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REASONABLE EXTREME BOUNDS ANALYSIS

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ABSTRACT
Leamer (1983) suggested to study the range of estimators β_0 in the model $y = X\beta + \epsilon$ when imposing linear constraints of the form $M (C\beta - c) = 0$ where only C and c are fixed. However the extremes may come from models with a bad R^2, say. In this paper we give the exact bounds when only considering models with $R^2 \geq (1-\delta) R^2_{max} + \delta R^2_{min}$. These exact bounds can be found from calculating only two regressions. We apply our techniques to study the velocity of money.

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1. Introduction

A modeller is faced with many different possible specifications for the model when there are several possible explanatory variables, each of which can enter with various lags. Leamer (1983) has suggested that certain essential features of the model can vary greatly between alternative specifications, thus making the interpretation of the model difficult or "fragile." An easily understood version of his argument has a single dependent variable y, a group of variables $\underline{X}_F$ that "should" be used as explanatory variables in any model of y when a particular question is under consideration, and a second group of variables $\underline{X}_D$ that may or may not enter the model as explanatory variables. A basic or "restricted" model would be

$$y = \beta_F \underline{X}_F + \text{residual} \tag{1.1}$$

a complete model would be

$$y = \beta_F \underline{X}_F + \beta_D \underline{X}_D + \text{residual} \tag{1.2}$$

with some linear constraints on β_D, such as requiring that certain variables be given a zero coefficient in (1.2). Thus, for example, if one was interested in the effect of an interest rate on velocity then the equation for velocity would certainly include this interest rate in $\underline{X}_F$ plus possibly also velocity lagged once, money lagged once and a price index lagged. These explanatory variables might be thought of as a minimum set of variables necessary to explain velocity, according to some theory. This would give the basic model. However, it may be thought necessary, by some modellers, to augment the basic model so that the model better explains the main features of the actual velocity series. This augmentation may include further lags of the variables already used plus other variables, such as variability of money base. As there are many possible ways to augment the basic model so there will be many different specifications. Suppose that we are most interested in the value of a particular coefficient, denoted β_0, such as the coefficients on interest rate in the velocity equation. The estimates of β_0 may vary considerably from one specification to another, and the extremes taken by the alternative estimates are called the "extreme bounds" by Leamer (1983). The extent of these bounds are viewed as measuring the fragility of the estimate of β_0 as alternative specifications are used. The value and interpretation of these bounds have been strongly criticized by McAleer, Pagan and Volker (1983, 1985) and also by Breusch (1985) and defended by Leamer (1985).

One criticism of the use of extreme bounds, which has some impact, in that the actual extremes may come from models that most economists would find unreasonable in some way, such as having low Durbin–Watson statistics, for example, in a time–series context. One way to express this problem is in terms of R^2 statistics. We are not defending R^2 as an ideal measure of the quality of a model, but it is possibly a relevant statistic and some exact results are achievable using it. Suppose that the maximum value achievable for R^2 is R^2_{max}, which is certainly found by using all of the variables X_F, X_D in (1.2) with no exclusions, which might be called the "full" model. Of course, other specifications may also achieve this R^2_{max}. The above worry about the virtues of using extreme bounds analysis is that the extreme may come from specifications that achieve R^2 values very much smaller than

R^2_{max} and these specifications might be considered irrelevant because of their relatively low goodness of fit so that estimates of β_o based on them would also be strongly discounted. It may be thought that specifications that achieve R^2 values not too far from R^2_{max} would produce much narrower extreme bounds for β_o. It is this possibility that we consider in this paper. Suppose that R^2_{max} is found from the full model and R^2_{min} from the basic model (1.1). Consider model specification achieving R^2 values equal to or greater than

$$R^2_\delta = (1-\delta)\, R^2_{max} + \delta\, R^2_{min}$$

where $0 < \delta < 1$[1]. For δ small these may be considered as being "reasonable" specifications as they are not far from the "best" model in terms of goodness of fit, as measured by R^2. In the next section of the paper an equation for the values of the extreme bounds of β_o is presented for any given δ. The proof is found in the appendix. A numerical example is presented in Section 3, concerning the modelling of velocity and using time series models. It is found that quite wide extreme bounds can occur using δ values as low as 0.1 or 0.2, relative to the extreme bounds found from the full set of possible specifications. This result strengthens Leamer's arguments about the difficulties that can arise when interpreting particular coefficients. Some further considerations are presented in the final section.

[1]See also Leamer(1981), in which similar ideas and results are obtained when constraining ridge estimates to achieve a given level of significance.

2. The Model and Results

The model being considered is

$$\underline{y} = \underline{X}\underline{\beta} + \underline{\epsilon} \tag{2.1}$$

where $\underline{y}$ is the vector of observations on the variable $\underline{y}$ and $\underline{X}$ is the matrix of observations on a vector of explanatory variables. At this stage, no distinction is being made between time–series or cross–section situations. It will be assumed that $\underline{\epsilon}$ is $N(0, \sigma^2\,\Omega)$ where the covarience matrix $\sigma^2\,\Omega$ is assumed known for the time being. The object of primary interest is the "focus" coefficient

$$\beta_o = \psi' \beta \tag{2.2}$$

so that β_o can be any individual coefficient or a weighted linear combination of coefficients. There will be a set of prior linear constraints

$$C\beta = c \tag{2.3}$$

It is convenient to use this general form for the constraints, but if the two types of variables X_F, X_D are considered, as in the first section, then the coefficients on X_F are free of restrictions (hence the notation β_F) and the coefficients on X_D are "doubtful" in that they may or may not appear in any particular specification. The restrictions would then be $C = (0, I)$ with appropriate sizes of the zero and unit vectors, and $c = 0$. In a particular specification any linear combination of these restrictions can be used, so the objective is to study the range of estimators for β_o when imposing linear constraints of the form

$$M(C\beta - c) = 0 \tag{2.4}$$

for some matrix M, which is assumed without loss of generating to be of full row rank. If no restrictions are placed on M, one gets the extreme bounds suggested by Leamer (1983). The following notation is used:

The generalized least square (GLS) estimates of β using the full model (2.1) with no exclusions is

$$b = (X' \Omega^{-1} X)^{-1} X' \Omega^{-1} y$$

which gives the estimate of β_o, $b_o = \psi' b$.

Let
$$D = \sigma^2 (X' \Omega^{-1} X)^{-1}$$

$$A = CDC'$$

and let $A^{-\frac{1}{2}}$ be the unique symmetric square root of A^{-1}.

For a given M in (2.4) define

$$W = A^{\frac{1}{2}} M',$$

noting that we get any full column rank W as M ranges over all full row rank matrices and vice versa.

Two important vectors are

(2.5) $$u = A^{-\frac{1}{2}} CD\psi$$

and

(2.6) $$v = A^{-\frac{1}{2}} (Cb-c)$$

The Euclidean norm of u is $\|u\| = (u'u)^{\frac{1}{2}}$. It is convenient to define an angle $\theta \in [0, \pi/2]$ by

(2.7) $$\cos 2\theta = \cos (u, v)$$

$$\equiv \frac{u'v}{\|u\| \; \|v\|}$$

The GLSE $\hat{\beta}$ of β_o under the restriction (2.4) is

$$\hat{\beta}_o = b_o - u'W(W'W)^{-1} W'v$$

Breusch (1985) proved:

THEOREM 1

The extreme values of $\hat{\beta}_o$ over all choices of (full column rank matrices) W are

$$b_o - \frac{1}{2}(\cos 2\theta \pm 1)\,\|u\|\,\|v\|$$

i.e.

$$b_o - \cos^2\theta\,\|u\|\,\|v\|$$

to

$$b_o + \sin^2\theta\,\|u\|\,\|v\|$$

The bounds can be attained for some W.

Suppose now that a value for δ is chosen, with $0 \leq \delta \leq 1$ and models are considered having R^2 values[2] greater or equal to

$$R^2_\delta = (1-\delta)\,R^2_{max} + \delta\,R^2_{min}$$

Define an angle $\lambda \in [0, \pi/2]$ such that

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[2]Alternatively, restrict M to be of rank m with $0 < m < \text{rank}(C)$. Consider all models where the F–statistic for testing the set of linear constraints $M(C\beta-c)$ is less or equal to

$$F_{\delta,m} = \delta F_{m,max} + (1-\delta)F_{m,min},$$

where $F_{m,max}$ is the maximum and $F_{m,min}$ is the minimum among the achievable F–statistics when there are m restrictions. Again, Theorem 2 gives the correct bounds.

Observe that the F–statistic is essentially R^2 except for the number of regressors. Furthermore, any R^2 can be achieved for any given number m, $0<m<\text{rank}(C)$ of restrictions: for the maximal R^2, simply include the regressor that arises as the linear combination of regressors with coefficients taken from the unrestricted regression. For the minimal R^2, choose regressors orthogonal to that regressor. Therefore fixing the number m is no restriction.

$$\sin^2 \lambda = \delta.$$

The upper and lower bounds on β_o will be given by

$$b_o - \phi_L \, \|u\| \, \|v\|$$

and

$$b_o + \phi_U \, \|u\| \, \|v\|$$

where ϕ_L and ϕ_U are always positive and depend on the chosen δ. The precise formula is given in:

THEOREM 2

(i) if $\lambda \leq \theta$ then $\phi_U = \sin^2 \theta - \sin^2 (\lambda - \theta)$

if $\lambda \geq \theta$ then $\phi_U = \sin^2 \theta$

(ii) if $\lambda \leq {}^{\pi}/2 - \theta$ then $\phi_L = \cos^2 \theta - \cos^2 (\lambda + \theta)$

if $\lambda \geq {}^{\pi}/2 - \theta$ then $\phi_L = \cos^2 \theta.$

The proof is in the appendix.

For given δ, and thus, λ, the extreme bounds can be found directly from the two regressions[3], the basic regression involving just the free variables X_F, which gives $\hat{b}_o$ as the estimate of β_o and also provides R^2_{min}, and the complete regression in which all variables enter, which gives b as the estimate of β, and b_o as the estimate of β_o and R^2_{max}. It follows that

$$u'v = b_o - \hat{b}_o$$

$$\|v\| = (Cb - c)' \; A^{-1} \, (b - c)$$

and

$$\|u\| = \mathrm{var}(b_o) - \mathrm{var}\,(\hat{b}_o)$$

From these quantities θ is determined as $\cos 2\theta = \cos\,(u,\, v)$. It may be suggested that it is good economic practice to report the values of $u'v$, $\|v\|$, $\|u\|$ as well as the extreme bounds on β_o for various values of δ, say 0.1 and 0.05. Table 1, at the end of the paper makes this latter task easy by tabulating the values of ϕ_L and ϕ_U for various values of $\cos 2\theta$.

3. The effect of interest rates and inflation on the velocity of money.

We want to analyze the effect of interest rates and inflation on the increase in the velocity of money. To this end, we consider two models in quarterly data and homoskedastic errors with unknown variance.

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[3]Observe that the bounds do not depend on σ^2. Thus one might therefore set $\sigma^2=1$ to simplify the calculation.

We used the following list of variables:

veloc	velocity of money, computed as GNP/M1,
dveloc	first differences in the velocity of money
tbill	3–month treasury bill rate,
infla	inflation, computed from the consumer price index CPI,
gnp	gross national product,
mbvariab	variability of money[4], computed as standard deviation within one year of the growth rate of the monetary base from its global mean and trend and 12 own lags, using monthly data. We multiply that number by 10000 for numerical reasons.

GNP, CPI, M1 and monetary base are seasonally adjusted. We use first differences in the velocity of money in our regression.[5] The time index t counts quarters.

[4]It has been argued that the recent decline in the velocity of money was caused by changes in the variability of money supply (see Friedman (1984) and Hall–Noble (1987)).

[5]It has been argued that the velocity of money follows a random walk, i.e. that there are unit roots in the corresponding regression equation in levels. Then, first differencing the velocity series is a reasonable procedure. But also in the level model the inference drawn from OLS (instead of using unit roots distribution theory) is valid, if the linear combination of regressors that we look at for our focus coefficient doesn't lie in the eigenspace of the unit root in the joint VAR (see Sims–Stock–Watson). Another way to justify using levels is the Bayesian point of view, in which the posterior distribution of the coefficient given the data is still almost normal even in the presence of unit roots, see Sims (1987). In fact, the results don't change much using levels instead of first differences except that the range of R^2 becomes much smaller. We chose a model in first differences because the results are more instructive and not because we believe the level model to yield incorrect results.

Model I:

a) unrestricted: ("equation 1")

$$\text{dveloc}(t) = \alpha + \beta_{0,0}t + \sum_{l=1}^{5} \beta_{1,1}\text{dveloc}(t-l) + \sum_{l=0}^{6} \beta_{1,2}\text{tbill}(t-l) + \sum_{l=0}^{5} \beta_{1,3}\text{infla}(t-l)$$

$$+ \sum_{l=0}^{6} \beta_{1,4}\text{gnp}(t-l) + \sum_{l=0}^{6} \beta_{1,5}\text{mbvariab}(t-l) + \epsilon_t$$

b) restricted: ("equation 2")

$$\text{dveloc}(t) = \alpha + \beta_{0,0}t + \sum_{l=1}^{2} \beta_{1,1}\text{dveloc}(t-l) + \sum_{l=0}^{1} \beta_{1,2}\text{tbill}(t-l) + \epsilon_t.$$

Model II:

a) unrestricted: ("equation 1"): same as in model I

b) restricted: ("equation 2")

$$\text{dveloc}(t) = \alpha + \beta_{0,0}t + \sum_{l=1}^{2} \beta_{1,1}\text{dveloc}(t-l) + \sum_{l=0}^{2} \beta_{1,2}\text{tbill}(t-l) + \sum_{l=0}^{2} \beta_{1,3}\text{infla}(t-l)$$

$$+ \sum_{l=0}^{1} \beta_{1,5}\text{mbvariab}(t-l) + \epsilon_t$$

We used ordinary least squares to estimate the models. The regression output is in appendix 2. Let us highlight a few results(the numbers for $\|u\|$,$\|v\|$ and cos(u,v) are calculated for $\sigma^2=1$, see remarks above):

Table 1:

Model I

focus coeff.		δ^2=1.0	δ^2=0.1	δ^2=.05	δ^2=0.0
$\beta_{1,2}$ (tbill – first lag)					
	upper	.08110	.07342	.06806	.05154
	lower	–.00540	.02420	.03230	.05154
		(‖u‖=.16853,‖v‖=.51325,cos(u,v)=.31675)			
$\beta_{0,0}$ (trend)					
	upper	.00300	.00161	.00107	–.00034
	lower	–.00342	–.00224	–.00172	–.00034
		(‖u‖=.01251,‖v‖=.51325,cos(u,v)=–.0392)			

Model II

focus coeff.		δ^2=1.0	δ^2=0.1	δ^2=.05	δ^2=0.0
$\beta_{1,2}$ (tbill – first lag)					
	upper	.07635	.07044	.06591	.05155
	lower	–.00108	.02709	.03441	.05155
		(‖u‖=.15999,‖v‖=.48395,cos(u,v)= .35931)			
$\beta_{2,2}$ (tbill – second lag)					
	upper	.00725	–.02404	–.03436	–.06019
	lower	–.10847	–.09251	–.08411	–.06019
		(‖u‖=.23911,‖v‖=.48395,cos(u,v)=–.16570)			
$\sum_{l=0}^{6} \beta_{1,2}$ (effect of permanent increase in interest rates)					
	upper	.01682	.01288	.01048	.00334
	lower	–.01911	–.00799	–.00469	.00334
		(‖u‖=.23911,‖v‖=.48395,cos(u,v)=–.16570)			
$\beta_{1,3}$ (inflation – first lag)					
	upper	.01156	.00918	.00795	.00448
	lower	–.00513	–.00072	.00076	.00448
		(‖u‖=.03449,‖v‖=.48395,cos(u,v)= .15132)			

These results contain several interesting aspects:

The range of possible estimates for the trend variable is bigger than the two coefficient estimates in the restricted and the unrestricted model suggest. This coincides with the uncertainty about the trend coefficient as indicated by the t–statistic (see McAleer et al (1985)).

The shape of the range of coefficient estimates for the first lag of tbills changes little in either model. In both cases it has a positive coefficient as long as we stay in the top 20% of R^2, say. Similarly, the second lag has a negative coefficient. Notice that the unrestricted EBA doesn't allow here for that conclusion (since coefficients of the opposite sign in these cases are included here when R^2 is not restricted). It is of course subject to debate whether it is reasonable to look at a fraction of the range of possible R^2. The point is, that we have to make the unrestricted model pretty much as bad as possible within the admissable range to arrive at coefficients of the opposite sign in either case. It is now up to the judgement of the individual researcher to decide whether (s)he wants to rule out these coefficients as unreasonable (because of the "bad" R^2) or not. This type of sensitivity analysis, putting the facts on the table, is of course the whole point of this approach.

We also see that not much can be said about the permanent effects of e.g a permanent rise in the nominal interest rate. Different models allow for different conclusions within the top 20%–range of R^2 and that might be all that can be said. The same applies to money base variability which we did not find to have a clear effect on velocity. Looking at a plot of real interest rates (computed as tbill(t)–inflation(t), where contemporaneous inflation is used as a crude substitute for the inflation expectation of agents in our economy), it seems that the recent change in the behavior of the velocity of money coincides with the shift of real interest rates from negative values to positive values.

An analysis as shown here helps to understand better the possibilities for the outcomes from different models and the type of restrictions we impose when passing from a large "bench–mark" model to a smaller model.

4. Conclusions:

Our result enables the researcher with a "continuum" of choices between classical econometrics and Leamers "extreme" EBA: if one only wants the maximal R^2, the theorem will give the coefficient of the classical analysis. If one allows for any R^2, the theorem gives the extreme bounds as e.g. given by Breuschs theorem (which is contained in our theorem as a special result). It seems reasonable, as explained above to give the extreme bounds of the coefficient of interest subject to restricting R^2 to be in the top 5% or top 10% of the range of possible R^2: using ordinary regression output, this can be done using the formulas of the theorem or the tables at the end of this paper. Of course, one would like extensions of our result: how can we deal with a vector of coefficients of interest? Is there a similar version of the theorem that controls for the Durbin–Watson statistic? How can we incorporate uncertainty about Ω in our model? Are there similar results for other inference procedures (e.g. probit models) or in the presence of nonnormal distributions (as e.g. in the unit roots case). How do we include coefficient–uncertainty arising from any of the specific models included in our range? How can we find bounds if we pose positivity constraints on certain coefficients? Are there interesting asymptotic results?

The theorem above allows us to get a feel of how much actually changes, if we proceed from a general "benchmark" model to a smaller, restricted model. The procedure described above is intended to append current practice in that way, not to replace it and can thus provide useful insights about the data.

Appendix 1:

Proof of the theorem:

Given a restriction matrix M as in (2.4), find the corresponding W. Write

$$\beta_0 = b_0 - \varphi(W)\|u\|\ \|v\| \quad \text{and}$$

$$R^2 = 1 - \frac{e'e + \gamma(W)\|v\|^2\sigma^2}{(y-\overline{y})'(y-\overline{y})},$$

where e is the vector of residuals under the unrestricted model, $\overline{y}$ is the mean of y[6] and $\gamma(W)$ is assumed to be positiv.

Assume that {u,v} is linear independent and that C is not trivial (i.e. there is at least one nonzero constraint).

Note that (i) and (ii) are equivalent to the following formulation, using the usual addition theorems for sin and cos. It is this version that we are going to prove:

[6]We have of course $\overline{y} = 0$, if the model is already in deviation form.

For a chosen $\delta \in [0,1]$ and matrix W, suppose that $\gamma(W) \leq \delta$.

(i') If $\delta \leq (1-\cos(u,v))/2$,

$$\text{then } \varphi(W) \geq \phi_L = \delta^{1/2}\sin(\lambda-2\theta)$$

$$\text{else } \varphi(W) \geq \phi_L = (\cos(u,v)-1)/2$$

(ii') If $\delta \leq (1+\cos(u,v))/2$,

$$\text{then } \varphi(W) \leq \phi_U = \delta^{1/2}\sin(\lambda+2\theta)$$

$$\text{else } \varphi(W) \leq \phi_U = (\cos(u,v)+1)/2.$$

These bounds can be achieved for properly chosen matrices W_L and W_U, i.e for these matrices we get the upper bound $\phi_L=\varphi(W_L)$ and the lower bound $\phi_U=\varphi(W_U)$.

The proof involves two steps. We consider a special class of W–matrices $\{W(t) \mid t \in \mathbb{R}\}$. First, we show, that any other W can do only "worse" than some W(t), i.e. at the same R^2, $\varphi(W)$ will be restricted to the range given by the corresponding $\varphi(W(t))$. Secondly, we show that we get the bounds mentioned in the Theorem within that class of W(t)'s, proving the theorem.

Consider any (full column rank) matrix W of size $k \times m$, say, where $1 \leq m \leq k$ and k is the number of possible restrictions, $k \geq 1$ by assumption. We find that

$$\varphi(W) = u'W(W'W)W'v/(\|u\| \; \|v\|) \text{ and}$$

$$\gamma(W) = v'W(W'W)W'v/(\|v\| \; \|v\|),$$

so assume w.l.o.g. (in order to avoid the norm–signs) that $\|u\|=\|v\|=1$. The special class of W that we want to consider are all W of size $k \times 1$ which are linear combinations of u and v.

Claim: For all W, there is a linear combination ζ of x and y and a real number η with $0 \leq \eta \leq 1$ such that $\gamma(W) = \gamma(\zeta)$ and $\varphi(W) = \eta\varphi(\zeta)$.

Proof of the Claim:

Recall from linear algebra, that W(W'W)W' is a matrix that maps any k–dimensional vector in an orthogonal way on the range of W. Hence $\gamma(W) \leq 1$. Fix a W. Let P be the orthogonal projector on the plane spanned by u and v. Let $z_W = PW(W'W)W'v$, we then have $\varphi(W) = u'z_W$ and similarly $\gamma(W) = v'z_W$. Assume $z_W \neq 0$ (otherwise we are done). Let x be the orthogonal projection of v on z_W, i.e. $x = \tau z_W$, where $\tau = (z_W'v)/(z_W'z_W)$. By Lemma 1 (below), we have $1 \leq \tau$. Now, find a linear combination ζ of u and v such that $\zeta'v = z_W'v$ and ζ is the orthogonal projection of v on ζ itself, i.e. $\zeta = \zeta(\zeta'\zeta)^{-1}\zeta'v$. By Lemma 2 (below), ζ exists and satisfies $\|\zeta\| \geq \|z_W\|$. Write $\zeta = (z_W'v)v + q$, where q is orthogonal to v. Notice that $\tilde{\zeta} = (z_W'v)v - q$ also satisfies $\tilde{\zeta}'v = z_W'v$, $\|\tilde{\zeta}\| \geq \|z_W\|$ and $\tilde{\zeta}$ is the orthogonal projection of v on $\tilde{\zeta}$ itself. Therefore, we can assume w.l.o.g. that $(u'q)(u'z_W) \geq 0$.

Write $z_W = (z_W'v)v + \alpha q$, where $|\alpha| \leq 1$. Now we are done, since $\gamma(\zeta) = v'\zeta = v'z_W = \gamma(W)$ and (for the case $u'z_W \neq 0$, otherwise the claim is now trivial) $\varphi(\zeta) = u'\zeta = u'z_W + (1-\alpha)u'q = \varphi(W)/\eta$, where $1/\eta = 1 + (1-\alpha)(u'q)/(u'z_W) \geq 1$. This finishes the first part of the proof.

We now proceed to the second part of the proof.

Consider $W(t) = u + tv$ (recall that we assumed w.l.o.g. $\|u\| = \|v\| = 1$). Observe that we exclude the matrix W=v among the class of all linear combinations of u and v. However, since $W(W'W)^{-1}W'$ is an orthogonal projector on the range of W, we can include W=v by considering $t \to \infty$. A look then at the algebraic results below shows that W=v is "irrelevant" for the bounds given in the theorem (actually, W=v gives the worst possible R^2 among our class).

Let $f(t) = \varphi(W(t))$ and $g^2(t) = \gamma(W(t))$. Let $\kappa=\cos(u,v)$. Calculate f and g^2 to be

$$f(t) = (\kappa t+1)(t+\kappa) / (t^2+2\kappa t+1) \text{ and}$$

$$g^2(t) = \quad (t+\kappa)^2/(t^2+2\kappa t+1)$$

so that we might set

$$g(t) = \quad (t+\kappa)/(t^2+2\kappa t+1)^{1/2}.$$

Observe that g vanishes only for $t = -\kappa$ and that also $f(-\kappa) = 0$. Define the ratio r(t) by $r(t) = f(t)/g(t)$, whenever $g(t)\neq 0$. We find

$$r(t) = (\kappa t+1)/(t^2+2\kappa t+1)^{1/2},$$

which is also defined for $t=-\kappa$. (For W=v, observe that $|r(t)|$ converges to κ, $|g(t)|\to 1$ and $f(t)\to\kappa$ as $t\to\infty$ and that $|g(t)|<1$ for all t). Observe that $r(-\kappa)=(1-\kappa^2)^{1/2}$. To get to the exact bounds, note that f has its absolute minimum at $t=-1$ and its absolute maximum at $t=1$, f is a strictly increasing function in between. Using the relationship between $(t-\kappa)^2$ and $g^2(t)$ as stated above, we find at $g^2(t) = \bar{\gamma}^2$:

$$\begin{aligned} r(t) &= \kappa g(t) + (1-\kappa^2)^{1/2}(1-g^2(t))^{1/2} \\ &= \mathrm{sign}(g(t))\cos(2\theta)\sin(\lambda) + \sin(2\theta)\cos(\lambda) \\ &= \mathrm{sign}(g(t))\sin(\lambda + \mathrm{sign}(g(t))2\theta). \end{aligned}$$

Using the fact that g is also a strictly increasing function on the whole real line with $g^2(-1)=-f(-1)=(1-\kappa)/2=\sin^2(\theta)$ and $g^2(1)=f(1)=(1+\kappa)/2=\cos^2(\theta)$ we obtain the sharp bounds as stated in (i') and (ii'). This concludes the proof of the Theorem.

In the proof of the Theorem we made use of the following two Lemmata:

Lemma 1:

Suppose for $u,v,w,x \in \mathbb{R}^k$, that $v=w+x$, that w and x are orthogonal to each other and that $\{u,v\}$ is linear independent. Project w in the plane spanned by u and v, call this projection z_w. Then $v'z_w \geq z_w'z_w$.

Proof:

Decompose w in $w=\alpha v+p+q$, where $\alpha v+p=z_w$ and p is orthogonal to v. Then $y=p+q$ is orthogonal to v and we have

$$z_w'z_w \leq \alpha^2 v'v + y'y = w'w = w'v = z_w'v.$$

Lemma 2:

Let $v,x,y,z \in \mathbb{R}^k$, $k \geq 2$ and $1 \leq \tau$ be such that $v=x+y$, x orthogonal to y and $x=\tau z$. Then there exists a linear combination ζ of x and v such that $\zeta'v=\zeta'x$ and $v-\zeta$ orthogonal to ζ and we have $\|z\| \leq \|\zeta\|$.

Proof:

W.l.o.g. $\|v\|=1$ (rescale every vector). Assume w.l.o.g. that $\{x,v\}$ is linear independent, otherwise the claim is Let q be orthogonal to v in the plane spanned by x and v and of unit length. Set $\zeta = (z'v)v + \alpha q$, where α is a solution to the equation $\alpha^2=(z'v)(1-z'v)$. Check that $\zeta'v=\zeta'\zeta=z'v$. Observe that with a similar argument, we can write x as $x=(x'v)v + aq$, where a now satisfies $a^2=(x'v)(1-x'v)$. Hence $|a| \leq \tau |\alpha|$, proving $\|z\| \leq \|\zeta\|$.

Appendix IIa)

Rats subroutine programs for performing the calculation of restricted extreme bounds.

```
*
*
*****************************************************************
*                                                               *
* RATS-procedures for the calculation of restricted extreme     *
* bounds acc. to Granger-Uhlig (1988)                           *
* ===================================                           *
*                                                               *
* Usage:                                                        *
* increase BMA COMPILE 500                                      *
* Define two equations:                                         *
* Equation 1 - the "unrestricted regression". It is assumed that*
*   a constant is included in the regression and that the sum of*
*   the coefficients lo1 to up1 (lo1 and up1 are options for    *
*   EBASET) is the focus coefficient.                           *
* Equation 2 - the "restricted regression". Again, a constant   *
*   has to be included and the sum of the coefficients lo2 to   *
*   up2 (lo2 and up2 are options for EBASET) is the focus       *
*   coefficient.                                                *
*   Equation 2 should define the regression equation for a      *
*   subset of the regressors of equation 1 and the sum of the   *
*   first coefficients lo2 to up2 should correspond to the sum  *
*   of the coefficients lo1 to up1 of equation 1 with           *
*   (up2-lo2+1)-(up1-lo1+1) zero restrictions.                  *
* Per default, lo1=up1=lo2=up2=1, i.e. the first coefficient is *
* the focus coefficient.                                        *
*                                                               *
* First execute the procedure EBASTART to do the initial        *
* calculation of the regressions etc. Then decide on your focus *
* coefficient and set the neccessary data for the calculations  *
* of the bounds with the procedure EBASET. Then, with each run  *
* of EBACALC, the bounds are calculated according to the given  *
* restriction on Rsquared. EBASET and EBACALC can be executed   *
* several times after an initial EBASTART.                      *
*                                                               *
* Reserved names used in these procedures are:                  *
* zzzru zzzrr zzzbu zzzbr zzzvaru zzzvarr zzzvlen zzzulen       *
* zzzuv zzzco_uv zzzg lo1 up1 lo2 up2 zzzxr zzzxu zzzbetau      *
* zzzbetar zzznobs zzzyvar                                      *
*                                                               *
* Output includes the lengths of the vectors u and v and their  *
* inner product and covariances, assuming sigma=1.0 (recall that*
* the calculated bounds do not depend on sigma).                *
*                                                               *
* Example: suppose, the effect of x1 on y is to be determined   *
*          Other possible regressors are const, trend, x2,..,x5,*
*          and we always want to include x2. Define             *
* SMPL firstper lastper                                         *
* *** allow for all lags in equation 1 !                        *
* EQUATION 1 y                                                  *
* # x1 const trend x2 x3 x4 x5                                  *
```

```
* EQUATION 2 y                                                  *
* # x1 const x2                                                 *
* EXEC EBASTART y                                               *
* EXEC EBASET(lo1=1,up1=1,lo2=1,up2=1)                          *
* (or simply EXEC EBASET )                                      *
* EVAL g2 = 0.1                                                 *
* EXEC EBACALC lower upper g2                                   *
* EVAL g2 = 0.05                                                *
* EXEC EBACALC lower upper g2                                   *
*                                                               *
*****************************************************************
*
PROCEDURE EBASTART y
*****************************************************************
* Instructions: y is the dependent variable                     *
*****************************************************************
TYPE EQV y
   DECLARE SYMM zzzxxu zzzxxr
   DECLARE VECTOR zzzbetau zzzbetar
   OLS(EQUATION=1)
   FETCH zzzru   = RSQUARED
   MATRIX zzzxxu = XX
   MATRIX zzzbetau = BETA
   OLS(EQUATION=2)
   FETCH zzzrr   = RSQUARED
   MATRIX zzzxxr = XX
   MATRIX zzzbetar = BETA
   STATISTIC y
   EVAL zzznobs = NOBS
   EVAL zzzyvar = VAR
END
*
*
PROCEDURE EBASET
OPTION LO1 INTEGER 1
OPTION UP1  INTEGER 1
OPTION LO2 INTEGER 1
OPTION UP2  INTEGER 1
*****************************************************************
* lo1,up1,lo2 and up2 are options.                              *
* the sum of the coefficients lo1 to up1 is the focus           *
*  coefficient in equation 1, the unrestricted equation.        *
* the sum of the coefficients lo2 to up2 is the focus           *
*  coefficient in equation 2, the unrestricted equation.        *
*****************************************************************
*
   LOCAL INTEGER counta countb
*
   EVAL zzzbu = 0.0
   EVAL zzzvaru = 0.0
   DO counta = lo1,up1
      EVAL zzzbu    = zzzbu + zzzbetau(counta)
      DO countb = lo1,up1
         EVAL zzzvaru  = zzzvaru + zzzxxu(counta,countb)
      END
   END
   EVAL zzzbr = 0.0
   EVAL zzzvarr = 0.0
   DO counta = lo2,up2
      EVAL zzzbr    = zzzbr + zzzbetar(counta)
```

```
        DO countb = lo2,up2
           EVAL zzzvarr  = zzzvarr + zzzxr(counta,countb)
        END
     END
     EVAL zzzvlen  = SQRT( (zzznobs-1.0)*zzzyvar*(zzzru-zzzrr) )
     EVAL zzzulen  = SQRT( zzzvaru - zzzvarr )
     EVAL zzzuv    = zzzbu - zzzbr
     EVAL zzzco_uv = zzzuv / (zzzulen*zzzvlen)
     DISPLAY
     DISPLAY
     DISPLAY 'RESTRICTED EXTREME BOUNDS.  DATA:'
     DISPLAY 'FOCUS ON SUM OF ' lo1 ' TO ' up1 ' IN EQUATION 1'
     DISPLAY 'FOCUS ON SUM OF ' lo2 ' TO ' up2 ' IN EQUATION 2'
     DISPLAY 'LENGTH OF U-VECTOR :' zzzulen
     DISPLAY 'LENGTH OF V-VECTOR :' zzzvlen
     DISPLAY 'INNER PRODUCT      : ' zzzuv '    COS(U,V)= ' zzzco_uv
     DISPLAY
     DISPLAY
END
*
*
PROCEDURE EBACALC lower upper g2
TYPE REAL lower upper g2
*****************************************************************
* Instructions: lower is the variable that is going to be set   *
*  to the lower extreme bound. upper is the variable that is    *
*  going to be set to the upper extreme bound. g2 is the para-  *
*  meter gamma-squared, determining the fraction of the range of*
*  admissible R-squares, 0.0<=g2<=1.0                           *
*****************************************************************
   EVAL zzzg = SQRT(g2)
   IF g2.LE.((1.0-zzzco_uv)/2.0)
      EVAL upper = zzzbu - zzzg*( zzzco_uv*zzzg -               $
                SQRT(1-zzzco_uv**2)*SQRT(1-g2) )*zzzulen*zzzvlen
   ELSE
      EVAL upper = zzzbu - ((zzzco_uv - 1.0)/2.0)*zzzulen*zzzvlen
   *
   IF g2.LE.((1.0+zzzco_uv)/2.0)
      EVAL lower = zzzbu - zzzg*( zzzco_uv*zzzg +               $
                SQRT(1-zzzco_uv**2)*SQRT(1-g2) )*zzzulen*zzzvlen
   ELSE
      EVAL lower = zzzbu - ((zzzco_uv + 1.0)/2.0)*zzzulen*zzzvlen
END
*
*****************************************************************
*                    End of EBA-Subroutines                     *
*****************************************************************
*
*
```

Appendix IIb

The following two tables list the lower and upper bounds as given in the theorem for different values of delta.

LOWER BOUND

cos(u,v)	delta=0.05	delta=0.1	delta=0.2	delta=1.0
0.9500	-0.0250	-0.0250	-0.0250	-0.0250
0.9000	-0.0500	-0.0500	-0.0500	-0.0500
0.8500	-0.0723	-0.0750	-0.0750	-0.0750
0.8000	-0.0908	-0.1000	-0.1000	-0.1000
0.7500	-0.1067	-0.1234	-0.1250	-0.1250
0.7000	-0.1206	-0.1442	-0.1500	-0.1500
0.6500	-0.1331	-0.1630	-0.1750	-0.1750
0.6000	-0.1444	-0.1800	-0.2000	-0.2000
0.5500	-0.1545	-0.1955	-0.2241	-0.2250
0.5000	-0.1637	-0.2098	-0.2464	-0.2500
0.4500	-0.1721	-0.2229	-0.2672	-0.2750
0.4000	-0.1797	-0.2350	-0.2866	-0.3000
0.3500	-0.1867	-0.2460	-0.3047	-0.3250
0.3000	-0.1929	-0.2562	-0.3216	-0.3500
0.2500	-0.1985	-0.2655	-0.3373	-0.3750
0.2000	-0.2035	-0.2739	-0.3519	-0.4000
0.1500	-0.2080	-0.2816	-0.3655	-0.4250
0.1000	-0.2119	-0.2885	-0.3780	-0.4500
0.0500	-0.2152	-0.2946	-0.3895	-0.4750
-0.0000	-0.2179	-0.3000	-0.4000	-0.5000
-0.0500	-0.2202	-0.3046	-0.4095	-0.5250
-0.1000	-0.2219	-0.3085	-0.4180	-0.5500
-0.1500	-0.2230	-0.3116	-0.4255	-0.5750
-0.2000	-0.2235	-0.3139	-0.4319	-0.6000
-0.2500	-0.2235	-0.3155	-0.4373	-0.6250
-0.3000	-0.2229	-0.3162	-0.4416	-0.6500
-0.3500	-0.2217	-0.3160	-0.4447	-0.6750
-0.4000	-0.2197	-0.3150	-0.4466	-0.7000
-0.4500	-0.2171	-0.3129	-0.4472	-0.7250
-0.5000	-0.2137	-0.3098	-0.4464	-0.7500
-0.5500	-0.2095	-0.3055	-0.4441	-0.7750
-0.6000	-0.2044	-0.3000	-0.4400	-0.8000
-0.6500	-0.1981	-0.2930	-0.4340	-0.8250
-0.7000	-0.1906	-0.2842	-0.4257	-0.8500
-0.7500	-0.1817	-0.2734	-0.4146	-0.8750
-0.8000	-0.1708	-0.2600	-0.4000	-0.9000
-0.8500	-0.1573	-0.2430	-0.3807	-0.9250
-0.9000	-0.1400	-0.2208	-0.3544	-0.9500
-0.9500	-0.1156	-0.1887	-0.3149	-0.9750

UPPER BOUND

cos(u,v)	delta=0.05	delta=0.1	delta=0.2	delta=1.0
0.9500	0.1156	0.1887	0.3149	0.9750
0.9000	0.1400	0.2208	0.3544	0.9500
0.8500	0.1573	0.2430	0.3807	0.9250
0.8000	0.1708	0.2600	0.4000	0.9000
0.7500	0.1817	0.2734	0.4146	0.8750
0.7000	0.1906	0.2842	0.4257	0.8500
0.6500	0.1981	0.2930	0.4340	0.8250
0.6000	0.2044	0.3000	0.4400	0.8000
0.5500	0.2095	0.3055	0.4441	0.7750
0.5000	0.2137	0.3098	0.4464	0.7500
0.4500	0.2171	0.3129	0.4472	0.7250
0.4000	0.2197	0.3150	0.4466	0.7000
0.3500	0.2217	0.3160	0.4447	0.6750
0.3000	0.2229	0.3162	0.4416	0.6500
0.2500	0.2235	0.3155	0.4373	0.6250
0.2000	0.2235	0.3139	0.4319	0.6000
0.1500	0.2230	0.3116	0.4255	0.5750
0.1000	0.2219	0.3085	0.4180	0.5500
0.0500	0.2202	0.3046	0.4095	0.5250
-0.0000	0.2179	0.3000	0.4000	0.5000
-0.0500	0.2152	0.2946	0.3895	0.4750
-0.1000	0.2119	0.2885	0.3780	0.4500
-0.1500	0.2080	0.2816	0.3655	0.4250
-0.2000	0.2035	0.2739	0.3519	0.4000
-0.2500	0.1985	0.2655	0.3373	0.3750
-0.3000	0.1929	0.2562	0.3216	0.3500
-0.3500	0.1867	0.2460	0.3047	0.3250
-0.4000	0.1797	0.2350	0.2866	0.3000
-0.4500	0.1721	0.2229	0.2672	0.2750
-0.5000	0.1637	0.2098	0.2464	0.2500
-0.5500	0.1545	0.1955	0.2241	0.2250
-0.6000	0.1444	0.1800	0.2000	0.2000
-0.6500	0.1331	0.1630	0.1750	0.1750
-0.7000	0.1206	0.1442	0.1500	0.1500
-0.7500	0.1067	0.1234	0.1250	0.1250
-0.8000	0.0908	0.1000	0.1000	0.1000
-0.8500	0.0723	0.0750	0.0750	0.0750
-0.9000	0.0500	0.0500	0.0500	0.0500
-0.9500	0.0250	0.0250	0.0250	0.0250

Appendix III

Application to the analysis of the velocity of money.
(See main text for the description of the model.)

The following plots of restricted extreme bounds show the range of coefficient estimates as a function of the fraction delta of the possible range of correlation coefficients.

The graphs contain lines at the 5%, 10% and 20% fraction of that range as well as a horizontal line for the coefficient estimates from the restricted model.

Model I and II - velocity in first differences

```
EQUATION     1
DEPENDENT VARIABLE    13       DVELOC
FROM     49:  3  UNTIL     87:  4
OBSERVATIONS              154      DEGREES OF FREEDOM     113
R**2               .79616921       RBAR**2             .72401672
SSR          .14906528             SEE               .36320267E-01
DURBIN-WATSON  1.99182778
Q(  36)=     32.4226          SIGNIFICANCE LEVEL  .639492
```

NO.	LABEL	VAR	LAG	COEFFICIENT	STAND. ERROR	T-STATISTIC
***	*******	***	***	***********	***********	***********
1	TBILL	4	0	-.1415851E-01	.6046652E-02	-2.341545
2	TBILL	4	1	.5154808E-01	.8464049E-02	6.090239
3	TBILL	4	2	-.6019413E-01	.9717375E-02	-6.194485
4	TBILL	4	3	.2764187E-01	.1107695E-01	2.495442
5	TBILL	4	4	-.1466998E-01	.1061456E-01	-1.382062
6	TBILL	4	5	.1365635E-01	.9594341E-02	1.423375
7	TBILL	4	6	-.4828803E-03	.7097789E-02	-.6803250E-01
8	CONST	1	0	.3657343E-01	.1214622E-01	3.011096
9	TREND	2	0	-.3399892E-03	.4729782E-03	-.7188264
10	NOMGNP	11	0	.2373780E-02	.2374066E-03	9.998798
11	NOMGNP	11	1	-.2806166E-02	.3948654E-03	-7.106641
12	NOMGNP	11	2	.8014433E-03	.5128168E-03	1.562826
13	NOMGNP	11	3	-.3222367E-05	.5336357E-03	-.6038515E-02
14	NOMGNP	11	4	-.3000919E-03	.5087625E-03	-.5898468
15	NOMGNP	11	5	-.8399701E-05	.5102534E-03	-.1646182E-01
16	NOMGNP	11	6	-.1401326E-03	.3668422E-03	-.3819969
17	INFLA	14	0	-.7337027E-03	.1798490E-02	-.4079549
18	INFLA	14	1	.4480048E-02	.2163098E-02	2.071125
19	INFLA	14	2	-.6417090E-04	.2147990E-02	-.2987486E-01
20	INFLA	14	3	-.5262394E-03	.2036806E-02	-.2583650
21	INFLA	14	4	-.1233611E-02	.2015203E-02	-.6121521
22	INFLA	14	5	-.1795641E-03	.1721894E-02	-.1042829
23	SP500	7	0	-.7737140E-03	.5121064E-03	-1.510846
24	SP500	7	1	.1459134E-02	.8364934E-03	1.744346
25	SP500	7	2	-.1538187E-02	.1096999E-02	-1.402178
26	SP500	7	3	.8227908E-03	.1186250E-02	.6936068
27	SP500	7	4	-.1571696E-02	.1356365E-02	-1.158756
28	SP500	7	5	.4283050E-02	.1347113E-02	3.179430
29	SP500	7	6	-.2404061E-02	.8705775E-03	-2.761455
30	MBVARIAB	10	0	-.4000235E-02	.1166322	-.3429785E-01
31	MBVARIAB	10	1	.2238494E-01	.1910302	.1171802
32	MBVARIAB	10	2	-.1565780E-01	.1762333	-.8884698E-01
33	MBVARIAB	10	3	.2782646E-01	.1583921	.1756808
34	MBVARIAB	10	4	.7859654E-01	.1644100	.4780521
35	MBVARIAB	10	5	-.7316997E-01	.1732389	-.4223644
36	MBVARIAB	10	6	-.1564823E-01	.1130170	-.1384591
37	DVELOC	13	1	.1589101	.9326050E-01	1.703938
38	DVELOC	13	2	-.9156270E-01	.1040530	-.8799620
39	DVELOC	13	3	-.1339621	.1018180	-1.315701
40	DVELOC	13	4	.4803443E-01	.1009783	.4756907
41	DVELOC	13	5	-.1157085	.9337908E-01	-1.239127

STATISTICS ON SERIES 13 DVELOC 154 OBSERVATIONS
FROM 49: 3 UNTIL 87: 4

SAMPLE MEAN	.2411900E-01	VARIANCE	.4779861E-0
STANDARD DEVIATION	.6913654E-01	STAN. DEV. OF MEAN	.5571181E-0
T-STAT FOR MEAN=0	4.329244	SIGNIFICANCE LEVEL	.1496223E-0

Model I - velocity in first differences

EQUATION 2
DEPENDENT VARIABLE 13 DVELOC
FROM 49: 3 UNTIL 87: 4
OBSERVATIONS 154 DEGREES OF FREEDOM 147
R**2 .43596533 RBAR**2 .41294350
SSR .41248914 SEE .52972149E-01
DURBIN-WATSON 1.79940473
Q(36)= 30.1241 SIGNIFICANCE LEVEL .743578

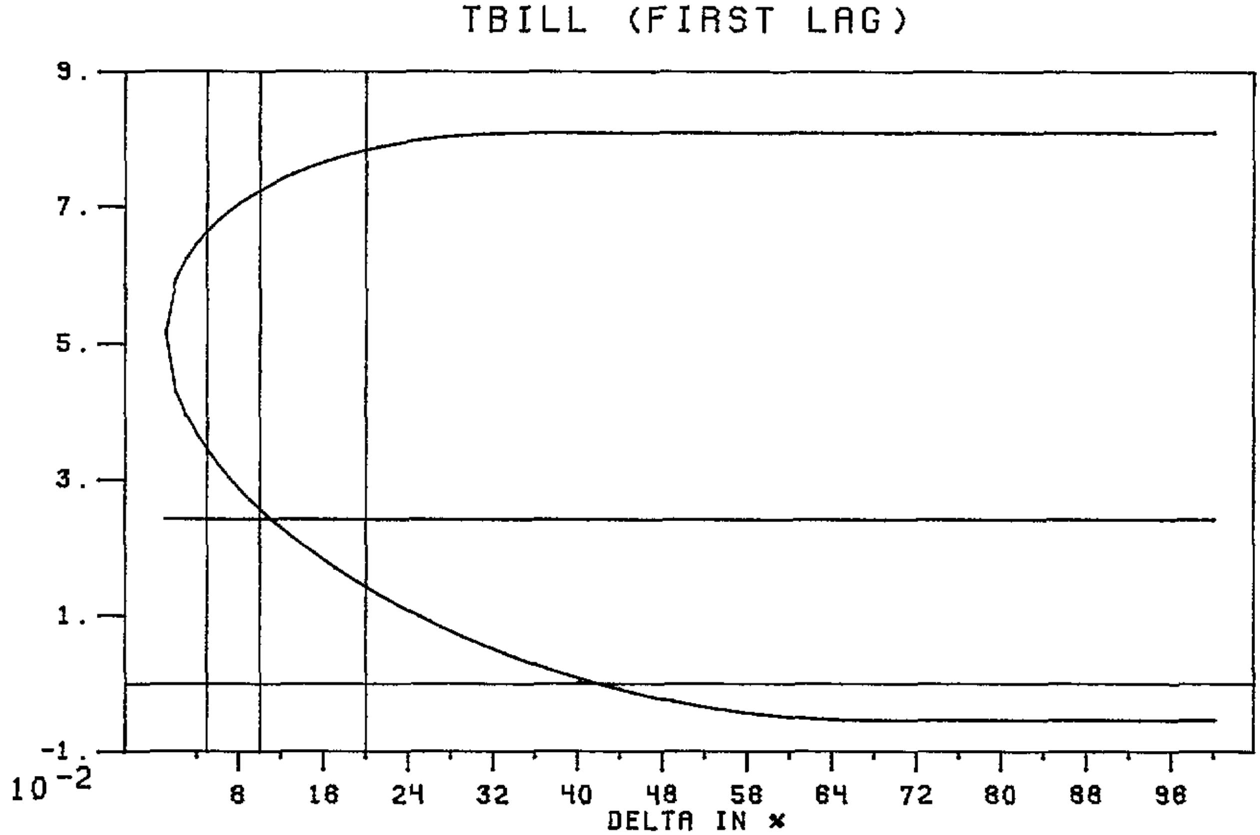
NO.	LABEL	VAR	LAG	COEFFICIENT	STAND. ERROR	T-STATISTIC
***	*******	***	***	************	************	************
1	TBILL	4	0	.1654119E-01	.5560119E-02	2.974970
2	TBILL	4	1	.2414939E-01	.8525709E-02	2.832538
3	TBILL	4	2	-.4186459E-01	.5846300E-02	-7.160870
4	CONST	1	0	.2961311E-01	.9758004E-02	3.034750
5	TREND	2	0	-.8813109E-04	.1908511E-03	-.4617793
6	DVELOC	13	1	.1759825	.7147110E-01	2.462289
7	DVELOC	13	2	.8176570E-01	.7137294E-01	1.145612

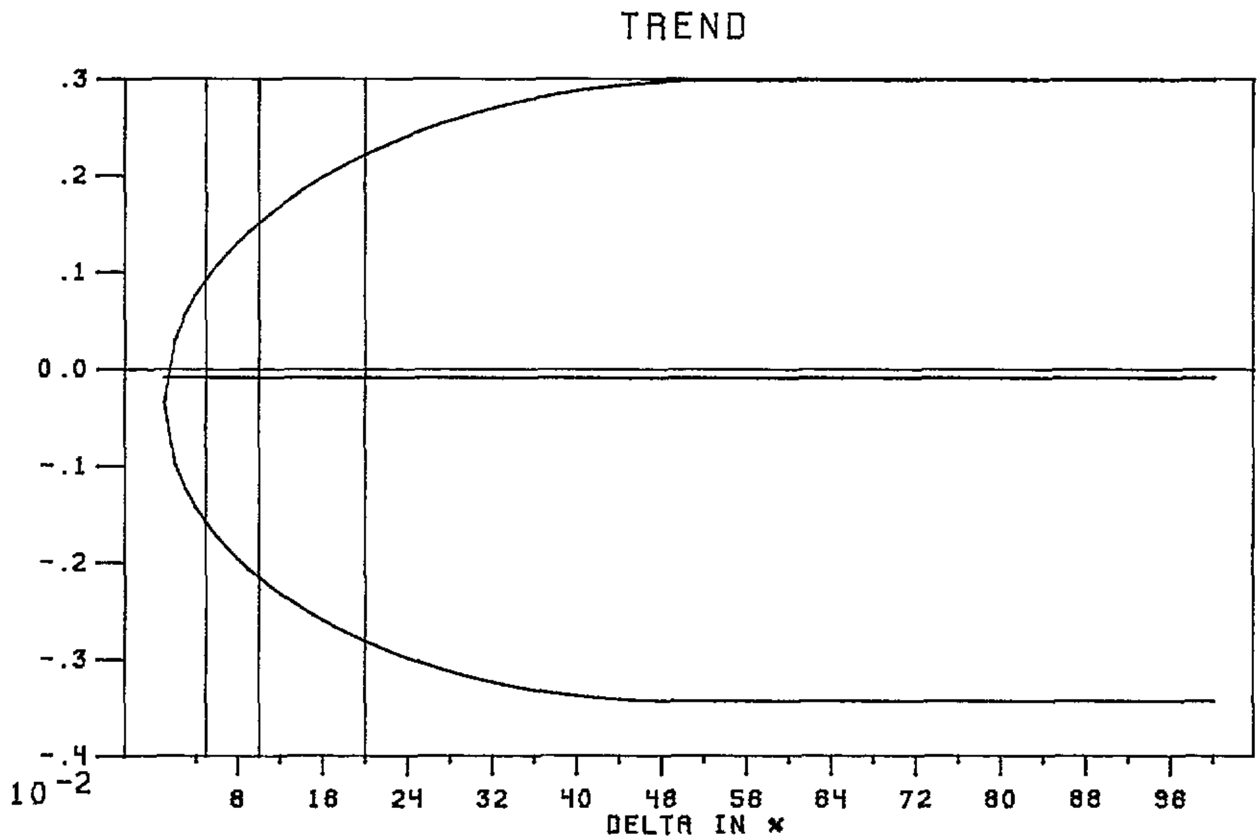
Model II - velocity in first differences

EQUATION 2
DEPENDENT VARIABLE 13 DVELOC
FROM 49: 3 UNTIL 87: 4
OBSERVATIONS 154 DEGREES OF FREEDOM 142
R**2 .47591604 RBAR**2 .43531799
SSR .38327243 SEE .51952879E-01
DURBIN-WATSON 1.78195047
Q(36)= 39.9104 SIGNIFICANCE LEVEL .300441

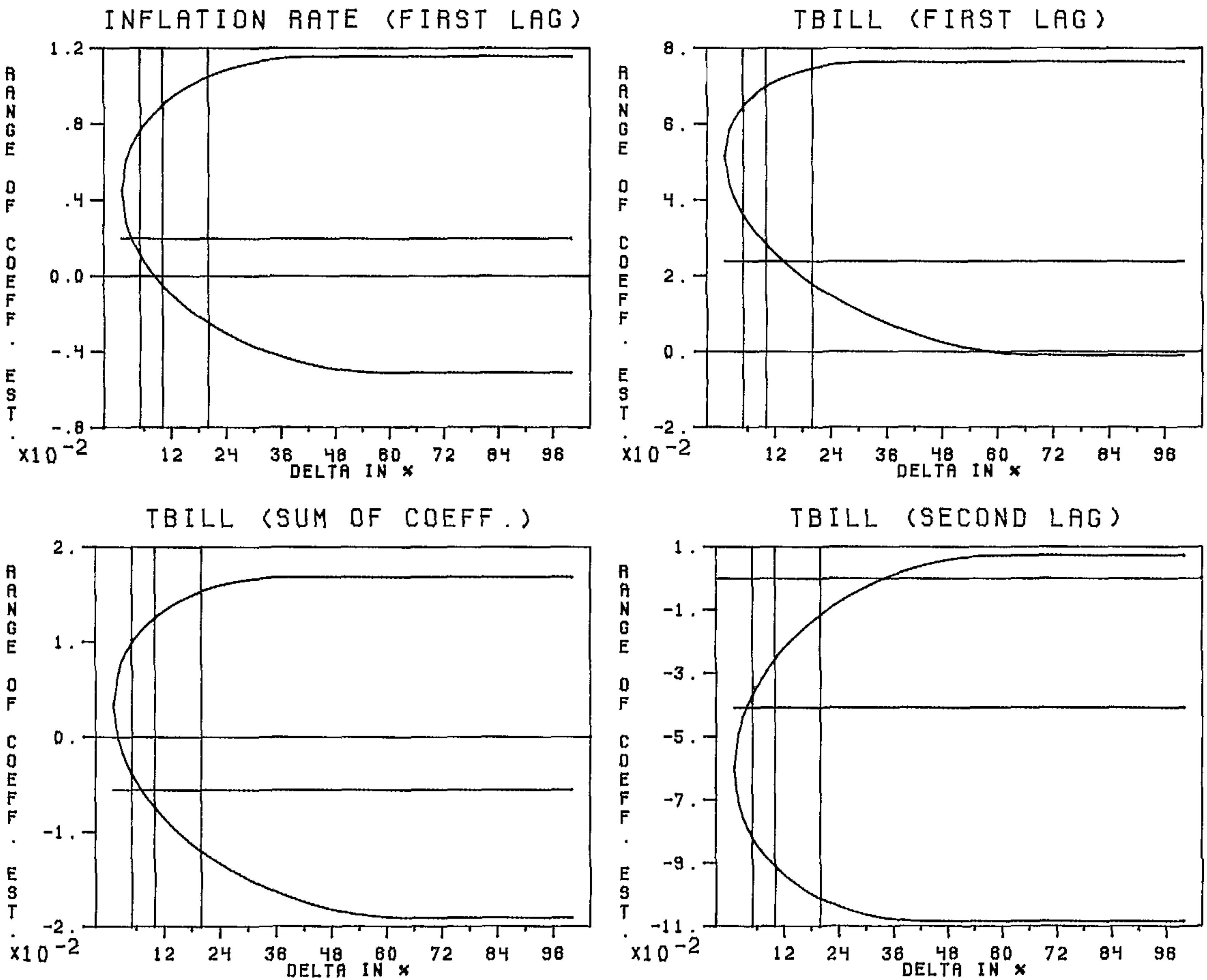
NO.	LABEL	VAR	LAG	COEFFICIENT	STAND. ERROR	T-STATISTIC
***	*******	***	***	************	************	************
1	INFLA	14	0	.2746374E-02	.2175553E-02	1.262380
2	INFLA	14	1	.1954262E-02	.2522484E-02	.7747373
3	INFLA	14	2	.5396981E-03	.2162828E-02	.2495335
4	TBILL	4	0	.1166279E-01	.5936678E-02	1.964532
5	TBILL	4	1	.2372811E-01	.8803060E-02	2.695439
6	TBILL	4	2	-.4102008E-01	.6235978E-02	-6.577970
7	MBVARIAB	10	0	.1715579	.1227636	1.397465
8	MBVARIAB	10	1	-.1202288	.1241446	-.9684580
9	CONST	1	0	.2572789E-01	.1118883E-01	2.299427
10	TREND	2	0	-.6990102E-04	.1896833E-03	-.3685144
11	DVELOC	13	1	.1209456	.7336283E-01	1.648596
12	DVELOC	13	2	.4979967E-01	.7244218E-01	.6874402

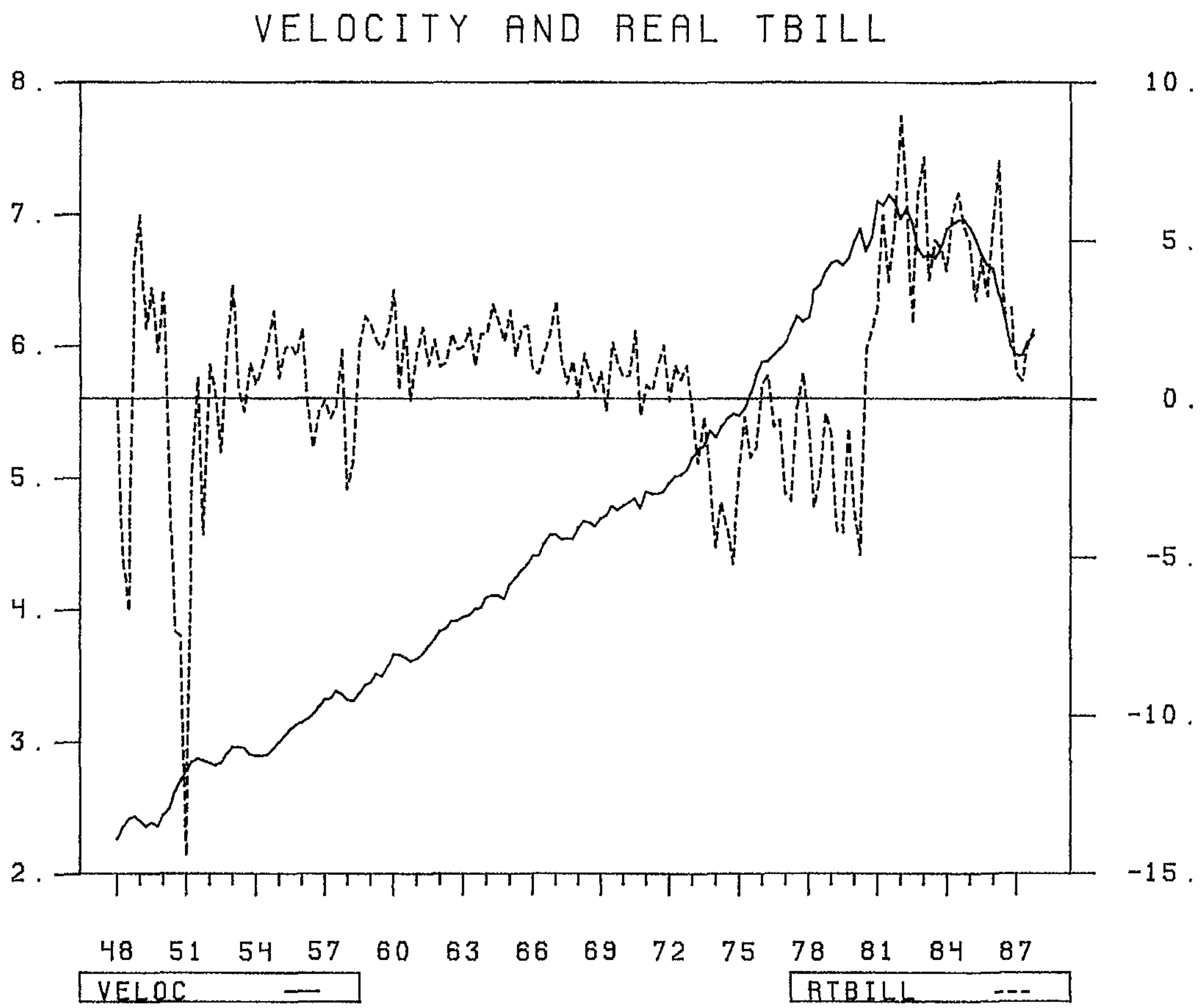
model I


TBILL (FIRST LAG)
9.
7.
5.
3.
1.
-1.
10^-2
8 16 24 32 40 48 56 64 72 80 88 96
DELTA IN %


TREND
.3
.2
.1
0.0
-.1
-.2
-.3
-.4
10^-2
8 16 24 32 40 48 56 64 72 80 88 96
DELTA IN %

model II


INFLATION RATE (FIRST LAG)
TBILL (FIRST LAG)
TBILL (SUM OF COEFF.)
TBILL (SECOND LAG)
RANGE OF COEFF. EST.
DELTA IN %
X10 -2


VELOCITY AND REAL TBILL
8.
7.
6.
5.
4.
3.
2.
10.
5.
0.
-5.
-10.
-15.
48 51 54 57 60 63 66 69 72 75 78 81 84 87
VELOC ——
RTBILL ---

References

M.McAleer, A.R.Pagan, P.A.Volker, "What will take the con out of econometrics", AER June 1985, Vol 75#3, pp 293 – 307.

T.S.Breusch, "Simplified extreme bounds", University of Southamption, U.K, Oct. 1985, working paper

M. Friedman "Lessons from the 1979–82 monetary policy experiment", the American Economic Review, papers and proceedings 74 (May 1984), 397–400

T.E. Hall, N.R. Noble "Velocity and the variability of money growth: evidence from Granger–causality tests", J.Money, Credit and Banking, Vol. 19, No.1, Feb. 1987, pp. 112 –116

E.E. Leamer, "Coordinate–free ridge regression bounds", J. Amer. Stat. Ass., Dec. 1981, Vol 76,#376, pp 842–849

E.E. Leamer, "Let's take the con out of econometrics", AER 1983

E.E. Leamer, "Sensitivity analysis would help", AER June 1985, Vol 75 #3, pp 308 – 313.

C.A.Sims, J.H.Stock, M.W.Watson, "Inference in time series models with some unit roots", mimeo Nov. 1986

C.A.Sims, Skeptical remarks on unit roots econometrics", mimeo Jan. 1988