at-risk approaches,[5] assume normality and serial independence and an absence of nonlinear positions such as options.[6] The dual assumption of normality and serial independence creates ease of use for two reasons. First, normality simplifies value-at-risk calculations because all percentiles are assumed to be known multiples of the standard deviation. Thus, the value-at-risk calculation requires only an estimate of the standard deviation of the portfolio's change in value over the holding period. Second, serial independence means that the size of a price move on one day will not affect estimates of price moves on any other day. Consequently, longer horizon standard deviations can be obtained by multiplying daily horizon standard deviations by the square root of the number of days in the longer horizon. When the assumptions of normality and serial independence are made together, a risk manager can use a single calculation of the portfolio's daily horizon standard deviation to develop value-at-risk measures for any given holding period and any given percentile.

The advantages of these assumptions, however, must be weighed against a large body of evidence suggesting that the tails of the distributions of daily percentage changes in financial market prices, particularly foreign exchange rates, will be fatter than predicted by the normal distribution.[7] This evidence calls into question the appealing features of the normality assumption, especially for value-at-risk measurement, which focuses on the tails of the distribution. Questions raised by the commonly used normality assumption are highlighted throughout the article.

In the sections below, we describe the individual features of the two variance-covariance approaches to value-at-risk measurement.

EQUALLY WEIGHTED MOVING AVERAGE APPROACHES

The equally weighted moving average approach, the more straightforward of the two, calculates a given portfolio's variance (and thus, standard deviation) using a fixed amount of historical data.[8] The major difference among equally weighted moving average approaches is the time frame of the fixed amount of data.[9] Some approaches employ just the most recent fifty days of historical data on the assumption that only very recent data are relevant to estimating potential movements in portfolio value. Other approaches assume that large amounts of data are necessary to estimate potential movements accurately and thus rely on a much longer time span—for example, five years.

The calculation of portfolio standard deviations using an equally weighted moving average approach is

$$(1) \qquad \sigma_t = \sqrt{\frac{1}{(k-1)} \sum_{s=t-k}^{t-1} (x_s - \mu)^2} \, ,$$

where σ_t denotes the estimated standard deviation of the portfolio at the beginning of day t. The parameter k specifies the number of days included in the moving average (the "observation period"), x_s, the change in portfolio value on day s, and μ, the mean change in portfolio value. Following the recommendation of Figlewski (1994), μ is always assumed to be zero.[10]

Consider five sets of value-at-risk measures with periods of 50, 125, 250, 500, and 1,250 days, or about two months, six months, one year, two years, and five years of historical data. Using three of these five periods of time, Chart 1 plots the time series of value-at-risk measures at biweekly intervals for a single fixed portfolio of spot foreign exchange positions from 1983 to 1994.[11] As shown, the fifty-day risk measures are prone to rapid swings. Conversely, the 1,250-day risk measures are more stable over long periods of time, and the behavior of the 250-day risk measures lies somewhere in the middle.

Exponentially Weighted Moving Average Approaches

Exponentially weighted moving average approaches emphasize recent observations by using exponentially weighted moving averages of squared deviations. In contrast to equally weighted approaches, these approaches attach different weights to the past observations contained in the observation period. Because the weights decline exponentially, the most recent observations receive much more weight than earlier observations. The formula for the portfolio standard deviation under an exponentially weighted moving average approach is

$$(2) \qquad \sigma_t = \sqrt{(1-\lambda) \sum_{s=t-k}^{t-1} \lambda^{t-s-1}(x_s - \mu)^2} \, .$$

The parameter λ, referred to as the "decay factor," determines the rate at which the weights on past observations decay as they become more distant. In theory, for the weights to sum to one, these approaches should use an infinitely large number of observations k. In practice, for the values of the decay factor λ considered here, the sum of the weights will converge to one, with many fewer observations than the 1,250 days used in the simulations. As with

the equally weighted moving averages, the parameter μ is assumed to equal zero.

Exponentially weighted moving average approaches clearly aim to capture short-term movements in volatility, the same motivation that has generated the large body of literature on conditional volatility forecasting models.[12] In fact, exponentially weighted moving average approaches are equivalent to the IGARCH(1,1) family of popular conditional volatility models.[13] Equation 3 gives an equivalent formulation of the model and may also suggest a more intuitive understanding of the role of the decay factor:

$$(3) \qquad \sigma_t = \sqrt{\lambda \sigma_{t-1}^2 + (1-\lambda)(x_{t-1} - \mu)^2} \, .$$

As shown, an exponentially weighted average on any given day is a simple combination of two components: (1) the weighted average on the previous day, which receives a weight of λ, and (2) yesterday's squared deviation, which receives a weight of $(1 - \lambda)$. This interaction means that the *lower* the decay factor λ, the *faster* the decay in the influence of a given observation. This concept is illustrated in Chart 2, which plots time series of value-at-risk measures using exponentially weighted moving aver-

Chart 1

Value-at-Risk Measures for a Single Portfolio over Time
Equally Weighted Moving Average Approaches
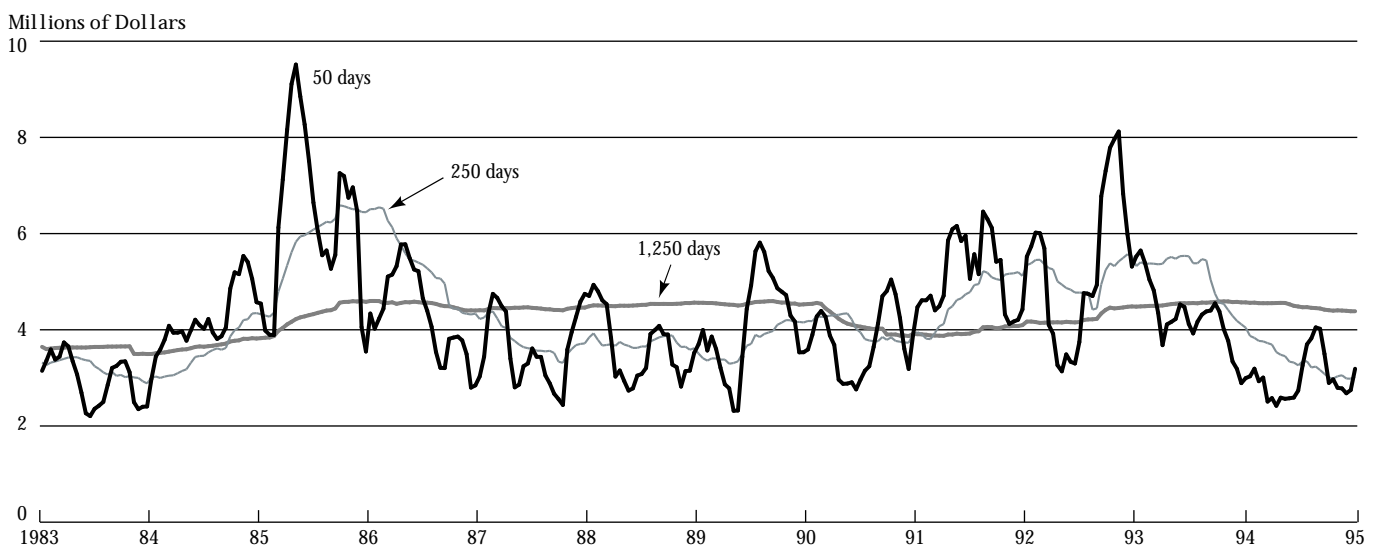
Source: Author's calculations.

ages with decay factors of 0.94 and 0.99. A decay factor of 0.94 implies a value-at-risk measure that is derived almost entirely from very recent observations, resulting in the high level of variability apparent for that particular series.

On the one hand, relying heavily on the recent past seems crucial when trying to capture short-term movements in actual volatility, the focus of conditional volatility forecasting. On the other hand, the reliance on recent data effectively reduces the overall sample size, increasing the possibility of measurement error. In the limiting case, relying only on yesterday's observation would produce highly variable and error-prone risk measures.

HISTORICAL SIMULATION APPROACHES

The third category of value-at-risk approaches is similar to the equally weighted moving average category in that it relies on a specific quantity of past historical observations (the observation period). Rather than using these observations to calculate the portfolio's standard deviation, however, historical simulation approaches use the actual percentiles of the observation period as value-at-risk measures. For example, for an observation period of 500 days, the 99th percentile historical simulation value-at-risk mea-

sure is the sixth largest loss observed in the sample of 500 outcomes (because the 1 percent of the sample that should exceed the risk measure equates to five losses).

In other words, for these approaches, the 95th and 99th percentile value-at-risk measures will not be constant multiples of each other. Moreover, value-at-risk measures for holding periods other than one day will not be fixed multiples of the one-day value-at-risk measures. Historical simulation approaches do not make the assumptions of normality or serial independence. However, relaxing these assumptions also implies that historical simulation approaches do not easily accommodate translations between multiple percentiles and holding periods.

Chart 3 depicts the time series of one-day 99th percentile value-at-risk measures calculated through historical simulation. The observation periods shown are 125 days and 1,250 days.[14] Interestingly, the use of actual percentiles produces time series with a somewhat different appearance than is observed in either Chart 1 or Chart 2. In particular, very abrupt shifts occur in the 99th percentile measures for the 125-day historical simulation approach.

Trade-offs regarding the length of the observation period for historical simulation approaches are similar to

Chart 2

Value-at-Risk Measures for a Single Portfolio over Time
Exponentially Weighted Moving Average Approaches
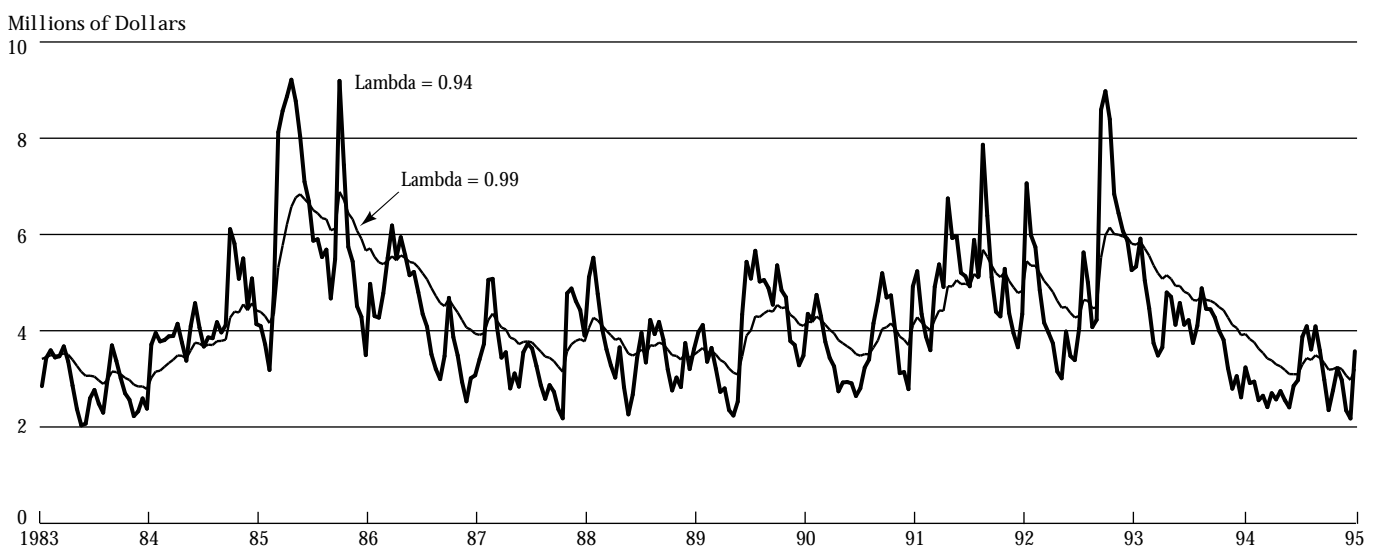
Source: Author's calculations.

those for variance-covariance approaches. Clearly, the choice of 125 days is motivated by the desire to capture short-term movements in the underlying risk of the portfolio. In contrast, the choice of 1,250 days may be driven by the desire to estimate the historical percentiles as accurately as possible. Extreme percentiles such as the 95th and particularly the 99th are very difficult to estimate accurately with small samples. Thus, the fact that historical simulation approaches abandon the assumption of normality and attempt to estimate these percentiles directly is one rationale for using long observation periods.

SIMULATIONS OF VALUE-AT-RISK MODELS

This section provides an introduction to the simulation results derived by applying twelve value-at-risk approaches to 1,000 randomly selected foreign exchange portfolios and assessing their behavior along nine performance criteria (see box). This simulation design has several advantages. First, by simulating the performance of each value-at-risk approach for a long period of time (approximately twelve years of daily data) and across a large number of portfolios, we arrive at a clear picture of how value-at-risk models would actually have performed for linear foreign exchange

portfolios over this time span. Second, the results give insight into the extent to which portfolio composition or choice of sample period can affect results.

It is important to emphasize, however, that neither the reported variability across portfolios nor variability over time can be used to calculate suitable standard errors. The appropriate standard errors for these simulation


The simulation results provide a relatively complete picture of the performance of selected value-at-risk approaches in estimating the market risk of a large number of portfolios.

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results raise difficult questions. The results aggregate information across multiple samples, that is, across the 1,000 portfolios. Because the results for one portfolio are not independent of the results for other portfolios, we cannot easily determine the total amount of information pro-

Chart 3

Value-at-Risk Measures for a Single Portfolio over Time
Historical Simulation Approaches
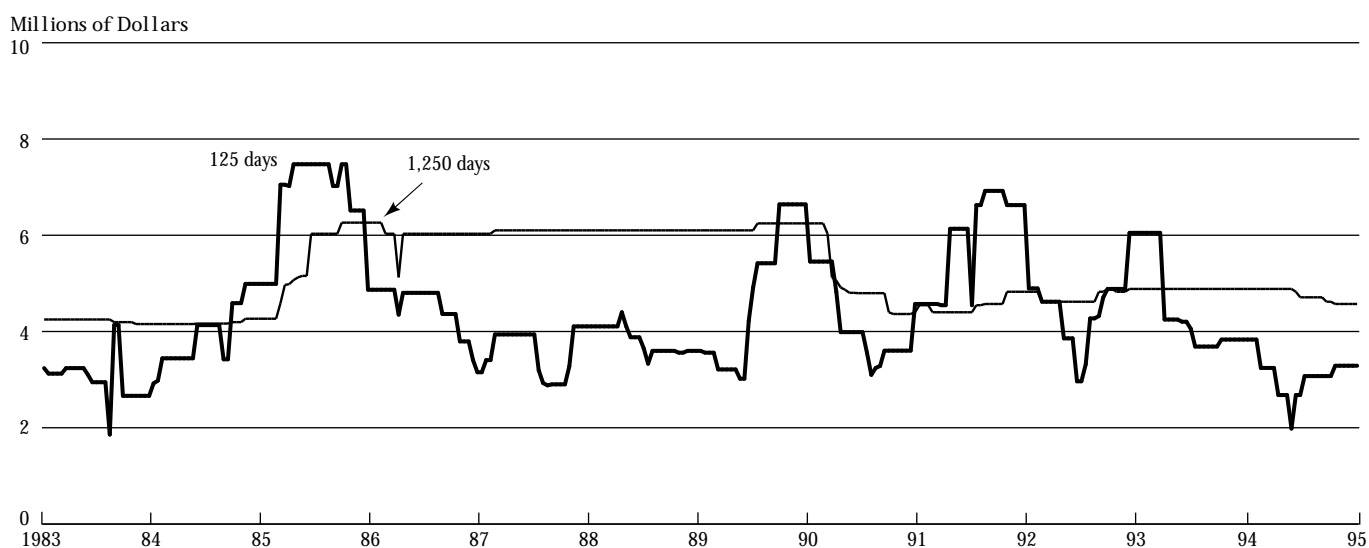
Source: Author's calculations.

DATA AND SIMULATION METHODOLOGY

This article analyzes twelve value-at-risk approaches. These include five equally weighted moving average approaches (50 days, 125 days, 250 days, 500 days, 1,250 days); three exponentially weighted moving average approaches (λ=0.94, λ=0.97, λ=0.99); and four historical simulation approaches (125 days, 250 days, 500 days, 1,250 days).

The data consist of daily exchange rates (bid prices collected at 4:00 p.m. New York time by the Federal Reserve Bank of New York) against the U.S. dollar for the following eight currencies: British pound, Canadian dollar, Dutch guilder, French franc, German mark, Italian lira, Japanese yen, and Swiss franc. The historical sample covers the period January 1, 1978, to January 18, 1995 (4,255 days).

Through a simulation methodology, we attempt to determine how each value-at-risk approach would have performed over a realistic range of portfolios containing the eight currencies over the sample period. The simulation methodology consists of five steps:

1. Select a random portfolio of positions in the eight currencies. This step is accomplished by drawing the position in each currency from a uniform distribution centered on zero. In other words, the portfolio space is a uniformly distributed eight dimensional cube centered on zero.[1]

2. Calculate the value-at-risk estimates for the random portfolio chosen in step one using the twelve value-at-risk approaches for each day in the sample—day 1,251 to day 4,255. In each case, we draw the historical data from the 1,250 days of historical data preceding the date for which the calculation is made. For example, the fifty-day equally weighted moving average estimate for a given date would be based on the fifty days of historical data preceding the given date.

3. Calculate the change in the portfolio's value for each day in the sample—again, day 1,251 to day 4,255. Within the article, these values are referred to as the ex post portfolio results or outcomes.

4. Assess the performance of each value-at-risk approach for the random portfolio selected in step one by comparing the value-at-risk estimates generated by step two with the actual outcomes calculated in step three.

5. Repeat steps one through four 1,000 times and tabulate the results.

[1] The upper and lower bounds on the positions in each currency are +100 million U.S. dollars and -100 million U.S. dollars, respectively. In fact, however, all of the results in the article are completely invariant to the scale of the random portfolios.

vided by the simulations. Furthermore, many of the performance criteria we consider do not have straightforward standard error formulas even for single samples.[15]

These stipulations imply that it is not possible to use the simulation results to accept or reject specific statistical hypotheses about these twelve value-at-risk approaches. Moreover, the results should not in any way be taken as indicative of the results that would be obtained for portfolios including other financial market assets, spanning other time periods, or looking forward. Finally, this article does not contribute substantially to the ongoing debate about the appropriate approach to or interpretation of "backtesting" in conjunction with value-at-risk modeling.[16] Despite these limitations, the simulation results do provide a relatively complete picture of the performance of selected value-at-risk approaches in estimating the market

risk of a large number of linear foreign exchange portfolios over the period 1983-94.

For each of the nine performance criteria, Charts 4-12 provide a visual sense of the simulation results for 95th and 99th percentile risk measures. In each chart, the vertical axis depicts a relevant range of the performance criterion under consideration (value-at-risk approaches are arrayed horizontally across the chart). Filled circles depict the average results across the 1,000 portfolios, and the boxes drawn for each value-at-risk approach depict the 5th, 25th, 50th, 75th, and 95th percentiles of the distribution of the results across the 1,000 portfolios.[17] In some charts, a horizontal line is drawn to highlight how the results compare with an important point of reference. Simulation results are also presented in tabular form in the appendix.

Mean Relative Bias

The first performance criterion we examine is whether the different value-at-risk approaches produce risk measures of similar average size. To ensure that the comparison is not influenced by the scale of each simulated portfolio, we use a four-step procedure to generate scale-free measures of the relative sizes for each simulated portfolio.

First, we calculate value-at-risk measures for each of the twelve approaches for the portfolio on each sample date. Second, we average the twelve risk measures for each date to obtain the average risk measure for that date for the portfolio. Third, we calculate the percentage difference between each approach's risk measure and the average risk measure for each date. We refer to these figures as daily relative bias figures because they are relative only to the average risk measure across the twelve approaches rather than to any external standard. Fourth, we average the daily relative biases for a given value-at-risk approach across all sample dates to obtain the approach's mean relative bias for the portfolio.

Intuitively, this procedure results in a measure of size for each value-at-risk approach that is relative to the average of all twelve approaches. The mean relative bias for

a portfolio is independent of the scale of the simulated portfolio because each of the daily relative bias calculations on which it is based is also scale-independent. This independence is achieved because all of the value-at-risk approaches we examine here are proportional to the scale of the portfolio's positions. For example, a doubling of the


Actual 99th percentiles for the foreign exchange portfolios considered in this article tend to be larger than the normal distribution would predict.

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scale of the portfolio would result in a doubling of the value-at-risk measures for each of the twelve approaches.

Mean relative bias is measured in percentage terms, so that a value of 0.10 implies that a given value-at-risk approach is 10 percent larger, on average, than the average of all twelve approaches. The simulation results suggest that differences in the average size of 95th percen-

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Chart 4a

Mean Relative Bias
95th Percentile Value-at-Risk Measures
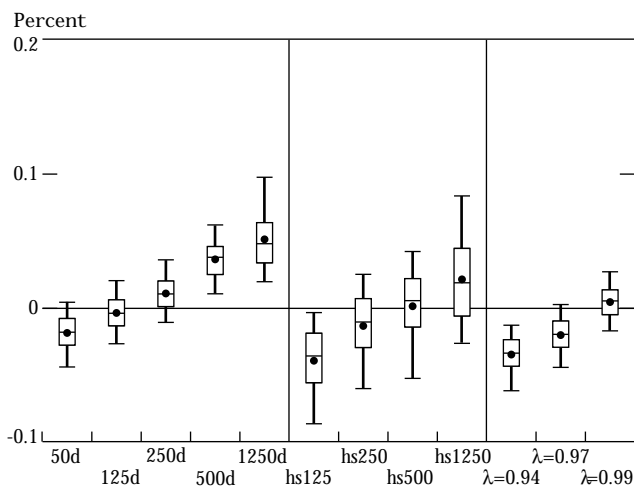
Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Chart 4b

Mean Relative Bias
99th Percentile Value-at-Risk Measures

Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Root Mean Squared Relative Bias
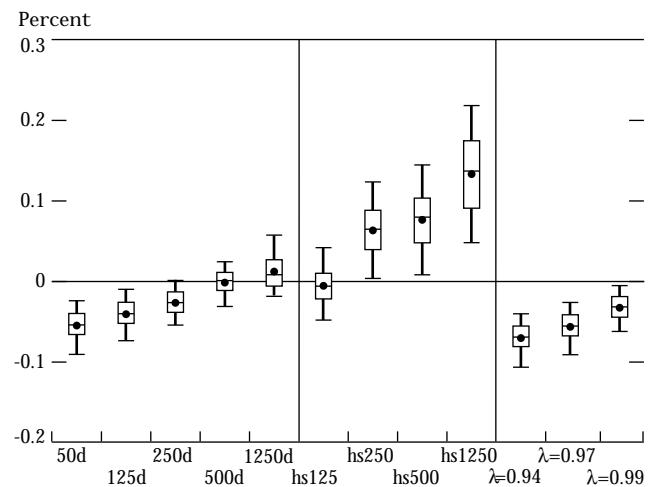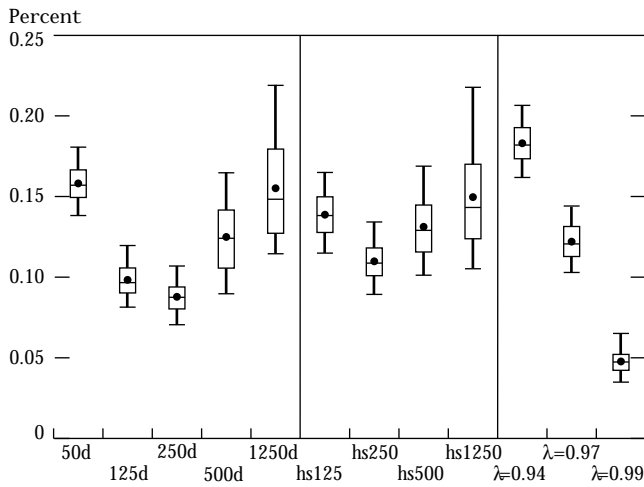95th Percentile Value-at-Risk Measures

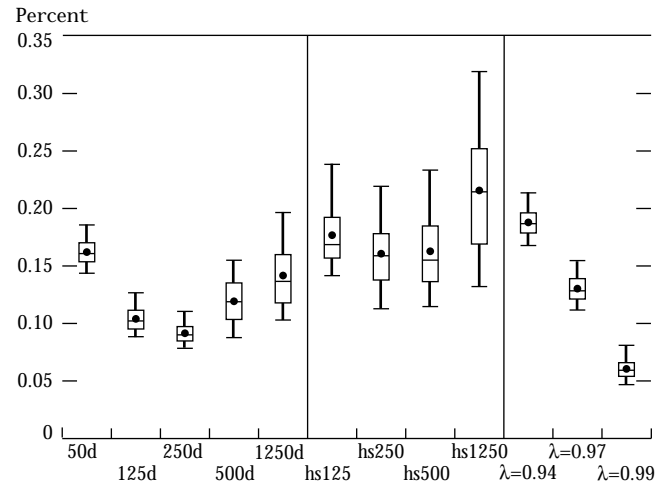Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Root Mean Squared Relative Bias
99th Percentile Value-at-Risk Measures

Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

tile value-at-risk measures are small. For the vast majority of the 1,000 portfolios, the mean relative biases for the 95th percentile risk measures are between -0.10 and 0.10 (Chart 4a). The averages of the mean relative biases across the 1,000 portfolios are even smaller, indicating that across approaches little systematic difference in size exists for 95th percentile value-at-risk measures.

For the 99th percentile value-at-risk measures, however, the results suggest that historical simulation approaches tend to produce systematically larger risk measures. In particular, Chart 4b shows that the 1,250-day historical simulation approach is, on average, approximately 13 percent larger than the average of all twelve approaches; for almost all of the portfolios, this approach is more than 5 percent larger than the average risk measure.

Together, the results for the 95th and 99th percentiles suggest that the normality assumption made by all of the approaches, except the historical simulations, is more reasonable for the 95th percentile than for the 99th percentile. In other words, actual 99th percentiles for the foreign exchange portfolios considered in this article tend to be larger than the normal distribution would predict.

Interestingly, the results in Charts 4a and 4b also

suggest that the use of longer time periods may produce larger value-at-risk measures. For historical simulation approaches, this result may occur because longer horizons provide better estimates of the tail of the distribution. The equally weighted approaches, however, may require a different explanation. Nevertheless, in our simulations the time period effect is small, suggesting that its economic significance is probably low.[18]

ROOT MEAN SQUARED RELATIVE BIAS

The second performance criterion we examine is the degree to which the risk measures tend to vary around the average risk measure for a given date. This criterion can be compared to a standard deviation calculation; here the deviations are the risk measure's percentage of deviation from the average across all twelve approaches. The root mean squared relative bias for each value-at-risk approach is calculated by taking the square root of the mean (over all sample dates) of the squares of the daily relative biases.

The results indicate that for any given date, a dispersion in the risk measures produced by the different value-at-risk approaches is likely to occur. The average root mean squared relative biases, across portfolios, tend to fall

largely in the 10 to 15 percent range, with the 99th percentile risk measures tending toward the higher end (Charts 5a and 5b). This level of variability suggests that, in spite of similar average sizes across the different value-at-risk approaches, differences in the range of 30 to 50 percent between the risk measures produced by specific approaches on a given day are not uncommon.

Surprisingly, the exponentially weighted average approach with a decay factor of 0.99 exhibits very low root mean squared bias, suggesting that this particular approach is very close to the average of all twelve approaches. Of course, this phenomenon is specific to the twelve approaches considered here and would not necessarily be true of exponentially weighted average approaches applied to other cases.

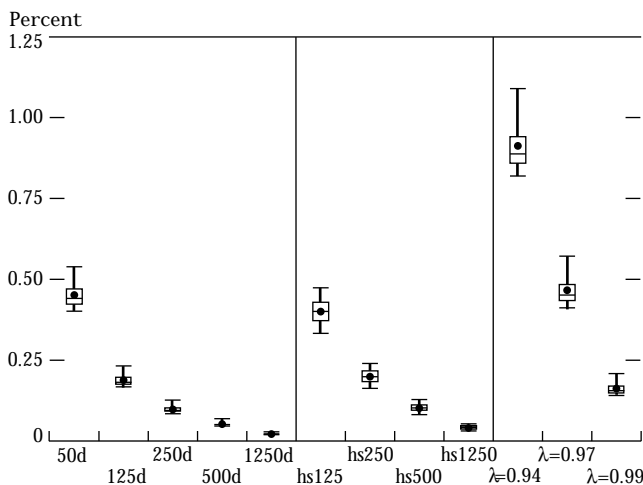### ANNUALIZED PERCENTAGE VOLATILITY

The third performance criterion we review is the tendency of the risk measures to fluctuate over time for the same portfolio. For each portfolio and each value-at-risk approach, we calculate the annualized percentage volatility by first taking the standard deviation of the day-to-day percentage changes in the risk measures over the sample

period. Second, we put the result on an annualized basis by multiplying this standard deviation by the square root of 250, the number of trading days in a typical calendar year. We complete the second step simply to make the results comparable with volatilities as they are often expressed in the marketplace. For example, individual foreign exchange rates tend to have annualized percentage volatilities in the range of 5 to 20 percent, although higher figures sometimes occur. This result implies that the value-at-risk approaches with annualized percentage volatilities in excess of 20 percent (Charts 6a and 6b) will fluctuate more over time (for the same portfolio) than will most exchange rates themselves.

Our major observation for this performance criterion is that the volatility of risk measures increases as reliance on recent data increases. As shown in Charts 6a and 6b, this increase is true for both the 95th and 99th percentile risk measures and for all three categories of value-at-risk approaches. This result is not surprising, and indeed it is clearly apparent in Charts 1-3, which depict time series of different value-at-risk approaches over the sample period. Also worth noting in Charts 6a and 6b is that for a fixed length of observation period, historical sim-

Chart 6a

Annualized Percentage Volatility
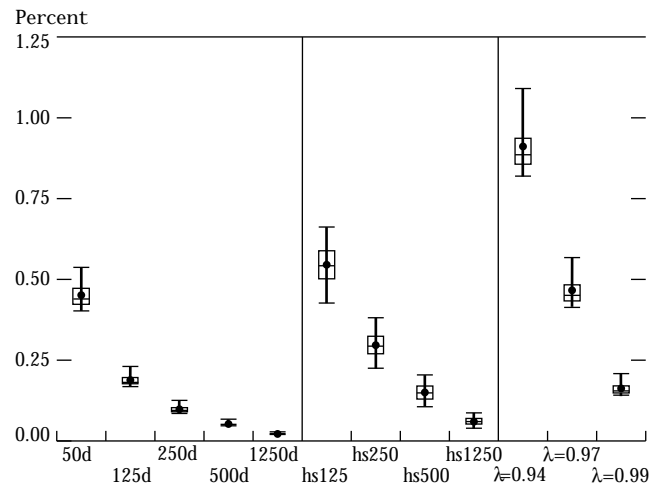95th Percentile Value-at-Risk Measures

Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Chart 6b

Annualized Percentage Volatility
99th Percentile Value-at-Risk Measures

Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

ulation approaches appear to be more variable than the corresponding equally weighted moving average approaches.

FRACTION OF OUTCOMES COVERED

Our fourth performance criterion addresses the fundamental goal of the value-at-risk measures—whether they cover the portfolio outcomes they are intended to capture. We calculate the fraction of outcomes covered as the percentage of results where the loss in portfolio value is less than the risk measure.

For the 95th percentile risk measures, the simulation results indicate that nearly all twelve value-at-risk approaches meet this performance criterion (Chart 7a). For many portfolios, coverage exceeds 95 percent, and only the 125-day historical simulation approach captures less than 94.5 percent of the outcomes on average across all 1,000 portfolios. In a very small fraction of the random portfolios, the risk measures cover less than 94 percent of the outcomes.

Interestingly, the 95th percentile results suggest that the equally weighted moving average approaches actually tend to produce excess coverage (greater than 95 percent) for all observation periods except fifty days. By contrast, the historical simulation approaches tend to pro-

vide either too little coverage or, in the case of the 1,250-day historical simulation approach, a little more than the desired amount. The exponentially weighted moving average approach with a decay factor of 0.97 produces exact 95 percent coverage, but for this approach the results

All twelve value-at-risk approaches either achieve the desired level of coverage or come very close to it on the basis of the percentage of outcomes misclassified.

are more variable across portfolios than for the 1,250-day historical simulation approach.

Compared with the 95th percentile results, the 99th percentile risk measures exhibit a more widespread tendency to fall short of the desired level of risk coverage. Only the 1,250-day historical simulation approach attains 99 percent coverage across all 1,000 portfolios, as shown in Chart 7b. The other approaches cover between 98.2 and

Chart 7a

Fraction of Outcomes Covered
95th Percentile Value-at-Risk Measures
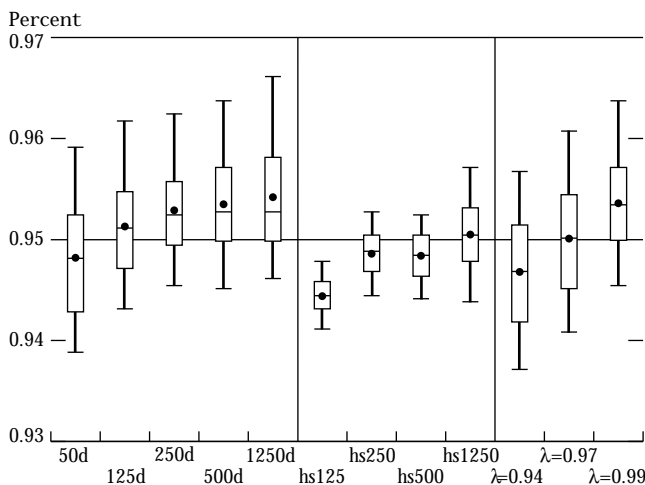
Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Chart 7b

Fraction of Outcomes Covered
99th Percentile Value-at-Risk Measures

Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

98.8 percent of the outcomes on average across portfolios. Of course, the consequences of such a shortfall in performance depend on the particular circumstances in which the value-at-risk model is being used. A coverage level of 98.2 percent when a risk manager desires 99 percent implies that the value-at-risk model misclassifies approximately two outcomes every year (assuming that there are 250 trading days per calendar year).

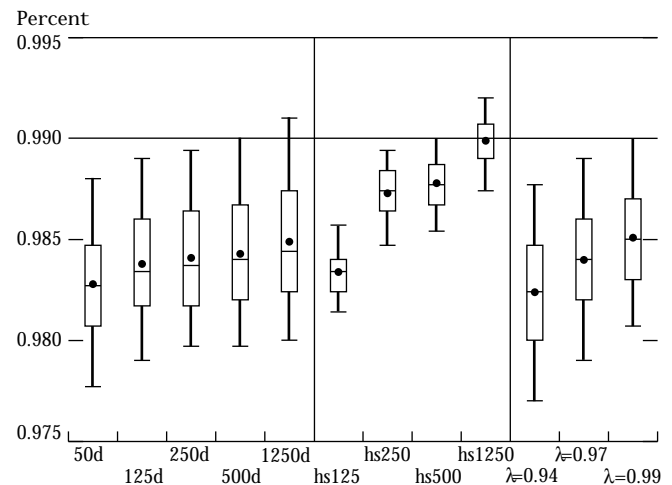Overall, the results in Charts 7a and 7b support the conclusion that all twelve value-at-risk approaches either achieve the desired level of coverage or come very close to it on the basis of the percentage of outcomes misclassified. Clearly, the best performer is the 1,250-day historical simulation approach, which attains almost exact coverage for both the 95th and 99th percentiles, while the worst performer is the 125-day historical simulation approach, partly because of its short-term construction.[19] One explanation for the superior performance of the 1,250-day historical simulation is that the unconditional distribution of changes in portfolio value is relatively stable and that accurate estimates of extreme percentiles require the use of long periods. These results underscore the problems associated with the assumption of normality for 99th percentiles and are consistent with findings in other recent studies of value-at-risk models.[20]

MULTIPLE NEEDED TO ATTAIN DESIRED COVERAGE

The fifth performance criterion we examine focuses on the size of the adjustments in the risk measures that would be needed to achieve perfect coverage. We therefore calculate on an ex post basis the multiple that would have been required for each value-at-risk measure to attain the desired level of coverage (either 95 percent or 99 percent). This performance criterion complements the fraction of outcomes covered because it focuses on the size of the potential errors in risk measurement rather than on the percentage of results captured.

For 95th percentile risk measures, the simulation results indicate that multiples very close to one are sufficient (Chart 8a). Even the 125-day historical simulation approach, which on average across portfolios is furthest

from the desired outcome, requires a multiple of only 1.04. On the whole, none of the approaches considered here appears to understate 95th percentile risk measures on a systematic basis by more than 4 percent, and several appear to overstate them by small amounts.

For the 99th percentile risk measures, most value-at-risk approaches require multiples between 1.10 and 1.15 to attain 99 percent coverage (Chart 8b). The 1,250-day historical simulation approach, however, is markedly superior to all other approaches. On average across all port-

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Shortcomings in value-at-risk measures that seem small in probability terms may be much more significant when considered in terms of the changes required to remedy them.

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folios, no multiple other than one is needed for this approach to achieve 99 percent coverage. Moreover, compared with the other approaches, the historical simulations in general exhibit less variability across portfolios with respect to this criterion.

The fact that most multiples are larger than one is not surprising. More significant is the fact that the size of the multiples needed to achieve 99 percent coverage exceeds the levels indicated by the normal distribution. For example, when normality is assumed, the 99th percentile would be about 1.08 times as large as the 98.4th percentile, a level of coverage comparable to that attained by many of the approaches (Chart 7b). The multiples for these approaches, shown in Chart 8b, are larger than 1.08, providing further evidence that the normal distribution does not accurately approximate actual distributions at points near the 99th percentile. More generally, the results also suggest that substantial increases in value-at-risk measures may be needed to capture outcomes in the tail of the distribution. Hence, shortcomings in value-at-risk measures that seem small in probability terms may be much more significant when considered in terms of the changes required to remedy them.

These results lead to an important question: what distributional assumptions other than normality can be used when constructing value-at-risk measures using a variance-covariance approach? The t-distribution is often cited as a good candidate, because extreme outcomes occur more often under t-distributions than under the normal distribution.[21] A brief analysis shows that the use of a t-distribution for the 99th percentile has some merit.

To calculate a value-at-risk measure for a single percentile assuming the t-distribution, the value-at-risk measure calculated with the assumption of normality is multiplied by a fixed multiple. As the results in Chart 8b suggest, fixed multiples between 1.10 and 1.15 are appropriate for the variance-covariance approaches. It follows that t-distributions with between four and six degrees of freedom are appropriate for the 99th percentile risk measures.[22] The use of these particular t-distributions, however, would lead to substantial overestimation of 95th percentile risk measures because the actual distributions near the 95th percentile are much closer to normality. Since the use of t-distributions for risk measurement involves a scaling up of the risk measures that are calculated assuming normality, the distributions are likely to be useful, although they may be more helpful for some percentiles than for others.

AVERAGE MULTIPLE OF TAIL EVENT TO RISK MEASURE

The sixth performance criterion that we review relates to the size of outcomes not covered by the risk measures.[23] To address these outcomes, we measure the degree to which events in the tail of the distribution typically exceed the value-at-risk measure by calculating the average multiple of these outcomes ("tail events") to their corresponding value-at-risk measures.

Tail events are defined as the largest percentage of losses measured relative to the respective value-at-risk estimate—the largest 5 percent in the case of 95th percentile risk measures and the largest 1 percent in the case of 99th percentile risk measures. For example, if the value-at-risk measure is $1.5 million and the actual portfolio outcome is a loss of $3 million, the size of the loss relative to the risk measure would be two. Note that this definition implies that the tail events for one value-at-risk approach may not be the same as those for another approach, even for the same portfolio, because the risk

Chart 8a

Multiple Needed to Attain 95 Percent Coverage
95th Percentile Value-at-Risk Measures
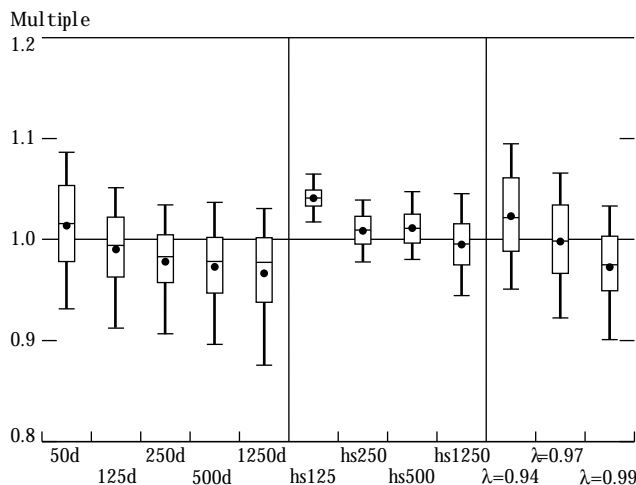
Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Chart 8b

Multiple Needed to Attain 99 Percent Coverage
99th Percentile Value-at-Risk Measures

Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

measures for the two approaches are not the same. Horizontal reference lines in Charts 9a and 9b show where the average multiples of the tail event outcomes to the risk measures would fall if outcomes were normally distributed and the value-at-risk approach produced a true 99th percentile level of coverage.

In fact, however, the average tail event is almost always a larger multiple of the risk measure than is predicted by the normal distribution. For most of the value-at-risk approaches, the average tail e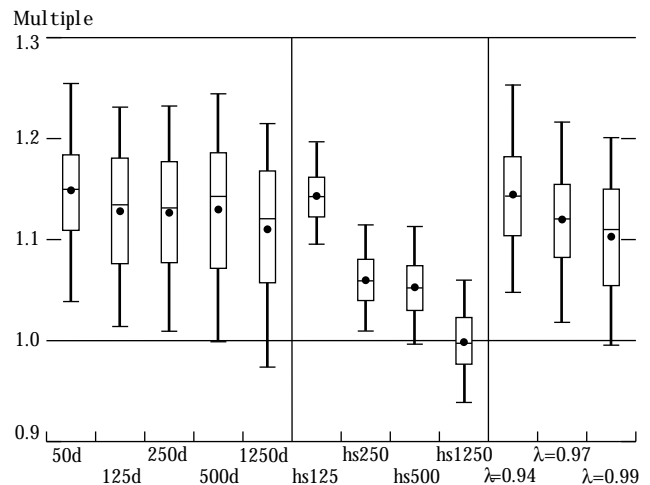vent is 30 to 40 percent larger than the respective risk measures for both the 95th percentile risk measures and the 99th percentile risk measures. This result means that approximately 1 percent of outcomes (the largest two or three losses per year) will exceed the size of the 99th percentile risk measure by an average of 30 to 40 percent. In addition, note that the 99th percentile results in Chart 9b are more variable across portfolios than the 95th percentile results in Chart 9a; the average multiple is also above 1.50 for a greater percentage of the portfolios for the 99th percentile risk measures.

The performance of the different approaches according to this criterion largely mirrors their performance in capturing portfolio outcomes. For example, the 1,250-day historical simulation approach is clearly supe-

rior for the 99th percentile risk measures. The equally weighted moving average approaches also do very well for the 95th percentile risk measures (Chart 7a).

MAXIMUM MULTIPLE OF TAIL EVENT TO RISK MEASURE

Our seventh performance criterion concerns the size of the maximum portfolio loss. We use the following two-step procedure to arrive at these measures. First, we calculate the multiples of all portfolio outcomes to their respective risk measures for each value-at-risk approach for a particular portfolio. Recall that the tail events defined above are those outcomes with the largest such multiples. Rather than average these multiples, however, we simply select the single largest multiple for each approach. This procedure implies that the maximum multiple will be highly dependent on the length of the sample period—in this case, approximately twelve years. For shorter periods, the maximum multiple would likely be lower.

Not surprisingly, the typical maximum tail event is substantially larger than the corresponding risk measure (Charts 10a and 10b). For 95th percentile risk measures, the maximum multiple is three to four times as large as the risk measure, and for the 99th percentile risk measure, it is

Chart 9a

Average Multiple of Tail Event to Risk Measure
95th Percentile Value-at-Risk Measures
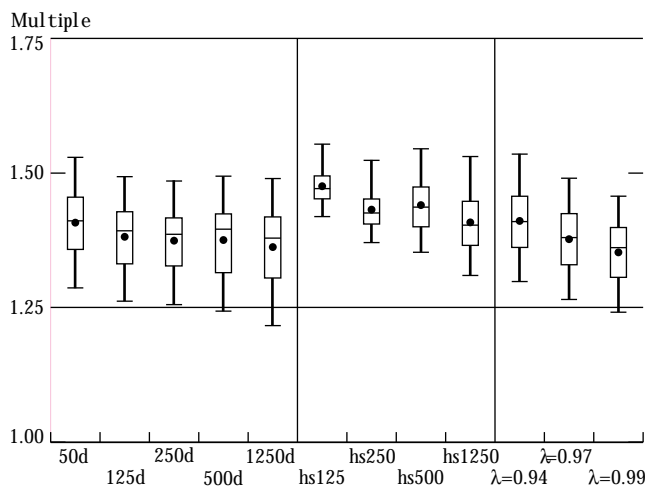
Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Chart 9b

Average Multiple of Tail Event to Risk Measure
99th Percentile Value-at-Risk Measures

Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

approximately 2.5 times as large. In addition, the results are variable across portfolios—for some portfolios, the maximum multiples are more than five times the 95th percentile risk measure. The differences among results for this performance criterion, however, are less pronounced than

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It is important not to view value-at-risk measures as a strict upper bound on the portfolio losses that can occur.

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for some other criteria. For example, the 1,250-day historical simulation approach is not clearly superior for the 99th percentile risk measure—as it had been for many of the other performance criteria—although it does exhibit lower average multiples (Chart 9b).

These results suggest that it is important not to view value-at-risk measures as a strict upper bound on the portfolio losses that can occur. Although a 99th percentile risk measure may sound as if it is capturing essentially all of the relevant events, our results make it clear that the other

1 percent of events can in extreme cases entail losses substantially in excess of the risk measures generated on a daily basis.

CORRELATION BETWEEN RISK MEASURE AND ABSOLUTE VALUE OF OUTCOME

The eighth performance criterion assesses how well the risk measures adjust over time to underlying changes in risk. In other words, how closely do changes in the value-at-risk measures correspond to actual changes in the risk of the portfolio? We answer this question by determining the correlation between the value-at-risk measures for each approach and the absolute values of the outcomes. This correlation statistic has two advantages. First, it is not affected by the scale of the portfolio. Second, the correlations are relatively easy to interpret, although even a perfect value-at-risk measure cannot guarantee a correlation of one between the risk measure and the absolute value of the outcome.

For this criterion, the results for the 95th percentile risk measures and 99th percentile risk measures are almost identical (Charts 11a and 11b). Most striking is the superior performance of the exponentially weighted moving average measures. This finding implies that these approaches tend to track changes in risk over time more accurately than the other approaches.
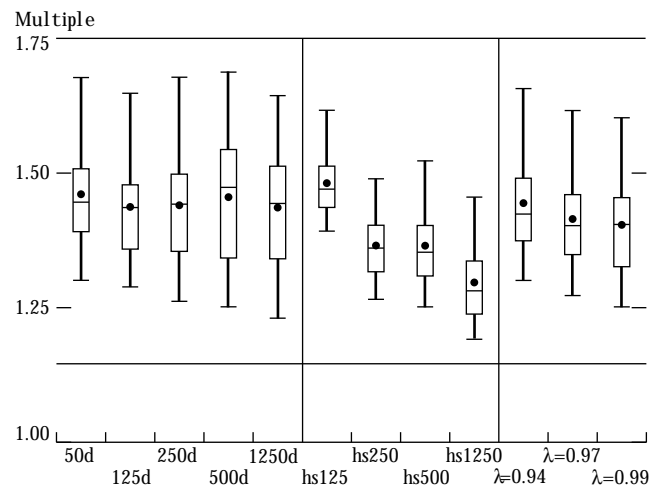
Chart 10a

Maximum Multiple of Tail Event to Risk Measure
95th Percentile Value-at-Risk Measures
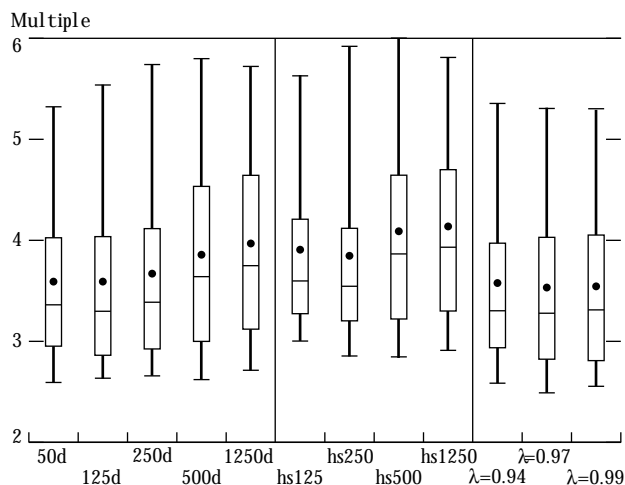
Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Chart 10b

Maximum Multiple of Tail Event to Risk Measure
99th Percentile Value-at-Risk Measures

Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Correlation between Risk Measure and Absolute Value of Outcome
95th Percentile Value-at-Risk Measures

Source: Author's calculations.
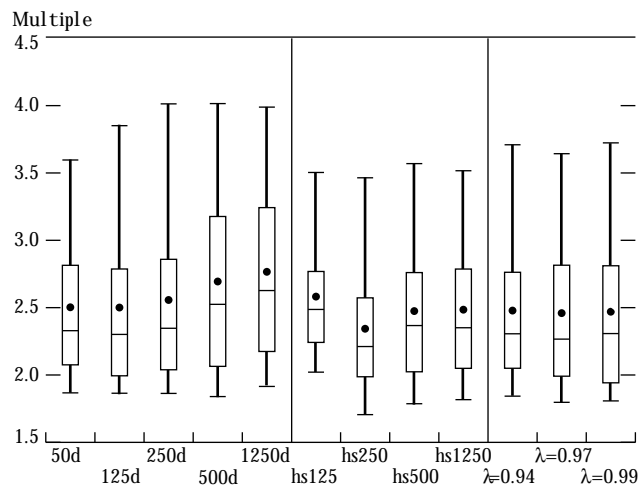
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Chart 11b

Correlation between Risk Measure and Absolute Value of Outcome
99th Percentile Value-at-Risk Measures

Source: Author's calculations.

Notes: d=days; hs=historical simulation; λ=exponentially weighted.

In contrast to the results for mean relative bias (Charts 4a and 4b) and the fraction of outcomes covered (Charts 7a and 7b), the results for this performance criterion show that the length of the observation period is inversely related to performance. Thus, shorter observation periods tend to lead to higher measures of correlation between the absolute values of the outcomes and the value-at-risk measures. This inverse relationship supports the view that, because market behavior changes over time, emphasis on recent information can be helpful in tracking changes in risk.

At the other extreme, the risk measures for the 1,250-day historical simulation approach are essentially uncorrelated with the absolute values of the outcomes. Although superior according to other performance criteria, the 1,250-day results here indicate that this approach reveals little about actual changes in portfolio risk over time.

MEAN RELATIVE BIAS FOR RISK MEASURES SCALED TO DESIRED LEVEL OF COVERAGE

The last performance criterion we examine is the mean relative bias that results when risk measures are scaled to either 95 percent or 99 percent coverage. Such scaling is accomplished on an ex post basis by multiplying the risk measures for each approach by the multiples needed to attain either exactly 95 percent or exactly 99 percent coverage (Charts 8a and 8b). These scaled risk measures provide

Because market behavior changes over time, emphasis on recent information can be helpful in tracking changes in risk.

the precise amount of coverage desired for each portfolio. Of course, the scaling for each value-at-risk approach would not be the same for different portfolios.

Once we have arrived at the scaled value-at-risk measures, we compare their relative average sizes by using the mean relative bias calculation, which compares the average size of the risk measures for each approach to the average size across all twelve approaches (Charts 4a and 4b). In this case, however, the value-at-risk measures have been scaled to the desired levels of coverage. The purpose of this criterion is to determine which approach, once suit-

ably scaled, could provide the desired level of coverage with the smallest average risk measures. This performance criterion also addresses the issue of tracking changes in portfolio risk—the most efficient approach will be the one that tracks changes in risk best. In contrast to the correlation statistic discussed in the previous section, however, this criterion focuses specifically on the 95th and 99th percentiles.

Once again, the exponentially weighted moving average approaches appear superior (Charts 12a and 12b). In particular, the exponentially weighted average approach with a decay factor of 0.97 appears to perform extremely well for both 95th and 99th percentile risk measures. Indeed, for the 99th percentile, it achieves exact 99 percent coverage with an average size that is 4 percent smaller than the average of all twelve scaled value-at-risk approaches.

The performance of the other approaches is similar to that observed for the correlation statistic (Charts 11a and 11b), but in this case the relationship between efficiency and the length of the observation period is not as pronounced. In particular, the 50-day equally weighted approach is somewhat inferior to the 250-day equally weighted approach—a finding contrary to what is observed

in Charts 11a and 11b—and may reflect the greater influence of measurement error on short observation periods along this performance criterion.

At least two caveats apply to these results. First, they would be difficult to duplicate in practice because the scaling must be done in advance of the outcomes rather than ex post. Second, the differences in the average sizes of the scaled risk measures are simply not very large. Nevertheless, the results suggest that exponentially weighted average approaches might be capable of providing desired levels of coverage in an efficient fashion, although they would need to be scaled up.

CONCLUSIONS

A historical examination of twelve approaches to value-at-risk modeling shows that in almost all cases the approaches cover the risk that they are intended to cover. In addition, the twelve approaches tend to produce risk estimates that do not differ greatly in average size, although historical simulation approaches yield somewhat larger 99th percentile risk measures than the variance-covariance approaches.

Despite the similarity in the average size of the risk estimates, our investigation reveals differences, some-

Chart 12a

Mean Relative Bias for Risk Measures Scaled to Cover Exactly 95 Percent
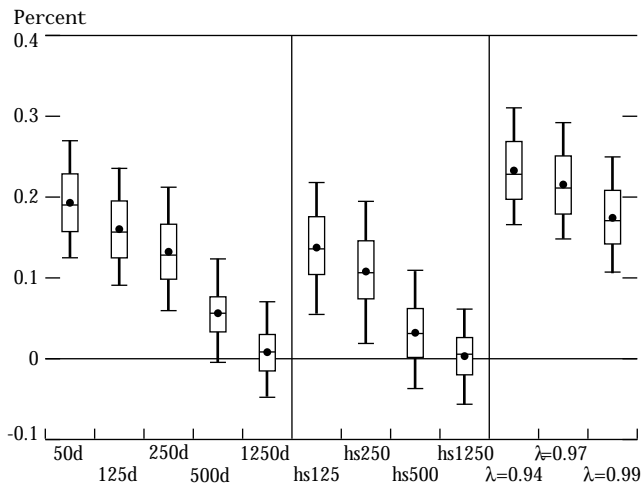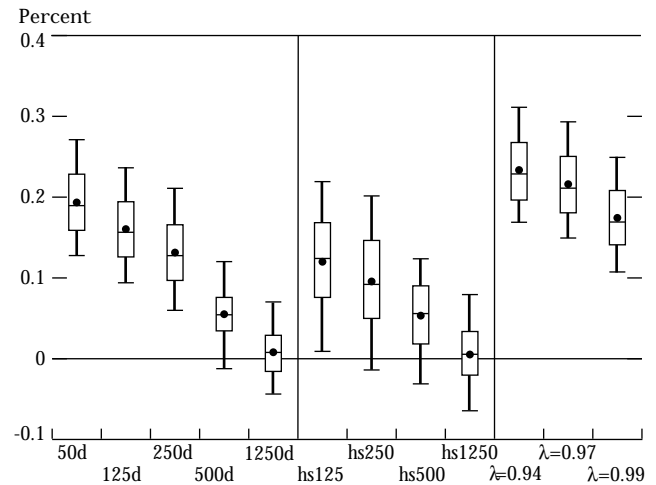95th Percentile Value-at-Risk Measures
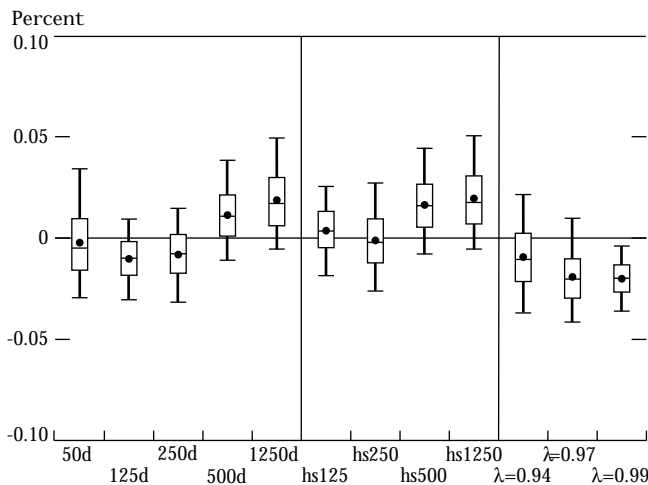
Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

Chart 12b

Mean Relative Bias for Risk Measures Scaled to Cover Exactly 99 Percent
99th Percentile Value-at-Risk Measures
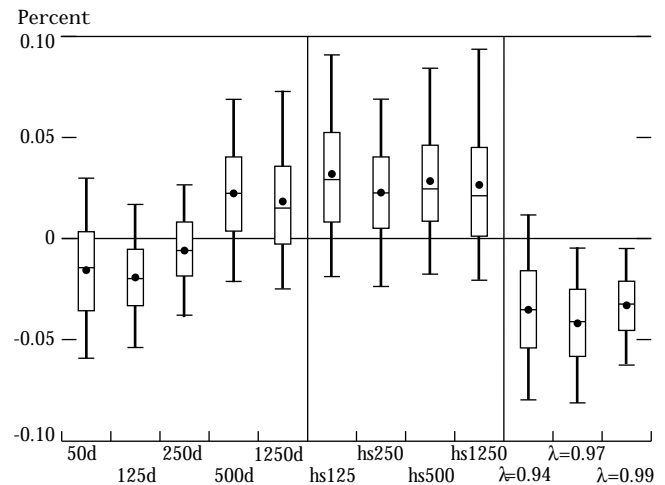
Source: Author's calculations.
Notes: d=days; hs=historical simulation; λ=exponentially weighted.

times substantial, among the various value-at-risk approaches for the same portfolio on the same date. In terms of variability over time, the value-at-risk approaches using longer observation periods tend to produce less variable results than those using short observation periods or weighting recent observations more heavily.

Virtually all of the approaches produce accurate 95th percentile risk measures. The 99th percentile risk measures, however, are somewhat less reliable and generally cover only between 98.2 percent and 98.5 percent of the outcomes. On the one hand, these deficiencies are small when considered on the basis of the percentage of outcomes misclassified. On the other hand, the risk measures would generally need to be increased across the board by 10 percent or more to cover precisely 99 percent of the outcomes. Interestingly, one exception is the 1,250-day historical simulation approach, which provides very accurate coverage for both 95th and 99th percentile risk measures.

The outcomes that are *not* covered are typically 30 to 40 percent larger than the risk measures and are also larger than predicted by the normal distribution. In some cases, daily losses over the twelve-year sample period are several times larger than the corresponding value-at-risk measures. These examples make it clear that value-at-risk measures—even at the 99th percentile—do not "bound" possible losses.

Also clear is the difficulty of anticipating or tracking changes in risk over time. For this performance criterion, the exponentially weighted moving average approaches appear to be superior. If it were possible to scale all approaches ex post to achieve the desired level of coverage over the sample period, these approaches would produce the smallest scaled risk measures.

What more general conclusions can be drawn from these results? In many respects, the simulation estimates clearly reflect two well-known characteristics of daily financial market data. First, extreme outcomes occur more often and are larger than predicted by the normal distribution (fat tails). Second, the size of market movements is not constant over time (conditional volatility). Clearly, constructing value-at-risk models that perform well by every measure is a difficult task. Thus, although we cannot recommend any single value-at-risk approach, our results suggest that further research aimed at combining the best features of the approaches examined here may be worthwhile.

APPENDIX: VALUE-AT-RISK SIMULATION RESULTS FOR EACH PERFORMANCE CRITERION

The nine tables below summarize for each performance criterion the simulation results for the 95th and 99th percentile risk measures. The value-at-risk approaches appear at the extreme left of each table. The first column reports the average simulation result of each approach across the 1,000 portfolios for the particular performance criterion. The next column reports the standard deviation of the results across the 1,000 portfolios, a calculation that provides information on the variability of the results across portfolios. To indicate the variability of results over time, the remaining four columns report results averaged over the 1,000 portfolios for four subsets of the sample period.

Table A1
MEAN RELATIVE BIAS

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	-0.02	0.01	-0.00	-0.05	0.01	-0.03
125-day equally weighted	-0.00	0.01	-0.00	-0.02	0.01	-0.00
250-day equally weighted	0.01	0.01	-0.01	0.03	0.00	0.03
500-day equally weighted	0.04	0.02	0.01	0.08	-0.01	0.07
1,250-day equally weighted	0.05	0.03	0.08	0.06	0.05	0.01
125-day historical simulation	-0.04	0.03	-0.04	-0.06	-0.03	-0.04
250-day historical simulation	-0.01	0.03	-0.03	0.00	-0.02	0.00
500-day historical simulation	0.00	0.03	-0.02	0.05	-0.05	0.03
1,250-day historical simulation	0.02	0.03	0.05	0.03	0.02	-0.02
Exponentially weighted ($\lambda = 0.94$)	-0.03	0.01	-0.02	-0.07	-0.01	-0.04
Exponentially weighted ($\lambda = 0.97$)	-0.02	0.01	-0.01	-0.05	0.00	-0.02
Exponentially weighted ($\lambda = 0.99$)	0.00	0.01	-0.00	0.00	0.01	0.01
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	-0.05	0.02	-0.03	-0.09	-0.03	-0.06
125-day equally weighted	-0.04	0.02	-0.03	-0.06	-0.03	-0.04
250-day equally weighted	-0.03	0.02	-0.04	-0.01	-0.04	-0.01
500-day equally weighted	-0.00	0.02	-0.02	0.04	-0.05	0.03
1,250-day equally weighted	0.01	0.03	0.04	0.02	0.01	-0.02
125-day historical simulation	-0.01	0.03	-0.03	-0.00	0.01	-0.00
250-day historical simulation	0.06	0.04	0.02	0.08	0.07	0.08
500-day historical simulation	0.08	0.04	0.04	0.11	0.05	0.11
1,250-day historical simulation	0.13	0.05	0.18	0.13	0.13	0.09
Exponentially weighted ($\lambda = 0.94$)	-0.07	0.02	-0.05	-0.10	-0.05	-0.08
Exponentially weighted ($\lambda = 0.97$)	-0.06	0.02	-0.04	-0.08	-0.04	-0.06
Exponentially weighted ($\lambda = 0.99$)	-0.03	0.02	-0.03	-0.04	-0.04	-0.03

Table A2
ROOT MEAN SQUARED RELATIVE BIAS

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	0.16	0.01	0.17	0.15	0.14	0.16
125-day equally weighted	0.10	0.01	0.10	0.10	0.08	0.11
250-day equally weighted	0.09	0.01	0.08	0.09	0.08	0.09
500-day equally weighted	0.13	0.02	0.13	0.13	0.08	0.13
1,250-day equally weighted	0.16	0.04	0.18	0.14	0.14	0.14
125-day historical simulation	0.14	0.02	0.15	0.13	0.13	0.14
250-day historical simulation	0.11	0.01	0.12	0.11	0.10	0.11
500-day historical simulation	0.13	0.02	0.14	0.13	0.10	0.14
1,250-day historical simulation	0.15	0.03	0.17	0.13	0.13	0.15
Exponentially weighted (λ=0.94)	0.18	0.01	0.20	0.17	0.17	0.19
Exponentially weighted (λ=0.97)	0.12	0.01	0.13	0.11	0.10	0.13
Exponentially weighted (λ=0.99)	0.05	0.01	0.05	0.04	0.04	0.05
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	0.16	0.01	0.17	0.16	0.14	0.16
125-day equally weighted	0.10	0.01	0.11	0.11	0.08	0.11
250-day equally weighted	0.09	0.01	0.09	0.09	0.09	0.09
500-day equally weighted	0.12	0.02	0.13	0.12	0.10	0.12
1,250-day equally weighted	0.14	0.03	0.16	0.13	0.13	0.14
125-day historical simulation	0.18	0.03	0.15	0.19	0.17	0.17
250-day historical simulation	0.16	0.03	0.14	0.15	0.16	0.16
500-day historical simulation	0.16	0.04	0.15	0.18	0.12	0.17
1,250-day historical simulation	0.22	0.06	0.24	0.20	0.19	0.19
Exponentially weighted (λ=0.94)	0.19	0.01	0.20	0.19	0.17	0.19
Exponentially weighted (λ=0.97)	0.13	0.01	0.14	0.13	0.11	0.13
Exponentially weighted (λ=0.99)	0.06	0.01	0.06	0.06	0.05	0.06

Table A3
ANNUALIZED PERCENTAGE VOLATILITY

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	0.45	0.05	0.49	0.42	0.44	0.45
125-day equally weighted	0.19	0.03	0.18	0.19	0.17	0.20
250-day equally weighted	0.10	0.02	0.10	0.09	0.09	0.11
500-day equally weighted	0.05	0.01	0.06	0.05	0.05	0.05
1,250-day equally weighted	0.02	0.00	0.03	0.02	0.02	0.02
125-day historical simulation	0.40	0.04	0.38	0.39	0.40	0.41
250-day historical simulation	0.20	0.02	0.20	0.19	0.19	0.21
500-day historical simulation	0.10	0.01	0.11	0.09	0.10	0.10
1,250-day historical simulation	0.04	0.01	0.04	0.04	0.04	0.04
Exponentially weighted (λ=0.94)	0.91	0.09	0.94	0.88	0.89	0.94
Exponentially weighted (λ=0.97)	0.47	0.06	0.49	0.43	0.44	0.49
Exponentially weighted (λ=0.99)	0.16	0.03	0.18	0.14	0.15	0.17
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	0.45	0.05	0.49	0.42	0.44	0.45
125-day equally weighted	0.19	0.03	0.18	0.19	0.17	0.20
250-day equally weighted	0.10	0.02	0.10	0.09	0.09	0.11
500-day equally weighted	0.05	0.01	0.06	0.05	0.05	0.05
1,250-day equally weighted	0.02	0.01	0.03	0.02	0.02	0.02
125-day historical simulation	0.55	0.07	0.49	0.55	0.51	0.57
250-day historical simulation	0.30	0.05	0.27	0.28	0.27	0.31
500-day historical simulation	0.15	0.03	0.16	0.13	0.14	0.15
1,250-day historical simulation	0.06	0.02	0.06	0.05	0.06	0.06
Exponentially weighted (λ=0.94)	0.91	0.10	0.94	0.88	0.88	0.94
Exponentially weighted (λ=0.97)	0.47	0.06	0.49	0.43	0.44	0.49
Exponentially weighted (λ=0.99)	0.16	0.03	0.18	0.14	0.15	0.17

Table A4
FRACTION OF OUTCOMES COVERED

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	0.948	0.006	0.948	0.947	0.949	0.948
125-day equally weighted	0.951	0.006	0.950	0.953	0.951	0.953
250-day equally weighted	0.953	0.005	0.946	0.960	0.950	0.956
500-day equally weighted	0.954	0.006	0.946	0.963	0.947	0.958
1,250-day equally weighted	0.954	0.006	0.954	0.959	0.954	0.950
125-day historical simulation	0.944	0.002	0.943	0.946	0.943	0.946
250-day historical simulation	0.949	0.003	0.943	0.955	0.945	0.952
500-day historical simulation	0.948	0.003	0.942	0.959	0.941	0.952
1,250-day historical simulation	0.951	0.004	0.951	0.956	0.951	0.945
Exponentially weighted (λ=0.94)	0.947	0.006	0.948	0.946	0.947	0.946
Exponentially weighted (λ=0.97)	0.950	0.006	0.950	0.950	0.950	0.950
Exponentially weighted (λ=0.99)	0.954	0.006	0.950	0.957	0.951	0.956
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	0.983	0.003	0.985	0.982	0.982	0.983
125-day equally weighted	0.984	0.003	0.984	0.984	0.982	0.984
250-day equally weighted	0.984	0.003	0.982	0.987	0.982	0.986
500-day equally weighted	0.984	0.003	0.981	0.989	0.981	0.987
1,250-day equally weighted	0.985	0.003	0.984	0.988	0.984	0.983
125-day historical simulation	0.983	0.001	0.983	0.985	0.982	0.984
250-day historical simulation	0.987	0.001	0.984	0.991	0.986	0.989
500-day historical simulation	0.988	0.001	0.985	0.991	0.986	0.990
1,250-day historical simulation	0.990	0.001	0.990	0.992	0.989	0.989
Exponentially weighted (λ=0.94)	0.982	0.003	0.984	0.981	0.982	0.983
Exponentially weighted (λ=0.97)	0.984	0.003	0.986	0.983	0.983	0.984
Exponentially weighted (λ=0.99)	0.985	0.003	0.985	0.986	0.983	0.986

Table A5

MULTIPLE NEEDED TO ATTAIN DESIRED COVERAGE LEVEL

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	1.01	0.05	1.01	1.02	1.01	1.02
125-day equally weighted	0.99	0.04	1.00	0.98	0.99	0.98
250-day equally weighted	0.98	0.04	1.02	0.93	1.00	0.95
500-day equally weighted	0.97	0.04	1.02	0.90	1.02	0.93
1,250-day equally weighted	0.97	0.05	0.95	0.93	0.97	1.00
125-day historical simulation	1.04	0.01	1.05	1.03	1.05	1.03
250-day historical simulation	1.01	0.02	1.05	0.96	1.03	0.98
500-day historical simulation	1.01	0.02	1.06	0.94	1.06	0.99
1,250-day historical simulation	1.00	0.03	0.98	0.95	0.99	1.04
Exponentially weighted (λ=0.94)	1.02	0.05	1.01	1.03	1.02	1.03
Exponentially weighted (λ=0.97)	1.00	0.04	0.99	1.00	1.00	1.00
Exponentially weighted (λ=0.99)	0.97	0.04	0.99	0.95	0.99	0.96
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	1.15	0.06	1.11	1.19	1.19	1.14
125-day equally weighted	1.13	0.07	1.12	1.11	1.17	1.13
250-day equally weighted	1.13	0.07	1.17	1.06	1.20	1.11
500-day equally weighted	1.13	0.08	1.22	1.03	1.20	1.10
1,250-day equally weighted	1.11	0.08	1.12	1.04	1.13	1.17
125-day historical simulation	1.14	0.03	1.15	1.13	1.18	1.16
250-day historical simulation	1.06	0.03	1.11	0.99	1.12	1.04
500-day historical simulation	1.05	0.03	1.13	0.98	1.10	1.02
1,250-day historical simulation	1.00	0.04	1.00	0.94	1.01	1.04
Exponentially weighted (λ=0.94)	1.14	0.06	1.12	1.19	1.14	1.16
Exponentially weighted (λ=0.97)	1.12	0.06	1.09	1.15	1.15	1.12
Exponentially weighted (λ=0.99)	1.10	0.06	1.11	1.08	1.17	1.09

Table A6
AVERAGE MULTIPLE OF TAIL EVENT TO RISK MEASURE

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	1.41	0.07	1.40	1.41	1.41	1.41
125-day equally weighted	1.38	0.07	1.39	1.35	1.39	1.39
250-day equally weighted	1.37	0.07	1.43	1.28	1.41	1.36
500-day equally weighted	1.38	0.08	1.46	1.24	1.43	1.34
1,250-day equally weighted	1.36	0.08	1.35	1.27	1.35	1.43
125-day historical simulation	1.48	0.04	1.47	1.45	1.49	1.50
250-day historical simulation	1.43	0.05	1.49	1.34	1.46	1.44
500-day historical simulation	1.44	0.06	1.53	1.29	1.48	1.43
1,250-day historical simulation	1.41	0.07	1.39	1.31	1.39	1.50
Exponentially weighted (λ=0.94)	1.41	0.07	1.39	1.42	1.41	1.42
Exponentially weighted (λ=0.97)	1.38	0.07	1.37	1.38	1.38	1.38
Exponentially weighted (λ=0.99)	1.35	0.07	1.38	1.30	1.38	1.34
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	1.46	0.12	1.48	1.45	1.48	1.47
125-day equally weighted	1.44	0.11	1.45	1.41	1.42	1.50
250-day equally weighted	1.44	0.13	1.49	1.34	1.44	1.50
500-day equally weighted	1.46	0.14	1.56	1.29	1.46	1.47
1,250-day equally weighted	1.44	0.14	1.43	1.31	1.39	1.55
125-day historical simulation	1.48	0.07	1.51	1.47	1.46	1.55
250-day historical simulation	1.37	0.07	1.44	1.28	1.37	1.41
500-day historical simulation	1.37	0.09	1.46	1.25	1.34	1.40
1,250-day historical simulation	1.30	0.10	1.28	1.20	1.25	1.40
Exponentially weighted (λ=0.94)	1.44	0.11	1.45	1.44	1.44	1.48
Exponentially weighted (λ=0.97)	1.42	0.11	1.43	1.40	1.41	1.45
Exponentially weighted (λ=0.99)	1.40	0.11	1.44	1.35	1.42	1.44

Table A7
MAXIMUM MULTIPLE OF TAIL EVENT TO RISK MEASURE

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	3.59	0.93	3.25	2.56	2.73	2.98
125-day equally weighted	3.59	0.98	3.01	2.54	2.56	3.09
250-day equally weighted	3.67	1.01	3.03	2.45	2.59	3.07
500-day equally weighted	3.86	1.08	3.25	2.33	2.66	3.04
1,250-day equally weighted	3.97	1.10	3.05	2.35	2.60	3.21
125-day historical simulation	3.91	1.02	3.13	2.84	2.78	3.49
250-day historical simulation	3.85	1.10	3.03	2.61	2.62	3.31
500-day historical simulation	4.09	1.16	3.35	2.44	2.73	3.30
1,250-day historical simulation	4.14	1.12	3.12	2.44	2.67	3.37
Exponentially weighted ($\lambda=0.94$)	3.58	0.99	3.16	2.55	2.75	3.03
Exponentially weighted ($\lambda=0.97$)	3.53	0.99	3.13	2.46	2.57	2.99
Exponentially weighted ($\lambda=0.99$)	3.55	0.96	3.03	2.40	2.55	2.96
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	2.50	0.61	2.26	1.83	1.91	2.08
125-day equally weighted	2.50	0.70	2.09	1.82	1.79	2.15
250-day equally weighted	2.56	0.73	2.11	1.75	1.81	2.14
500-day equally weighted	2.70	0.78	2.27	1.66	1.85	2.13
1,250-day equally weighted	2.77	0.77	2.14	1.67	1.81	2.24
125-day historical simulation	2.58	0.52	2.18	1.97	1.86	2.25
250-day historical simulation	2.34	0.57	2.00	1.66	1.72	2.02
500-day historical simulation	2.48	0.63	2.08	1.60	1.70	2.05
1,250-day historical simulation	2.49	0.65	1.89	1.54	1.63	2.02
Exponentially weighted ($\lambda=0.94$)	2.48	0.64	2.20	1.83	1.92	2.10
Exponentially weighted ($\lambda=0.97$)	2.46	0.66	2.18	1.76	1.79	2.08
Exponentially weighted ($\lambda=0.99$)	2.47	0.68	2.11	1.72	1.78	2.06

Table A8
CORRELATION BETWEEN RISK MEASURES AND ABSOLUTE VALUE OF OUTCOME

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	0.19	0.05	0.21	0.15	0.12	0.19
125-day equally weighted	0.16	0.05	0.17	0.13	0.07	0.14
250-day equally weighted	0.13	0.05	0.12	0.15	0.02	0.13
500-day equally weighted	0.06	0.04	0.01	0.07	0.05	0.05
1,250-day equally weighted	0.01	0.03	0.05	0.05	-0.04	-0.02
125-day historical simulation	0.14	0.05	0.16	0.11	0.04	0.12
250-day historical simulation	0.11	0.05	0.10	0.12	0.02	0.10
500-day historical simulation	0.03	0.04	-0.00	0.06	0.03	0.01
1,250-day historical simulation	0.00	0.04	0.06	0.05	-0.03	-0.05
Exponentially weighted (λ =0.94)	0.23	0.05	0.26	0.18	0.15	0.24
Exponentially weighted (λ =0.97)	0.22	0.05	0.23	0.17	0.14	0.21
Exponentially weighted (λ =0.99)	0.17	0.04	0.17	0.15	0.09	0.17
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	0.19	0.04	0.21	0.15	0.12	0.19
125-day equally weighted	0.16	0.05	0.17	0.12	0.07	0.15
250-day equally weighted	0.13	0.05	0.12	0.15	0.02	0.13
500-day equally weighted	0.06	0.04	0.02	0.07	0.05	0.06
1,250-day equally weighted	0.01	0.04	0.06	0.04	-0.04	-0.02
125-day historical simulation	0.12	0.06	0.16	0.07	0.06	0.13
250-day historical simulation	0.10	0.07	0.10	0.09	0.01	0.12
500-day historical simulation	0.05	0.05	0.03	0.04	0.06	0.06
1,250-day historical simulation	0.01	0.04	0.05	0.04	-0.02	0.00
Exponentially weighted (λ =0.94)	0.23	0.05	0.26	0.18	0.15	0.24
Exponentially weighted (λ =0.97)	0.22	0.05	0.23	0.17	0.14	0.22
Exponentially weighted (λ =0.99)	0.17	0.04	0.17	0.15	0.09	0.17

Table A9

MEAN RELATIVE BIAS FOR RISK MEASURES SCALED TO DESIRED COVERAGE LEVELS

	Entire Sample Period		1983-85	1986-88	1989-91	1992-94
	Mean across Portfolios	Standard Deviation across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios	Mean across Portfolios
PANEL A: 95TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	-0.00	0.02	-0.00	-0.00	0.00	-0.00
125-day equally weighted	-0.01	0.01	-0.01	-0.00	-0.00	-0.01
250-day equally weighted	-0.01	0.01	-0.00	-0.01	-0.00	-0.01
500-day equally weighted	0.01	0.02	0.02	0.01	-0.01	0.01
1,250-day equally weighted	0.02	0.02	0.01	0.01	0.01	0.02
125-day historical simulation	0.00	0.01	0.00	0.01	0.01	0.01
250-day historical simulation	-0.00	0.02	0.01	-0.00	0.00	-0.01
500-day historical simulation	0.02	0.02	0.03	0.01	-0.00	0.02
1,250-day historical simulation	0.02	0.02	0.01	0.01	0.01	0.03
Exponentially weighted (λ=0.94)	-0.01	0.02	-0.02	-0.01	0.01	-0.01
Exponentially weighted (λ=0.97)	-0.02	0.01	-0.02	-0.02	-0.01	-0.02
Exponentially weighted (λ=0.99)	-0.02	0.01	-0.02	-0.02	-0.02	-0.02
PANEL B: 99TH PERCENTILE VALUE-AT-RISK MEASURES						
50-day equally weighted	-0.02	0.03	-0.03	0.02	0.00	-0.03
125-day equally weighted	-0.02	0.02	-0.03	-0.02	-0.00	-0.02
250-day equally weighted	-0.01	0.02	0.00	-0.02	0.01	-0.01
500-day equally weighted	0.02	0.03	0.06	0.00	-0.00	0.02
1,250-day equally weighted	0.02	0.03	0.04	-0.01	-0.01	0.03
125-day historical simulation	0.03	0.03	-0.00	0.06	0.05	0.05
250-day historical simulation	0.02	0.03	0.02	-0.00	0.05	0.02
500-day historical simulation	0.03	0.03	0.05	0.01	0.00	0.03
1,250-day historical simulation	0.03	0.04	0.04	-0.01	0.00	0.03
Exponentially weighted (λ=0.94)	-0.04	0.03	-0.05	0.01	-0.05	-0.04
Exponentially weighted (λ=0.97)	-0.04	0.02	-0.06	-0.01	-0.03	-0.05
Exponentially weighted (λ=0.99)	-0.03	0.02	-0.04	-0.03	-0.01	-0.04

ENDNOTES

1. See, for example, the so-called G-30 report (1993), the U.S. General Accounting Office study (1994), and papers outlining sound risk management practices published by the Board of Governors of the Federal Reserve System (1993), the Basle Committee on Banking Supervision (1994), and the International Organization of Securities Commissions Technical Committee (1994).

2. Work along these lines is contained in Jordan and Mackay (1995) and Pritsker (1995).

3. Results for ten-day holding periods are contained in Hendricks (1995). This paper is available from the author on request.

4. The 99th percentile *loss* is the same as the 1st percentile *gain* on the portfolio. Convention suggests using the former terminology.

5. Variance-covariance approaches are so named because they can be derived from the variance-covariance matrix of the relevant underlying market prices or rates. The variance-covariance matrix contains information on the volatility and correlation of all market prices or rates relevant to the portfolio. Knowledge of the variance-covariance matrix of these variables for a given period of time implies knowledge of the variance or standard deviation of the portfolio over this same period.

6. The assumption of linear positions is made throughout the paper. Nonlinear positions require simulation methods, often referred to as Monte Carlo methods, when used in conjunction with variance-covariance matrices of the underlying market prices or rates.

7. See Fama (1965), a seminal paper on this topic. A more recent summary of the evidence regarding foreign exchange data and "fat tails" is provided by Hsieh (1988). See also Taylor (1986) and Mills (1993) for general discussions of the issues involved in modeling financial time series.

8. The portfolio variance is an equally weighted moving average of squared deviations from the mean.

9. In addition, equally weighted moving average approaches may differ in the frequency with which estimates are updated. This article assumes that all value-at-risk measures are updated on a daily basis. For a comparison of different updating frequencies (daily, monthly, or quarterly), see Hendricks (1995). This paper is available from the author on request.

10. The intuition behind this assumption is that for most financial time series, the true mean is both close to zero and prone to estimation error.

Thus, estimates of volatility are often made worse (relative to assuming a zero mean) by including noisy estimates of the mean.

11. Charts 1-3 depict 99th percentile risk measures and are derived from the same data used elsewhere in the article (see box). For Charts 1 and 2, the assumption of normality is made, so that these risk measures are calculated by multiplying the portfolio standard deviation estimate by 2.33. The units on the y-axes are millions of dollars, but they could be any amount depending on the definition of the units of the portfolio's positions.

12. Engle's (1982) paper introduced the autoregressive conditional heteroskedastic (ARCH) family of models. Recent surveys of the literature on conditional volatility modeling include Bollerslev, Chou, and Kroner (1992), Bollerslev, Engle, and Nelson (1994), and Diebold and Lopez (1995). Recent papers comparing specific conditional volatility forecasting models include West and Cho (1994) and Heynen and Kat (1993).

13. See Engle and Bollerslev (1986).

14. For obvious reasons, a fifty-day observation period is not well suited to historical simulations requiring a 99th percentile estimate.

15. Bootstrapping techniques offer perhaps the best hope for standard error calculations in this context, a focus of the author's ongoing research.

16. For a discussion of the statistical issues involved, see Kupiec (1995). The Basle Committee's recent paper on backtesting (1996b) outlines a proposed supervisory backtesting framework designed to ensure that banks using value-at-risk models for regulatory capital purposes face appropriate incentives.

17. The upper and lower edges of the boxes proper represent the 75th and 25th percentiles, respectively. The horizontal line running across the interior of each box represents the 50th percentile, and the upper and lower "antennae" represent the 95th and 5th percentiles, respectively.

18. One plausible explanation relies solely on Jensen's inequality. If the true conditional variance is changing frequently, then the average of a concave function (that is, the value-at-risk measure) of this variance will tend to be less than the same concave function of the average variance. This gap would imply that short horizon value-at-risk measures should on average be slightly smaller than long horizon value-at-risk measures. This logic may also explain the generally smaller average size of the exponentially weighted approaches.

19. With as few as 125 observations, the use of actual observations inevitably produces either upward- or downward-biased estimates of most specific percentiles. For example, the 95th percentile estimate is taken to be the seventh largest loss out of 125, slightly lower than the 95th percentile. However, taking the sixth largest loss would yield a bias upward. This point should be considered when using historical simulation approaches together with short observation periods, although biases can be addressed through kernel estimation, a method that is considered in Reiss (1989).

20. In particular, see Mahoney (1995) and Jackson, Maude, and Perraudin (1995).

21. See, for example, Bollerslev (1987) and Baillie and Bollerslev (1989).

22. The degrees of freedom, d, are chosen to solve the following equation, $a^*z(0.99) = t(0.99,d) / \sqrt{\frac{d}{d-2}}$, where a is the ratio of the observed 99th percentile to the 99th percentile calculated assuming normality, $z(0.99)$ is the normal 99th percentile value, and $t(0.99,d)$ is the t-distribution 99th percentile value for d degrees of freedom. The term under the square root is the variance of the t-distribution with d degrees of freedom.

23. This section and the next were inspired by Boudoukh, Richardson, and Whitelaw (1995).

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