St. Petersburg State University
Institute of Management

DISCUSSION PAPPER

Nicolay Zenkevich
Leon Petrosjan

TIME-CONSISTENCY OF COOPERATIVE SOLUTIONS

#20(E)–2006

Saint Petersburg
2006

Zenkevich Nicolay, Petrosjan Leon Time-consistency of cooperative solutions. Discussion Paper # 20(E)–2006. Institute of Management, St. Petersburg State University: SPb., 2006.

Management in complex social and economics networks includes elements of cooperative behavior or full cooperation between agents involved in decision making process. The most appropriate mathematical tool for modeling in this case is the mathematical theory of cooperative games. Unfortunately the classical cooperative game theory considers cooperation as one-shot interaction between the decision makers and for this reason can not be used for modeling of dynamic interactions arising in long term strategic management. The theory of cooperative differential games is the most adequate tool for modeling strategic management development on a given time interval. The use of this theory from the beginning poses the problems connected with dynamic stability (time-consistency) of optimal cooperative solutions. The consideration of optimality principles taken from the classical cooperative game theory shows the time-inconsistency and thus non applicability of these principles in strategic management. In this paper the methods of construction of time consistent solutions is proposed for the problems of strategic management in social and economic networks. The authors tried to present a rather complicated material on acceptable level. Theory is illustrated with a number of examples and a more comprehensive analysis of joint venture.

Nikolay A. Zenkevich, Associate Professor, Ph. D, Strategic Management Department, School of Management, St. Petersburg State University
zenkevich@som.pu.ru

Leon A. Petrosjan, Professor, Dean, D. Sc., Head of Mathematical Game Theory and Statistical Decisions Department, School of Applied Mathematics, St. Petersburg State University
spbuoasis7@peterlink.ru

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INTRODUCTION

Advances in technology, communications, industrial organization, regulation methodology, international trade, economic integration and political reform have created rapidly expanding social and economic networks incorporating cross-personal and cross-country activities and interactions. From a decision- and policy-maker's perspective, it has become increasing important to recognize and accommodate the interdependencies and interactions of human decisions under such circumstances. The strategic aspects of decision making are often crucial in areas as diverse as trade negotiation, foreign and domestic investment, multinational pollution planning, market development and integration, technological R&D, resource extraction, competitive marketing, regional cooperation, military policies, and arms control.

Game theory has greatly enhanced our understanding of decision making. As socioeconomic and political problems increase in complexity, further advances in the theory's analytical content, methodology, techniques and applications as well as case studies and empirical investigations are urgently required. In the social sciences, economics and finance are the fields which most vividly display the characteristics of games. Not only would research be directed towards more realistic and relevant analysis of economic and social decision-making, but the game-theoretic approach is likely to reveal new and interesting questions and problems, especially in management science.

The origin of differential games traces back to the late 1940s. Rufus Isaacs modeled missile versus enemy aircraft pursuit schemes in terms of descriptive and navigation variables (state and control), and formulated a fundamental principle called the tenet of transition. For various reasons, Isaacs's work did not appear in print until 1965. In the meantime, control theory reached its maturity in the Optimal Control Theory of Pontryagin et al. (1962) and Bellman's Dynamic Programming (1957). Research in differential games focused in the first place on extending control theory to incorporate strategic behavior. In particular, applications of dynamic programming improved Isaacs' results. Berkovitz (1964) developed a variation approach to differential games, and Leitmann and Mon (1967) investigated the geometry of differential games. Pontryagin (1966) solved differential games in open-loop solution in terms of the maximum principle.

First paper about differential games in Soviet Union appeared in 1965 [Krasovsky, 1966; Petrosjan, 1965; Pontryagin,1967].

Research in differential game theory continues to appear over a large number of fields and areas. Applications in economics and management science are surveyed in Dockner et al. (2000). In the general literature, derivation of open-loop equilibria in nonzero-sum deterministic differential games first appeared in Petrosjan, Murzov (1967); Case (1967, 1969) and Starr and Ho (1969a, 1969b) were the first to study open-loop and feedback Nash equilibria in nonzero-sum deterministic differential games. While open-loop solutions are relatively tractable and easy-to-apply, feedback solutions avoid time inconsistency at the expense of reduced intractability. In following research, differential games solved in feedback Nash format were presented by Clemhout and Wan (1974), Fershtman (1987), Jorgensen (1985), Jorgensen and Sorger (1990), Leitmann and Schmitendorf (1978), Lukes (1971a, 1971b), Sorger (1989), and Yeung (1987, 1989, 1992, 1994).

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. Formulation of optimal behavior for players is a fundamental element in this theory. In dynamic cooperative games, a stringent condition on cooperation and agreement is required: In the solution, the optimality principle must remain optimal throughout the game, at any instant of time along the optimal state trajectory determined at the outset. This condition is known as dynamic stability or time consistency. In other words, dynamic stability of solutions to any cooperative differential game involved the property that, as the game proceeds along an optimal trajectory, players are guided by the same optimality principle at each instant of time, and hence do not possess incentives to deviate from the previously adopted optimal behavior throughout the game.

The question of dynamic stability in differential games has been rigorously explored in the past three decades. Haurie (1976) raised the problem of instability when the Nash bargaining solution is extended to differential games. Petrosjan (1977) formalized the notion of dynamic stability (time consistency) in solutions of differential games. Kydland and Prescott (1977) found time inconsistency of optimal plans (Nobel Prize 2005). Petrosjan and Danilov (1982) introduced the notion of "imputation distribution procedure" for cooperative solution. Tolwinski et al. (1986) investigated cooperative equilibria in differential games in which memory-dependent strategies and threats are introduced to maintain the agreed-upon control path. Petrosjan (1993) and Petrosjan and Zenkevich (1996) presented a detailed analysis of dynamic stability in cooperative differential games, in which the method of regularization was introduced to construct time-consistent solutions. Yeung and Petrosjan (2001) designed time-consistent solutions in differential games and characterized the conditions that the allocation-distribution procedure must satisfy. Petrosjan (2003) employed the regularization method to construct time-consistent bargaining procedures. Petrosjan and Zaccour (2003) presented time-consistent Shapley value allocation in a differential game of pollution cost reduction.

In the field of cooperative stochastic differential games, little research has been published to date, mainly because of difficulties in deriving tractable time-consistent solutions. Haurie et al. (1994) derived cooperative equilibria in a stochastic differential game of fishery with the use of monitoring and memory strategies. In the presence of stochastic elements, a more stringent condition – that of *subgame consistency* – is required for a credible cooperative solution. In particular, a cooperative solution is subgame-consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behavior would remain optimal.

As pointed out by Jorgensen and Zaccour (2002) conditions ensuring time consistency of cooperative solutions are generally stringent and intractable. A significant breakthrough in the study of cooperative stochastic differential games can be found in the recent work of Yeung and Petrosjan (2004). In particular, these authors developed a generalized theorem for the derivation of an analytically tractable "payoff distribution procedure" which would lead to subgame-consistent solutions. In offering analytical tractable solutions, Yeung and Petrosjan’s work is not only theoretically interesting in itself, but would enable hitherto insurmountable problems in cooperative stochastic differential games to be fruitfully explored.

When payoffs are nontransferable in cooperative games, the solution mechanism becomes extremely complicated and intractable. Recently, a subgame-consistent solution was constructed by Yeung and Petrosjan (2005) for a class of cooperative stochastic differential games with nontransferable payoffs. The problem of obtaining subgame-consistent cooperative solutions has been rendered tractable for the first time.

Stochastic dynamic cooperation represents perhaps decision-making in its most complex form. Interactions between strategic behavior, dynamic evolution and stochastic elements have to be considered simultaneously in the process, thereby leading to enormous difficulties in the way of satisfactory analysis. Despite urgent calls for cooperation in the politics, environmental control, the global economy and arms control, the absence of formal solutions has precluded rigorous analysis of this problem.

1. COOPERATIVE SOLUTIONS

It is essential to begin with basic definitions. Since the main subject of the paper is game theory applications in management studies, corresponding models and solutions will be considered.

In *general* we treat *cooperative solution* as solution of participants (players) joined by will to make decision about actual problem. Suppose that such decision requires players' behavior coordination guarantied by an agreement. Thus, cooperation means any coordinated agreement of parties involved. Consideration of time consistency problems is directly connected with cooperative solutions in such general context.

Cooperative decision problems appear in various fields of management and management science. Note the problem of signing of contract as a result of a given agreement. In strategic management, this could be merger and takeover, strategic alliance agreements and other type's inter-firm cooperation. In financial management – long term investment decisions. On a firm level this is a long term agreement between owners and managers about profit distribution. There are many other examples. At the same time cooperative solutions are possible in legal contracts or agreement forms with legal or not, obvious or secret aims. More complicated cooperative agreement forms are possible also.

In analyzing cooperative decision making some important aspects are usually considered. Firstly, what are participants' motivations to make cooperative decision? If such motivations exist, are they sufficient? Often categories of utility and equity of coordinated agreement serve as such motivation. Secondly, what coordinated agreement is to be chosen as optimal (what optimality principle is to be chosen)? How to choose optimal solution (what is algorithm of decision making)? Thirdly, how to realize the cooperation solutions? In this paper we will be interested in behavior of cooperative solutions in time, so the third question of decision making will be a key problem.

Cooperative solutions in general are divided to static and dynamic. In static case solution is made once, instantaneously realized and players get the outcomes right away. In spite of seeming simplicity of such an approach, classic game theory deals with static models. However, management and management science deals with control, and therefore – with processes evolving in time (with conflict processes in our case). To understand cooperative solution concept, it is necessary to begin with consideration of static game.

Game in normal form Γ is defined as:

where is the set of players, – the set of strategies ( ), – the payoff function of player .

What is the solution of the game ? The answer to the question is given by concepts (principles) of optimality, formulated in the definitions below. In general the solution is the set of - tuples of strategies , satisfying required optimality conditions. Nash equilibrium is most widespread optimality concept in nonzero sum game theory.

Definition 1. [Nash, 1951] The - tuple of strategies is called *Nash equilibrium* if for all and the following inequalities holds

.

Nash Equilibrium (NE – solution) is cooperative solution in general, because the choice of such solution requires coordinated players’ behavior. If there is more than one NE – solution, the following notice is especially important. In such case players also have to agree what NE – solution they would realize, since the payoffs in different NE – solutions are different in general.

Definition 2. The - tuple of strategies is called *Pareto optimal* if there is no such - tuple of strategies that the following inequalities hold for all :

and for at least one :

.

There may be many Pareto optimal solutions with different payoffs for players. This is the reason why Pareto optimal solution (PO - solution) is also a cooperative solution, because choosing such solution requires co-ordinated players' behavior and contains the property of group rationality.

Typical representative of Pareto optimal solution is *Nash bargaining solution.* Nash bargaining solution is the solution of the optimization problem [Nash, 1950]:

subject to

.

Here is given “reference solution”, defining the “status quo” point , . Nash bargaining solution (*NB - solution*) is cooperative solution which selects a special Pareto optimal solution.

Another representative of Pareto optimal solution is *Kalai-Smorodinskiy bargaining solution.*

Kalai-Smorodinskiy bargaining solution is the solution of the following optimization problem [Kalai, Smorodinskiy, 1975]:

subject to

,

,

where is given “reference solution”, defined by the “status quo” point , and by “ideal” point ,

.

Usually it is not possible to obtain ideal point in any solution (else, this point would be optimal solution), i. e. it doesn’t belong to the set of feasible estimates. Geometrically Kalai-Smorodinsky solution is defined intersection point of the line segment connecting “status quo” and “ideal” points with the set of feasible estimates. Note that Kalai-Smorodinsky bargaining solution (*KS - solution*) is cooperative solution in general as special case of Pareto optimal solutions.

All mentioned above optimality principles are strategic in sense that they are constructed on based of coordinated or joint strategy choice.

Consider now a special type of cooperative solution. Such cooperative solution concept assumes two-stage cooperation: selection of - tuple of strategies, which maximize the sum of players’ payoffs, and allocation of the aggregate maximal cooperation payoff.

Recall, that the *cooperative game* in characteristic function form is defined as a system:

,

where is the set of players, is characteristic function, possessing *the superadditivity property:*

The characteristic function value is often interpreted as *maximal guarantied payoff of coalition* , . From the superadditivity property of characteristic function we have , when . Therefore it is advantageous to create maximal coalition to obtain maximum possible aggregate payoff during game evolution.

Let be a cooperative game, constructed on the game structure (with transferable payoffs), where players play according to some accepted in advance optimality principle [Petrosjan, Zenkevich, 1996]. Then, as mentioned above, the value is interpreted as *maximal guarantied payoff of coalition* , i.e. maximum payoff of coalition in worst case, when other players create coalition to play against the coalition .

The agreement about how exactly realize cooperation and share the gain of joined cooperative payoff is *optimality principle of cooperative game solution*. In particular, a solution of cooperative game is

- Agreement about the cooperative - tuple of strategies, oriented on receiving maximal cooperative payoff
- Method for share of aggregate maximal payoff between participants.

The set of all allocations of maximal aggregate payoff is called *imputation set.* Denote player's payoff under cooperation, when aggregate cooperation payoff is .

Vector (aggregate payoff allocation)

,

is called *imputation* in game , if the following conditions are satisfied:

(i) , ,

(ii) ,

where is the value of characteristic function computed for singleton coalition .

The condition guaranties individual rationality, i.e. every player obtains at least as much as the maximal payoff in case she plays against all other players. The condition guaranties Pareto optimality for the imputation and therefore group rationality.

Denote imputation set in game by . *Cooperative optimality principle* in the game is a fixed subset of imputation set . If optimality principle is chosen, then the imputation is called *optimal* according to given optimality principle .

Definition 3. The imputation belongs to the *core* of game , if for every coalition the following inequalities are satisfied:

The core is denoted by . The sense of cooperative solution from the core is obvious: if the imputation from the core is chosen, then every coalition of players gets at least as much as he could get playing independently.

Definition 4. [Shapley, 1953]. The imputation

is called Shapley value, if it is obtained as

There exist many others cooperative optimality principles, for example: Neyman-Morgenshtern solution, N-core, nucleus. In all cases they are some subsets of the game imputation set.

2. TIME CONSISTENCY OF COOPERATIVE SOLUTION PROBLEM.

In previous section we considered static concepts of cooperative solutions. However, management and management science deals with control, and therefore – with processes (with a conflict evaluation of a large system in time). Control is chosen at initial moment and realized on a given time interval.

2.1 Dynamic stability (time consistency) of optimal control problems.

Illustrate the time consistency property of optimal control on a classical example.

Let is a given point which defines in some sense “ideal” state of the system under consideration. Consider the following classical management (control) problem. Let

(1)

be the system of differential equations, where is state variable, control (management) variable which is selected continuously at each time instant .

The system develops on a given time interval . The aim of the management is to bring the initial point (initial state of the system) as close as possible to a given point at the terminal moment . Mathematically this means that the aim of the management is to find such an open-loop control which minimizes the distance between the terminal point and the point .

Construct the reachability set of the system (1) denoted by from the initial state at the terminal moment . is the set of such points which can arise exactly at the terminal moment when all possible open-loop controls are used from the initial position according to the system (1).

Denote our minimization problem by to underline the dependence of the problem from the initial condition and the duration of the process .

For simplicity reasons suppose that the point does not belong to , i.e. .

This means that the point can not be reached from the initial state during the time . The optimality principle in this optimal management problem is to minimize the distance between the point and the point .

It is clear that the "optimal motion" or "optimal trajectory" has to bring the initial point to the point ( ) – the closest point of the reachability set to the point . Denote by the trajectory connecting and realized under optimal (fixed) open loop control , i.e.

Suppose that the process is evaluating along the trajectory as shown on the fig.1. Consider an intermediate time instant and suppose that at this time instant we want to check will the point remain the closest to the point in the sub-problem with the initial condition on the optimal trajectory and duration ? It is evident that the answer will be “yes”. This means that the continuation of the optimal motion along on the time interval will remain optimal in the subproblem (see fig.1). This means time consistency or dynamic stability of the optimal trajectory . This was first formulated by R. Bellman (1957) and lies in the bases of dynamic programming. Time consistency nearly always holds in the classical optimal control problems.

At the same time we can see that in this case a stronger condition holds (this was not mentioned by R. Bellman). In the subproblem a new optimal trajectory and a corresponding optimal control can arise leading from the initial point in the subproblem to the point . It is interesting to mention that the open-loop control of the form transfers the point to the point in the problem and thus is optimal in the problem .

Fig. 1 Dynamic stability of optimal control

So, we get that any optimal prolongation in the subproblem together with initially selected optimal motion on the time-interval in is also optimal in . This property we call “strong dynamic stability”.

The notion of strong dynamic stability was first introduced practically simultaneously and independently by L. Petrosyan (1979) and S. Chistyakov (1981).

2.2 Time consistency (dynamic stability) of Pareto optimal solutions in multycriterial control problems.

As in the previous section the management is described by the system of differential equations

Here as in the previous case is the state variable and control variable selected by the manager continuously at each time-instant from a given set .

The system develops on the time interval . The difference with the previous problem is in the fact that in this case the quality of the management is evaluated by a number of parameters (in the previous optimal control problem the quality of the management was evaluated only by one single parameter – the distance from a given fixed point). The aim of the management in this case is to bring the initial point as close as possible to a finite number of fixed points

. Mathematically the problem is to minimize the vector criteria

,

where and is the terminal state of the management process.

Since we have here a multycriterial optimization problem, as optimality principal we have to consider a Pareto optimal set.

As before let be the reachibility set of the system (1) and denote our optimization problem by to underline the dependence of the problem from the initial condition and duration

. Denote by the convex hull of the points . For simplicity reasons suppose that

It can be shown that the set of all Pareto optimal trajectories coincides with those with endpoint on the projection of the convex hull on the reachibility set.

Denote by a trajectory connecting the initial state with the some fixed point on the projection of the convex hull on the reachibility set , and let be the corresponding open-loop control.

We shall call optimal trajectory. It is clear that in our problem we may have an infinite number of optimal trajectories with non comparable outcomes in the sense of the different values for distances to the aim-points, since in general the projection of the set on is closed and may contain infinite number of points.

Consider now an intermediate time instant and ask ourselves will the continuation of the optimal trajectory be optimal in subproblem starting in the state on the optimal trajectory with duration . By other words: will the point remain Pareto-optimal in the subproblem .

Fig. 2 Strong dynamic instability for Pareto optimal solution

As in an optimal control problem considered in the previous section the answer will be “yes”, thus the prolongation of the optimal trajectory in the subproblem remains optimal (Pareto-optimal) in this subproblem.

In the same time as it is seen from the fig. 2 the Pareto optimal set in problem coincides with the arc (projection of the set on the reachibility set) and differs from the Pareto-optimal set in the subproblem which coincides with the arc (projection of the set on the reachibility set).

But both sets have one point in common. Thus we see that in the subproblem there are new optimal (Pareto-optimal) trajectories with endpoints out of the Pareto-optimal set of the previous problem .

Consider the following open-loop control

where is the segment of the optimal Pareto-optimal control in the problem and is the Pareto-optimal control in the subproblem which transfers the initial point in the subproblem to the point .

Since the point does not belong to the arc the open-loop control is not Pareto-optimal in the problem .

And we come to the conclusion that not any Pareto-optimal prolongation in the subproblem with initial conditions on the Pareto-optimal motion in the previous problem (problem) is Pareto-optimal in .

This means that Pareto-optimal solutions in general are not strongly dynamic stable or strongly time-consistent.

We see that by transition to multycriterial control problems we loose strong time consistency of optimal solutions. This arise difficulties in the practical implementation of optimal solutions in multycriterial control problems, because in some intermediate time instant the manager can change to another Pareto-optimal solution (considering this solution for some reason as more attractive) and loose Pareto-optimality of the hole process. This implies instability in long term management and is unacceptable for practical use.

2.3 TIME INCONSISTENCY OF SPECIALLY SELECTED COOPERATIVE SOLUTION

The problem of choosing specific Pareto optimal solution is more complicated, than in the case considered above. Most of optimality principles (even in non-game theoretical problems), determining the choice of specific Pareto optimal solution from the set of all Pareto optimal solutions are not only strongly dynamically unstable (not strongly time-consistent), but even dynamically unstable (time-inconsistent).

There are a number of approaches to choose a specific Pareto optimal solution from the set of all Pareto optimal solutions. Unfortunately, most complicated and well-defined of them are dynamically unstable (time-inconsistent). Illustrate it with an example. Consider the choice of Pareto optimal solution according to Kalai-Smorodinsky bargaining procedure. Pareto optimal solution chosen in such way, as we noticed earlier, is called Kalai-Smorodinsky solution, or - solution.

Now we suppose that the long term management process depends from the decisions made by different agents (players). Thus we shall consider the case when the right side of the differential equations (1) depend upon a number of parameters each one of them under control of corresponding agent (player) acting in his own interests. So we have the motion equations

, (2)

where the parameters (control variables) are chosen continuously in time by players.

For simplicity we shall suppose that each of the players is interested in a payoff which has the form

,

where is the solution of the system (2) corresponding to the choice of controls as functions of current state and time (strategies, feedback controls) and initial condition . As a result we have a differential game, which we shall denote by .

Denote by the set of all possible values of vectors

for all possible - tuples of strategies chosen by players.

Let be the Pareto frontier of the set . There are different ways for selection of a particular Pareto optimal point from the whole Pareto frontier. In this selection the so-called “status quo” plays an important role. Usually the status-quo point is vector with components , where each is equal to the maximal payoff the player can get in the worst case, when all other players are playing against him (not for themselves). Let

.

be the status-quo point. It is clear that this point depends from the initial state of the system and duration of the process . Denote by

.

The point

is called “ideal” point and has the meaning of maximal possible gains of the players. In general we have otherwise the ideal point will be the “solution” of the problem.

To define the KS-solution, draw a line segment connecting the status-quo point and the ideal point. Since the ideal point does not belong to the set , exist a point on the intersection of the set and this line segment closest to ideal point (we suppose that the set is closed and bounded). This point is called - solution. If the set is convex the solution is always Pareto optimal. It is easy to see that even in the simplest cases the - solution is not time-consistent (dynamic stable). For the illustration of this property consider the following very trivial example.

Suppose that the system (2) has the form

,

Show the time inconsistency of the – solution. Here the status-quo point in the problem is equal to and corresponds to the initial state of the system . The ideal point is , since . The reachibility set is a circle with the center and radius 4. The optimal trajectory (leading to

-solution) corresponds to the motion along the line segment from the initial point in direction to the point until the intersection with the circumference of the circle . This point of intersection defines the - solution of the problem (see fig.3).

On this figure we see that the - solution in a subproblem with the initial conditions on the optimal trajectory is different from the solution of the previously considered problem

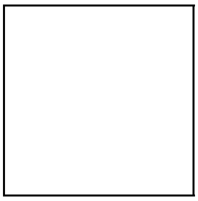

Fig. 3 Time inconsistency of - solution

This implies time inconsistency (dynamic instability) of -solution.

It is necessary to mention that not only -solution is time-inconsistent, but so are all nontrivial bargaining solutions based on the selection of status-quo points. This is also true for Nash bargaining solution.

3. REGULARIZATION OF COOPERATIVE OPTIMALITY PRINCIPLE.

Previous considerations imply that the majority of cooperative solutions are not time-consistent. Therefore, there are serious difficulties for their practical implementation and ultimately it is not possible to get stable solution results. Only classical optimal control solutions and Nash equilibrium with constant discount rate are dynamically stable (time-consistent).

Is there a way out of this problem? Yes. We shall explain this in the case of cooperative differential game.

3.1 Definition of cooperative differential game

We begin with the basic formulation of cooperative differential games in characteristic function form and the solution imputations.

Consider a general -person differential game in which the state dynamics has the form:

, (3)

The payoff of player is:

(4)

for

where denotes the state variables of game, and is the control of Player , for . In particular, the players' payoffs are transferable. A feedback Nash equilibrium solution can be characterized if the players play no cooperatively.

Now consider the case when the players agree to cooperate. Let denote a cooperative game with the game structure of in which the players agree to act according to an agreed upon optimality principle. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for a cooperative game includes

(i) an agreement on a set of cooperative strategies/controls, and

(ii)a mechanism to distribute total payoff among players.

The solution optimality principle will remain in effect along the cooperative state trajectory path . Moreover, group rationality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. In addition, the allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

To fulfill group rationality in the case of transferable payoffs, the players have to maximize the sum of their payoffs:

(5)

subject to (3).

A set of optimal controls is possible to be found using Pontryagin's maximum principle or Bellman equation. Substituting this set of optimal controls into (3) yields the optimal trajectory , where

. (6)

For notational convenience in subsequent exposition, we use and interchangeably.

We denote

by . Let and stands for a characteristic function reflecting the payoff of coalition . The quantity yields the maximized payoff to coalition as a rest of the players form a coalition to play against . Calling on the superadditivity property of characteristic functions, for . Hence, it is advantageous for the players to form a maximal coalition and obtain a maximum total payoff that is possible in the game.

One of the integral parts of cooperative game is to explore the possibility of forming coalitions and offer an "agreeable" distribution of the total cooperative payoff among players. In fact, the characteristic function framework displays the possibilities of coalitions in an effective manner and establishes a basis for formulating distribution schemes of the total payoffs that are acceptable to participating players.

We can use to denote a *cooperative differential game in characteristic function form*.

Denote

an arbitrary imputation, – core, Shapley value in the game .

3.2 IMPUTATION IN A DYNAMIC CONTEXT.

In dynamic games, the solution imputation along the cooperative trajectory would be of concern to the players. Now we focus our attention on the dynamic imputation brought about by the solution optimality principle.

Let an optimality principle be chosen in the game . The solution of this game constructed in the initial state based on the chosen principle of optimality contains the solution imputation set and the conditionally optimal trajectory which maximizes

.

Assume that .

Definition 5. Any trajectory of the system (3) such that

is called *a conditionally optimal trajectory in the game* .

Definition 5 suggests that along the conditionally optimal trajectory the players obtain the largest total payoff. For exposition sake, we assume that such a trajectory exits. Now we consider the behavior of the set along the conditionally optimal trajectory . At time with state , we define the current subgame with characteristic function and the set of imputations .

Consider the family of current games

,

and their solutions generated by the same principle of optimality that yields the initially solution .

Obviously, the set is the solution of current game at the moment and consist of single imputation

3.3 Principle of Dynamic Stability

Formulation of optimal behaviors for players is a fundamental element in the theory of cooperative games. The players' behaviors satisfying some specific optimality principles constitute a solution of the game. In other words, the solution of a cooperative game is generated by a set of optimality principles (for instance, the Shapley value (1953), the von Neumann Morgenstern solution (1944) and the Nash bargaining solution (1953)). For dynamic games, an additional stringent condition on their solutions is required: the specific optimality principle must remain optimal at any instant of time throughout the game along the optimal state trajectory chosen at the outset. This condition is known as *dynamic stability or time consistency*. Assume that at the start of the game the players adopt an optimality principle (which includes the consent to maximize the joint payoff and an agreed upon payoff distribution principle). When the game proceeds along the "optimal" trajectory, the state of the game changes and the optimality principle may not be feasible or remain optimal to all players. Then, some of the players will have an incentive to deviate from the initially chosen trajectory. If this happens, instability arises. In particular, the dynamic stability of a solution of a cooperative differential game is the property that, when the game proceeds along an "optimal" trajectory, at each instant of time the players are guided by the same optimality principles, and yet do not have any ground for deviation from the previously adopted "optimal" behavior throughout the game.

The question of dynamic stability in differential games has been explored in the past three decades. Haurie (1976) discussed the problem of Stability in extending the Nash bargaining solution to differential games.

Petrosyan (1977) formalized mathematically the notion of dynamic stability in solutions of differential games. Petrosyan and Danilov (1979 and 1982) introduced the notion of "imputation distribution procedure" for cooperative solution. Tolwinski et al. (1986) considered cooperative equilibria in differential games in which memory-dependent strategies and threats are introduced to maintain the agreed-upon control path. Petrosyan and Zenkevich (1996) provided a detailed analysis of dynamic stability in cooperative differential games. In particular, the method of regularization was introduced to construct time-consistent solutions. Yeung and Petrosyan (2001) designed a time-consistent solution in differential games and characterized the conditions that the allocation distribution procedure must satisfy. Petrosyan (2003) used regularization method to construct time-consistent bargaining procedures.

Let there exist solutions , along the conditionally optimal trajectory . If this condition is not satisfied, it is impossible for the players to adhere to the chosen principle of optimality, since at the very first instant ,

, the players have no possibility to follow this principle. Assume that at time to when the initial state is the players agree on the imputation

This means that the players agree on an imputation of the gain in such a way that the share of the player over the time interval is equal to . If according to player is supposed to receive a payoff equaling over the time interval then over the remaining time interval according to the player is supposed to receive:

. (7)

For the original imputation agreement (that is the imputation) to remain in force at the instant , it is essential that the vector

(8)

and is indeed a solution of the current game . If such a condition is satisfied at each instant of time along the trajectory , then the imputation is dynamical stable.

Dynamic stability or time consistency of the solution imputation guarantees that the extension of the solution policy to a situation with a later starting time and along the optimal trajectory remains optimal. Moreover, group and individual rationalities are satisfied throughout the entire game interval.

A payment mechanism leading to the realization of this imputation scheme must be formulated. This will be done in the next section.

3.4 PAYOFF DISTRIBUTION PROCEDURE

A payoff distribution procedure (PDP) proposed by Petrosyan (1997) will be formulated so that the agreed upon dynamically stable imputations can be realized. Let the payoff Player receives over the time interval be expressed as:

(9)

where

, for

Therefore

(10)

This quantity may be interpreted as the instantaneous payoff of the Player at the moment . Hence it is clear the vector prescribes distribution of the total gain among the members of the coalition . By properly choosing , the players can ensure the desirable outcome that at each instant there will be no objection against realization of the original agreement (the imputation)) as shown on fig 4, i.e. the imputation is dynamic stable.

Cooperative differential game has dynamically stable solution , if all imputations are dynamically stable. Conditionally optimal trajectory, on which dynamically stable solution of the game exists, is *called optimal trajectory.*

We have proved under general conditions that the procedure , (PDP) leading to dynamic stable cooperative solution exist and realizable [Petrosjan, Zenkevich, 1996].

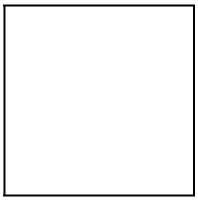

Fig. 4 Dynamically stable cooperative solution.

4. A DYNAMIC MODEL OF JOINT VENTURE

Consider a dynamic joint venture in which there are firms. The state dynamics of the firm is characterized by the set of vector-valued differential equations:

$$, \qquad (11)$$

where denotes the state variables of player , is the control vector of firm . The state of firm i include its capital stock, level of technology, special skills and productive resources. The objective of firm is:

,

where is the discount factor, the instantaneous profit, and the terminal payment. In particular, and are positively related to the level of technology .

Consider a joint venture consisting of a subset of companies . There are firms in the subset . The participating firms can gain core skills and technology that would be very difficult for them to obtain on their own, and hence the state dynamics of firm in the coalition becomes

, , (12)

where is the concatenation of the vectors for . In particular, for . Thus positive effects on the state of firm could be derived from the technology of other firms within the coalition. Again, without much loss of generalization, the effect of on remains the same for all possible coalitions containing firms and .

4.1. COALITION PAYOFFS

At time , the profit to the joint venture becomes:

(13)

To compute the profit of the joint venture we have to consider the optimal control problem which maximizes (13) subject to (12).

For notational convenience, we express (12) as:

, , (14)

where is the set of for , ; is a column vector containing for .

Using Bellman's technique of dynamic programming the solution of the problem can be characterized as follows.

Using the dynamic programming approach, it is possible to describe the solution at the following form. Denote firm's optimal control (in terms of maximizing the coalition payoff).

In the case when all the firms are in the joint venture, that is , the optimal control is

The dynamics of the optimal state trajectory of the grand coalition can be obtained as:

,

which can also be expressed as

, . (15)

Let denote the solution to (15). The optimal trajectories characterizes the states of the participating firms within the venture period. We use to denote the value of at time .

Consider the above joint venture involving n firms. The member firms would maximize their joint profit and share their cooperative profits according to the Shapley value (1953). The problem of profit sharing is inescapable in virtually every joint venture. The Shapley value is one of the most commonly used sharing mechanism in static cooperation games with transferable payoffs. Besides being individually rational and group rational, the Shapley value is also unique. The uniqueness property makes a more desirable cooperative solution relative to other solutions like the Core or the Stable Set. Specifically, the Shapley value gives an imputation rule for sharing the cooperative profit among the members in a coalition as:

, , (16)

where is the relative complement of in , is the profit of coalition , and is the marginal contribution of firm to the coalition .

To maximize the joint venture's profits the firms would adopt the control vector over the time interval, and the corresponding optimal state trajectory in (15) would result. At time with the state , the firms agree that firm 's share of profits be:

, (17)

However, the Shapley value has to be maintained throughout the venture horizon . In particular, at time with the state being the following imputation principle has to be maintained:

, (18)

where and .

Note that , as specified in (18) satisfies the basic properties of an imputation vector.

Moreover, if condition (18) can be maintained, the solution optimality principle - sharing profits according to the Shapley value - is in effect at any instant of time throughout the game along the optimal state trajectory chosen at the outset. Hence time consistency is satisfied and no firms would have any incentive to depart the joint venture. Therefore a dynamic imputation principle leading to (18) is dynamically stable or time-consistent.

Crucial to the analysis is the formulation of a profit distribution mechanism that would lead to the realization of condition (18).

4.2. TRANSITORY COMPENSATION

In this section, a profit distribution mechanism will be developed to compensate transitory changes so that the Shapley value principle could be maintained throughout the venture horizon. First, an imputation distribution procedure (similar to those in Petrosyan and Zaccour (2003) and Yeung and Petrosyan (2004)) must be now formulated so that the imputation scheme in condition (18) can be realized. Let denote the payment received by firm at time dictated by . In particular,

. (19)

The following formula describes the rule for distribution Shapley value in the time, providing time consistency of Shapley value.

, (20)

or

,

where is a column vector containing , .

The vector serves as a form equilibrating transitory compensation that guarantees the realization of the Shapley value imputation throughout the game horizon. Note that the instantaneous profit offered to Player at time is conditional upon the current state and current time . One can elect to express as . Hence an instantaneous payment to player yields a dynamically stable solution to the joint venture.

4.3. An Application in Joint Venture

Consider the case when there are 3 companies involved in joint venture. The planning period is . Company profit is

, (21)

where , and are positive constants, is the discount rate, is the level of technology of company at time , and is its physical investment in technological advancement. The term reflects the net operating revenue of company at technology level and is the cost of investment, gives the salvage value of company 's technology at time .

The evolution of the technology level of company follows the dynamics:

, , (22)

where is the addition to the technology brought about by amount of physical investment, and is the rate of obsolescence.

Consider the case when all these three firms agree to form a joint venture and share their joint profit according to the dynamic Shapley. Through knowledge diffusion participating firms can gain core skills and technology that would be very difficult for them to obtain on their own.

The evolution of the technology level of company under joint venture becomes:

for and , (23)

where and are non-negative constants. In particular, represents the technology transfer effect under joint venture on firm brought about by firm 's technology.

The profit of the joint venture is the sum of the participating firms' profits:

. (24)

The firms in the joint venture then act cooperatively to maximize (24) subject to (23). Giving up technical calculation, we have

$$=\qquad (25)$$

for .

Denoting by , we can write

$$=$$

for and ,

=

. (26)

After analytical transformation we have

,

for and .

,

for .

, (27)

for .

Note that coefficients are the solutions of linear differential equation system. The explicit solution is not stated here because of its lengthy expressions.

Using eq. (25) to (27) and (20) we obtain the form for . A payment offered to player at time will lead to the realization of the dynamic Shapley value. Hence a dynamically stable solution to the joint venture will result.

CONCLUSION

Long term cooperative solutions based on interest coordination are considered. It is shown, that basic cooperative optimality principles haven't dynamic stability (time consistency) property. This property requires saving optimality property along the optimal trajectory. We have proposed regularization procedure (PDP), introducing a new control variable. Applying the method of regularization for dynamic cooperation problem, we constructed the control in the form of special payments, paid at each time instant on the optimal trajectory. As special case the joint venture dynamic model is investigated. For this problem the dynamic stable solution is obtained.

REFERENCES

Chistyakov S. V. 1981. On nonzero-sum differential games. *Dokladi AN USSR* **259** (5).

Krasovskii N. N. 1967. On the problem pursuit. *Dokladi AN USSR* **27 (**2).

Petrosjan L. A. 1965. Differential survival games with many participants. *Dokladi AN USSR* **161** (2): 285-287.

Petrosjan L. A. 1977. Stable solutions of differential games with many participants, *Viestnik of Leningrad University* **19**: 46-52.

Petrosjan L. A. 1978. Nonzero-sum differential games. In: *Problems of mechanic control processes. Dynamic systems control*. L.: 173-181

Petrosjan L. A., Danilov N. N. 1979. Stability of solutions in non-zero sum differential games with transferable payoffs. *Viestnik of Leningrad University* **1**: 52-59.

Petrosjan L. A. 1979. Solutions of nonzero-sum differential games. In: *Proceedings of Conference "Dynamic control"*. Sverdlovsk: 208-210

Petrosjan L. A., Danilov N. N. 1985. *Cooperative differential games and their applications.* Izd. Tomskogo University, Tomsk.

Petrosjan L. A., Murzov N. V. 1967. Lugging games with many participants. *Viestnik of Leningrad University* **13**: 125-129.

Petrosjan L. A., Zakharov V. V. 1997. *Mathematical models in ecology*. SPb., Izd-vo SPbSU.

Pontryagin, L.S. 1966. On the theory of differential games. *Uspekhi Mat. Nauk* **21**: 219-274.

Bellman R. 1957. *Dynamic programming*. Princeton University Press: Princeton, NJ.

Case J. H. 1967. *Equilibrium points of n-person differential games.* Ph. D. Thesis. Department of Industrial Engineering, University of Michigan: Ann Arbor, MI; Tech. Report No. 1967-1

Haurie A. 1976. A note on nonzero-sum differential games with bargaining solutions *Journal of optimization theory and application* **18**: 31-39.

Haurie A., Krawczyk J. B., Roche M. 1976. Monitoring cooperative equilibria in a stochastic differential game. *Journal of optimization Theory and Applications* **81**: 73-95.

Isaacs R. 1965. *Differential Games*. Wiley: N. Y.

Jorgensen S. 1985. An exponential differential games which admits a simple Nash solutions. *Journal of Optimization Theory and Applications* **45**: 383-396.

Jorgensen S., Sorger G. 1990. Feedback Nash equilibria in a problem of optimal fishery management. *Journal of Optimization Theory and Applications* **64**: 293-310.

Jorgensen S., Zaccour G. 2001. Time-consistent side payment in a dynamic game in downstream pollution. *Journal of economic dynamics and control* **25**: 1973-1987.

Jorgensen S., Zaccour G. 2002. Time consistency in cooperative differential games. In: Zaccour G. (ed.). *Decision and control in management sciences: essays in honor of Alan Haurie.* Kluwer Science Publisher: London; pp. 349-366.

Kaitala V. 1993. Equilibria in a stochastic resource management game under imperfect information. *European Journal of Operational Research* **71**: 439-453.

Kalai E., Smorodinskiy M. 1975. Other solutions to Nash's bargaining problem. *Econometrica* **43**: 513-518.

Kydland F. E., Prescott E. C. 1977. Rules rather than discretion: the inconsistency of optimal plans. *Journal of political economy* **85**: 473-490

Nash J. F. 1950. The bargaining problem. *Econometrica* **18** (2): 155-162.

Nash J. F. 1951. Non-cooperative games. *Ann. Math.* **54** (2): 286-295.

Neumann J. von, Morgenstern O. 1994. *Theory of games and economic behavior*. Princeton, Princeton University Press: Princeton, NJ.

Petrosjan L. A. 1993. *Differential games of pursuit.* World Scientific Publishing Co. Pte. Ltd.: Singapore.

Petrosjan L. A. 2003. Bargaining in dynamic games. In: Petrosjan L. A., Yeung D. (eds.). *ICM Millennium Lectures on Games*. Springer-Verlag: Berlin; 139-143.

Petrosjan L. A., Zaccour G. 2003. Time-consistent Shapley value allocation of pollution cost reduction. *Journal of economic dynamics and control* **27** (3): 381-398.

Petrosjan L. A., Zenkevich N. A. 1996. *Game Theory*. World Scientific Publishing Co. Pte. Ltd.: Singapore.

Shapley L. S. 1953. A value for n-person games. In: *Contributions to the Theory of Games II*. Princeton University Press: Princeton; 307-317.

Sorger G. 1989. Competitive dynamic advertising: A modification of the case games. *Journal of Economic Dynamics and Control* **13**: 55-80.

Starr A. W., Ho Y. C. 1969a. Further properties of nonzero-sum differential games. *Journal of Optimization Theory and Applications* **3**: 207-219

Starr A. W., Ho Y. C. 1969b. Nonzero-sum differential games. *Journal of Optimization Theory and Applications* **3**: 184-206.

Tolwinski B., Haurie A., Leitmann G. 1986. Cooperative equilibria in differential games, *Journal of Mathematical Analysis and Applications* **119**: 182-202.

Yeung D. W. K. 1992. A differential game of industrial pollution management, *Annals of Operation Research* **37**: 297-311.

Yeung D. W. K. 1994. On differential games with a feedback Nash equilibrium, *Journal of Optimization Theory and Applications* **82** (1): 181-188.

Yeung D. W. K., Petrosyan L. A. 2006. *Cooperative stochastic differential games*. Springer.

Zenkevich N. A. 2001. Auction games and integrative imputations. In: *International yearbook on game theory and applications,* Vol. 6, Nova Science Publ.: N. Y.; 192 – 203.