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On the discrete lot-sizing and scheduling problem with Wagner–Whitin costs

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Abstract

We consider the single-item discrete lot-sizing and scheduling problem. We present a partial linear description of the convex hull of feasible solutions that solves this problem in the presence of Wagner–Whitin costs.

Keywords: Lot-sizing; Polyhedral combinatorics

1. Introduction

In recent years a great number of lot-sizing problems have been studied from a polyhedral point of view (cf. [4]). Most of the results concern the polyhedral structure of single-item models. Valid inequalities derived for these models have been successfully used in cutting plane algorithms for multi-item problems. Hence, (partial) linear descriptions of the convex hull of feasible solutions of single-item models are a valuable aid in solving lot-sizing problems by methods based on polyhedral combinatorics.

In [3], Pochet and Wolsey study four single-item lot-sizing problems in the presence of Wagner–Whitin costs, i.e., when the unit inventory cost h_t and the unit production cost p_t satisfy $h_t + p_t \geqslant p_{t+1}$ for every period t of the planning interval. For each of these problems, they give a partial linear description of the convex hull of feasible solutions that solves the problem when the costs satisfy the Wagner–Whitin property. These polyhedra involve considerably fewer constraints than in the general cost case.

In this paper we derive a similar result for the single-item discrete lot-sizing and scheduling problem (DLSP). The proof, however, differs from the proofs in [3]. In the following section we formulate the problem and discuss a partial linear description of the convex hull of feasible solutions that solves the problem in the presence of Wagner–Whitin costs. This result is proved in Section 3.

2. The DLSP with Wagner–Whitin costs

We consider a single-item single-machine production planning problem with a planning horizon of T periods in each of which the production is either zero

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or at full capacity, say, one unit. This is often a reasonable assumption in short-term production planning, when the time periods are small. The demand in period t, denoted by d_t, is either zero or one, and has to be satisfied in time, i.e., backlogging is not allowed. Furthermore, if production occurs in period t, but not in period $t-1$, then a startup has to take place in period t, which incurs a cost f_t. In multi-item problems startup costs also arise when the machine switches from the production of one item to the production of another item. Apart from startup costs, a production cost p_t and a unit inventory cost h_t are given for each period t. Now DLSP is the problem of determining a production schedule that satisfies the above restrictions at minimum costs.

The problem can be mathematically formulated using two types of binary variables: x_t, which indicates whether production occurs in period t or not, and y_t, which equals one if a startup occurs in period t and zero otherwise. For notational convenience we write x_{t_1,t_2} instead of $\sum_{t=t_1}^{t_2} x_t$, d_{t_1,t_2} instead of $\sum_{t=t_1}^{t_2} d_t$, etc. Now DLSP is modelled as follows:

$$\text{(DLSP)} \quad \min \quad \sum_{t=1}^{T} (f_t y_t + c_t x_t) \tag{1}$$

$$\text{s.t.} \quad x_{1,t} \geqslant d_{1,t} \quad (1 \leqslant t < T) \tag{2}$$

$$x_{1,T} = d_{1,T} \tag{3}$$

$$x_t \leqslant x_{t-1} + y_t \quad (1 \leqslant t \leqslant T, x_0 = 0) \tag{4}$$

$$x_t, y_t \in \{0,1\} \quad (1 \leqslant t \leqslant T). \tag{5}$$

In (1) we have $c_t = p_t + h_{t,T}$. The latter term is obtained by expressing the inventory costs as $\sum_{t=1}^{T} h_t \cdot (x_{1,t} - d_{1,t}) = \sum_{t=1}^{T} h_{t,T} x_t$ minus a constant, which is omitted from the objective function. Inequalities (2) yield that the total production up to period t equals at least the total demand up to this period. Furthermore, overproduction is prohibited by (3). Constraints (4) force that a startup takes place in period t if production occurs in this period but not in the preceding one.

Although the single-item DLSP is polynomially solvable, the convex hull of the set of feasible solutions of (2)–(5) is not known explicitly. Van Hoesel [7] discusses several classes of facet-defining inequalities. Magnanti and Vachani [2] and Sastry [5] derive inequalities for a slightly more general problem in which also setup costs are involved.

The following inequalities are adapted from the *interval left supermodular inequalities* derived by Constantino [1, Section 2.2], for the capacitated lot-sizing problem with startup costs. This problem is a generalization of DLSP in which the production in period t can attain any value between zero and the available capacity in this period.

Lemma 1. *Let $t \in \{1,\dots,T\}$ and $j \in \{0,\dots,d_{t+1,T}\}$. Then all feasible solutions of DLSP satisfy*

$$x_{1,t} + \sum_{i=1}^{j} (x_{t+i} + y_{t+i+1,s_{d_{1,t}+i}}) \geqslant d_{1,t} + j. \tag{6}$$

Throughout, we denote by s_k, $k \in \{1,\dots,d_{1,T}\}$, the kth demand period. Thus, $s_{d_{1,t}+i}$ denotes the ith demand period after period t. Before proving the validity of (6), let us briefly explain the idea behind these inequalities. Observe that $x_{t+i} + y_{t+i+1,s_{d_{1,t}+i}}$ is nonnegative and integral for any feasible solution (y,x) of DLSP. Moreover, $x_{t+i} + y_{t+i+1,s_{d_{1,t}+i}} = 0$ if and only if no production occurs in the interval $\{t+i,\dots,s_{d_{1,t}+i}\}$. Rewrite (6) as $x_{1,t} - d_{1,t} \geqslant \sum_{i=1}^{j} (1 - x_{t+i} - y_{t+i+1,s_{d_{1,t}+i}})$ and observe that the left-hand side of this inequality denotes the stock at the end of period t. Now one immediately sees that this constraint forces an increase of the stock at the end of period t by one for each index i for which no production occurs in the interval $\{t+i,\dots,s_{d_{1,t}+i}\}$. Note that there exist only $O(Td_{1,T})$ constraints of the form (6).

Proof of Lemma 1. First, note that inequalities (2) are a special case of (6) (take $j = 0$). Hence, for every t and $j = 0$, (6) is valid. Consider an arbitrary (integral) feasible solution of DLSP, say (y,x). By r_k, $k \in \{1,\dots,d_{1,T}\}$, we denote the kth production period in this solution. By definition, we have $s_1 < s_2 < \cdots < s_{d_{1,T}-1} < s_{d_{1,T}}$, $r_1 < r_2 < \cdots < r_{d_{1,T}-1} < r_{d_{1,T}}$. Moreover, by (2) $r_k \leqslant s_k$, $k \in \{1,\dots,d_{1,T}\}$.

Let i_0 be the highest index $i \in \{0,\dots,j\}$ such that $r_{d_{1,t}+i} < t+i$. Then, obviously, $r_{d_{1,t}+i} < t+i$

for all $i \leqslant i_0$ and $x_{t+i} + y_{t+i+1,s_{d_{1,t}+i}} \geqslant 1$ for $i > i_0$. Hence,

$$x_{1,t} + \sum_{i=1}^{j} (x_{t+i} + y_{t+i+1,s_{d_{1,t}+i}})$$

$$\geqslant x_{1,t} + x_{t+1,t+i_0} + \sum_{i=i_0+1}^{j} (x_{t+i} + y_{t+i+1,s_{d_{1,t}+i}})$$

$$\geqslant d_{1,t} + i_0 + j - i_0 = d_{1,t} + j.$$

This establishes the validity of (6). □

Recall that the costs are said to satisfy the Wagner–Whitin property if $h_t + p_t \geqslant p_{t+1}$, or, equivalently, $c_t \geqslant c_{t+1}$ for all t. With Wagner–Whitin costs there always exists an optimal solution satisfying the *zero-inventory* property, i.e., when a new production batch is started in period t, the stock at the end of period $t-1$ is zero.

Denote by RDLSP the LP-relaxation of DLSP where inequalities (6) replace (2). We claim that the following holds:

Theorem 2. *For cost functions that satisfy the Wagner–Whitin property, the objective value of RDLSP equals the objective value of DLSP.*

When c_t strictly decreases in t, an even stronger result can be proven, namely, that RDLSP solves DLSP.

Theorem 3. *If the objective function* (1) *satisfies* $c_t > c_{t+1}$ *for every period* t, *then any optimal solution of RDLSP is a convex combination of feasible solutions of DLSP, i.e., the set of optimal solutions of RDLSP has integral extreme points.*

The proof of the above theorems is postponed until the following section.

Our purpose is to develop a branch-and-cut algorithm for solving multi-item problems. Due to the above result, the $O(Td_{1,T})$ constraints of type (6) are expected to yield strong cutting planes. Furthermore, we will study the effectiveness of these inequalities for problems with more complicating features such as startup times.

3. Proof of the theorems

The major part of this section deals with the proof of Theorem 3. Therefore, assume that $c_t > c_{t+1}$ for every t. The proof uses a partitioning of a solution (y,x) of RDLSP into a set of batches $\mathscr{B}$, where a batch $B = \{p^B, \ldots, q^B\}$ is identified with the partial solution (y^B, x^B) defined by

$$\begin{aligned} y^B &= (0\ldots0\;1\;0\ldots0\;0\ldots0) \\ x^B &= \underline{(0\ldots0\;1\;1\ldots1\;0\ldots0)} \\ &\quad\; 1 \quad\; p^B \quad\; q^B \quad\; T \end{aligned}$$

The construction of the batches B from the solution (y,x) will be discussed in more detail later. Furthermore, a *value* b^B, $0 < b^B \leqslant 1$, is attached to every batch B such that $y \geqslant \sum_{B\in\mathscr{B}} b^B y^B$ and $x = \sum_{B\in\mathscr{B}} b^B x^B$. We say that $\mathscr{B}$ satisfies the *partitioning condition* if

$$\forall_{i\in\{1,\ldots,d_{1,T}\}} \sum_{B\in\mathscr{B}: s_i\in I^B} b^B = 1, \tag{7}$$

where I^B consists of the first $|B|$ demand periods in $\{p^B, \ldots, T\}$. The partioning condition can be interpreted as follows: if for every $B \in \mathscr{B}$ we produce an amount b^B in each of the periods in $\{p^B, \ldots, q^B\}$ and use this production to satisfy an amount b^B of the demand in each of the first $|B|$ demand periods in $\{p^B, \ldots, T\}$, then all demand is completely satisfied. Note in particular that $\sum_{B\in\mathscr{B}} b^B(y^B, x^B)$ is a feasible solution of DLSP for which the zero-inventory property holds if (7) is satisfied by a set of batches $\mathscr{B}$ with $b^B = 1$ for all $B \in \mathscr{B}$. Thus, the partioning condition is equivalent to the zero-inventory property for integer solutions.

The proof of Theorem 3 consists of the following two steps. First, we prove that the partitioning condition is a sufficient condition for (y,x) to be a convex combination of solutions of DLSP that satisfy the zero-inventory property (Lemma 4 and Corollary 5). Second, we present a greedy algorithm that partitions any optimal solution (y^*, x^*) of RDLSP into a set of batches $\mathscr{B}$ with values b^B, $B \in \mathscr{B}$, such that $y^* \geqslant \sum_{B\in\mathscr{B}} b^B y^B$, $x^* = \sum_{B\in\mathscr{B}} b^B x^B$, and the partitioning condition is satisfied. Combining these results yields that all extreme points of the set of optimal solutions of RDLSP are integral (and satisfy the zero-inventory property).

Lemma 4. *Given a set of batches $\mathscr{B}$ with values b^B, $0<b^B \leqslant 1$, $B \in \mathscr{B}$, such that (7) is satisfied. Then $(y,x) := \sum_{B\in\mathscr{B}} b^B(y^B,x^B)$ is a convex combination of solutions of DLSP that satisfy the zero-inventory property.*

Proof. The lemma is proven by induction on the number of batch-pairs (B,D) in $\mathscr{B}$ with intersecting demand sets I^B and I^D, which is denoted by v. Thus, $v = |\{(B,D): B,D \in \mathscr{B}, B \neq D, \text{ and } I^B \cap I^D \neq \emptyset\}|$.

If $v = 0$, i.e., if no two batches have an intersecting demand set, then, by (7), each batch B in $\mathscr{B}$ has value $b^B = 1$. We already observed that in this case (y,x) is a feasible solution of DLSP that satisfies the zero-inventory property.

Now let $v > 0$ and suppose that the result has been established for sets of batches that satisfy the partitioning condition and for which at most $v-1$ batch-pairs have intersecting demand sets. In order to show that (y,x) can be written as a convex combination of solutions of DLSP, we introduce the following definition: a subset $\mathscr{D}$ of $\mathscr{B}$ is said to yield a partition of the set of demand periods $\{s_1,\ldots,s_j\}$, $1 \leqslant j \leqslant d_{1,T}$, if $\bigcup_{B\in\mathscr{D}} I^B = \{s_1,\ldots,s_j\}$ and no two batch-pairs in $\mathscr{D}$ have intersecting demand set. We will construct a subset $\mathscr{D}$ of $\mathscr{B}$ that yields a partition of the set $\{s_1,\ldots,s_{d_{1,T}}\}$. First, we take a batch B whose demand set contains the first demand period s_1 and set $\mathscr{D} = \{B\}$. Suppose that we have found a set of batches $\mathscr{D}$ that yields a partition of the first $i < d_{1,T}$ demand periods. Then there exists a batch $D \in \mathscr{B}\backslash\mathscr{D}$ such that the demand set I^D contains s_{i+1} but not s_i. This follows from

$$\sum_{B\in\mathscr{B}\backslash\mathscr{D}:s_{i+1}\in I^B} b^B = \sum_{B\in\mathscr{B}:s_{i+1}\in I^B} b^B = 1$$

$$= \sum_{B\in\mathscr{B}:s_i\in I^B} b^B > \sum_{B\in\mathscr{B}\backslash\mathscr{D}:s_i\in I^B} b^B.$$

The demand set of D is $\{s_{i+1},\ldots,s_{i'}\}$ for some $i' \in \{i+1,\ldots,d_{1,T}\}$. Adding D to $\mathscr{D}$ gives a partition of the demand periods $\{s_1,\ldots,s_{i'}\}$. We proceed in this way until $\mathscr{D}$ yields a partition of $\{s_1,\ldots,s_{d_{1,T}}\}$. By construction, the integral vector $(y',x') := \sum_{B\in\mathscr{D}}(y^B,x^B)$ is a feasible solution of DLSP that satisfies the zero-inventory property.

Set $\bar{b} = \min\{b^B : B \in \mathscr{D}\}$ and define $\overline{\mathscr{B}} = \mathscr{B}\backslash\{B \in \mathscr{D} : b^B = \bar{b}\}$. Note that, by (7) and the assumption that $v > 0$, we have $\bar{b} < 1$. Set $\bar{b}^B = (b^B - \bar{b})/(1-\bar{b})$ for $B \in \overline{\mathscr{B}} \cap \mathscr{D}$ and $\bar{b}^B = b^B/(1-\bar{b})$ for $B \in \overline{\mathscr{B}}\backslash\mathscr{D}$. Let $i \in \{1,\ldots,d_{1,T}\}$. Since there is exactly one batch $B \in \mathscr{D}$ such that $s_i \in I^B$, we have

$$\sum_{B\in\overline{\mathscr{B}}:s_i\in I^B} \bar{b}^B = \sum_{B\in\mathscr{D}:s_i\in I^B} \frac{b^B-\bar{b}}{1-\bar{b}} + \sum_{B\in\mathscr{B}\backslash\mathscr{D}:s_i\in I^B} \frac{b^B}{1-\bar{b}}$$

$$= \frac{\sum_{B\in\mathscr{B}:s_i\in I^B} b^B - \bar{b}}{1-\bar{b}} = 1.$$

Hence, $\overline{\mathscr{B}}$ satisfies the partitioning condition. Since $\bar{b} < 1$, there is at least one batch-pair (B,D) with $B \in \overline{\mathscr{B}}$ and $D \in \mathscr{B}\backslash\overline{\mathscr{B}}$ such that $I^B \cap I^D \neq \emptyset$. This implies that the number of pairwise intersecting demand sets in $\overline{\mathscr{B}}$ is less than v, the number of pairwise intersecting demand sets in $\mathscr{B}$. Now the induction hypothesis yields that $(y'',x'') := \sum_{B\in\overline{\mathscr{B}}} \bar{b}^B(y^B,x^B)$ is a convex combination of integral solutions satisfying the zero-inventory property. Thus, so is $(y,x) = \bar{b}(y',x') + (1-\bar{b})(y'',x'')$. □

Using the above lemma it is not hard to show the following:

Corollary 5. *If (y,x) is a feasible solution of RDLSP and $\mathscr{B}$ a set of batches B with values $b^B, B\in\mathscr{B}$, such that $x = \sum_{B\in\mathscr{B}} b^B x^B$, $y \geqslant \sum_{B\in\mathscr{B}} b^B y^B$, and the partitioning condition is satisfied, then (y,x) is a convex combination of solutions of DLSP that satisfy the zero-inventory property.*

In the sequel, let (y^*,x^*) denote an optimal solution of RDLSP. From the above results it follows that, in order to prove Theorem 3, it suffices to show that (y^*,x^*) can be partitioned into a set of batches $\mathscr{B}$ with values b^B, $B \in \mathscr{B}$, such that $y^* \geqslant \sum_{B\in\mathscr{B}} b^B y^B, x^* = \sum_{B\in\mathscr{B}} b^B x^B$, and the partitioning condition is satisfied. We claim that the following algorithm provides a set of batches $\mathscr{B}$ with the desired properties.

```
begin CONSTRUCT_BATCHES
    for t = 1 to T do
    begin
        x̄_t := x*_t; ȳ_t := y*_t; d̄_t := d_t
    end;
        {x̄_t is called the residual production, etc.}
    𝒟 := ∅;
```

while $\bar{x}_{1,T} > 0$ **do**
begin
q^D := last period with positive residual production;
p^D := last period in $\{1,\ldots,q^D\}$ with positive residual startup;
$D := \{p^D,\ldots,q^D\}$;
J^D := set of demand periods with positive residual demand in $\{p^D,\ldots,T\}$;
$b^D := \min\{\bar{y}_{p^D}, \min_{t\in D}\ \bar{x}_t, \min_{t\in J^D} \bar{d}_t\}$;
$\bar{y}_{p^D} := \bar{y}_{p^D} - b^D$;
for $t \in D$ **do** $\bar{x}_t := \bar{x}_t - b^D$;
for $t \in J^D$ **do** $\bar{d}_t := \bar{d}_t - b^D$;
$\mathscr{D} := \mathscr{D} \cup \{D\}$
end;
end.

Observe that $\bar{x}_t$, $\bar{y}_t$, and $\bar{d}_t$ are non-increasing and nonnegative during the execution of the algorithm. Moreover, the residual demands $\bar{d}_{s_i}$ are non-increasing in i. It is also easily seen that $\bar{x}_t \leqslant \bar{y}_t + \bar{x}_{t-1}$ and $\bar{y}_t \leqslant \bar{x}_t$ hold for all t. Therefore, $\bar{x}_{q^D} = \min_{t\in D}\ \bar{x}_t$, and if $J^D \neq \emptyset$, then $\bar{d}_{\bar{s}} = \min_{t\in J^D} \bar{d}_t$, where $\bar{s}$ denotes the last period with positive residual demand.

To prove that the batches constructed by the algorithm satisfy the partitioning condition, we show that during the execution of the algorithm the following invariants hold:

(I_1) $\forall_{t\in\{1,\ldots,T\}}\ x^*_t = \bar{x}_t + \sum_{B\in\mathscr{D}:t\in B} b^B$;
(I_2) $\forall_{t\in\{1,\ldots,T\}}\ y^*_t = \bar{y}_t + \sum_{B\in\mathscr{D}:t=p^B} b^B$;
(I_3) $\forall_{i\in\{1,\ldots,d_{1,T}\}}\ 1 = \bar{d}_{s_i} + \sum_{B\in\mathscr{D}:s_i\in J^B} b^B$;
(I_4) $\forall_{B\in\mathscr{D}}\ |J^B| = |B|$;
(I_5) $\forall_{t\in\{1,\ldots,T-1\}}\ \bar{x}_{1,t} \geqslant \bar{d}_{1,t}$ and $\bar{x}_{1,T} = \bar{d}_{1,T}$.

Note that (I_1)–(I_3) relate the residuals of the variables and parameters to the batch sizes, whereas (I_5) states that (4) remains valid during the algorithm. Finally, (I_4) relates the production periods to the demand periods, in a way similar to the zero-inventory property if integrality of the variables were true.

Suppose that (I_1)–(I_5) hold during the execution of the algorithm. At termination of the algorithm we have $\bar{x}_t = 0, \bar{y}_t \geqslant 0$, and, by ($I_5$), $\bar{d}_t = 0$ for all t. Hence, by (I_1) and (I_2), the set of batches $\mathscr{D}$ provided by CONSTRUCT_BATCHES satisfies $x^* = \sum_{B\in\mathscr{D}} b^B x^B$ and $y^* \geqslant \sum_{B\in\mathscr{D}} b^B y^B$. Since $\bar{d}_{s_i} \geqslant \bar{d}_{s_{i+1}}$ during the execution of the algorithm, (I_4) implies that J^B is the set of the first $|B|$ demand periods in $\{p^B,\ldots,T\}$. Thus, if (I_3) and (I_4) hold, then the set $\mathscr{D}$ satisfies the partitioning condition with $I^B = J^B$ for all constructed batches $B\in\mathscr{D}$ at the end of the algorithm. Now Corollary 5 yields that (y^*,x^*) is a convex combination of feasible solutions of DLSP. Hence, the validity of the invariant during the execution of the algorithm implies the validity of Theorem 3.

The invariant is easily checked to hold initially. We will prove that if the invariant holds at the beginning of an iteration, then it also holds at the end of that iteration. In the sequel the *current* iteration is the one for which validity of the invariant is proven. We denote the batch constructed in the current iteration by D. The set of batches that are constructed in previous iterations is denoted by $\mathscr{D}$. Now (I_1)–(I_3) are easily checked to hold at the end of the current iteration, and (I_5) follows from (I_4). The latter holds at the end of the current iteration if $|J^D| = |D|$. Hence, we are left with the proof of $|J^D| = |D|$.

Proof of $|J^D| = |D|$**.** We first show that $|J^D| > |D|$ implies that (y^*,x^*) is not optimal. Next, we show that if $|J^D| < |D|$, then (y^*,x^*) violates a constraint of type (6). Both results contradict the assumption that (y^*,x^*) is an optimal solution of RDLSP, which leads to the conclusion that $|J^D| = |D|$.

Part 1: $|J^D| \leqslant |D|$. Assume that $|J^D| > |D|$. We claim that in this case we can move an amount $\varepsilon > 0$ from the production in period q^D to period $q^D + 1$ while maintaining feasibility. Since $c_{q^D} > c_{q^D+1}$, this yields a cheaper solution than (y^*,x^*), which contradicts the optimality of (y^*,x^*). In order to prove our claim, it suffices to show that the following constraints have positive slack, i.e., they are not satisfied at equality:

(i) $x^*_{q^D} \geqslant 0$;
(ii) $x^*_{q^D+1} \leqslant 1$;
(iii) $x^*_{q^D+1} \leqslant y^*_{q^D+1} + x^*_{q^D}$;
(iv) $\forall_{t,j:t+j=q^D}$
$$x^*_{1,t} + \sum_{i=1}^{j}(x^*_{t+i} + y^*_{t+i+1,s_{d_{1,t+i}}}) \geqslant d_{1,t} + j.$$

By definition of q^D, we have $x^*_{q^D} \geqslant \bar{x}_{q^D} > 0$. For the proof of $x^*_{q^D+1} < 1$, we use the following important observation: if period s has positive residual demand in the current iteration, then $s \in J^B$ for every batch $B\in\mathscr{D}$ with $p^B \leqslant s$. From the assumption that $|J^D| > |D|$ it follows that there is at least one demand period after q^D with positive residual demand. Thus, $\bar{d}_{s'} > 0$, where s' denotes the first demand period after q^D. Hence, if $B \in \mathscr{D}$ satisfies $q^D + 1 \in B$, then $s' \in J^B$. Together with $\bar{x}_{q^D+1} = 0$, this yields

$$x^*_{q^D+1} = x^*_{q^D+1} - \bar{x}_{q^D+1} \overset{(\mathrm{I}_1)}{=} \sum_{B: q^D+1 \in B} b^B \leqslant \sum_{B: s' \in J^B} b^B$$
$$= 1 - \bar{d}_{s'} < 1.$$

In order to show that (iii) is not satisfied at equality, notice that whenever $\bar{x}_{q^D+1}$ decreases in an iteration, one of the variables $\bar{x}_{q^D}$ or $\bar{y}_{q^D+1}$ decreases by the same amount. At the beginning of the current iteration, strict inequality holds since $0 = \bar{x}_{q^D+1} < \bar{x}_{q^D}$.

We omit the proof that (iv) has positive slack, since it is rather technical and does not provide any further insight. The interested reader is referred to [6].

We conclude that none of the constraints (i)–(iv) is satisfied at equality, which establishes the validity of $|J^D| \leqslant |D|$.

Part 2: $|J^D| \geqslant |D|$. Suppose that $|J^D| < |D|$. We claim that in this case constraint (6) with $t = p^D - 1$ and $j = |J^D|$ is violated by (y^*, x^*), i.e.,

$$x^*_{1,p^D-1} + \sum_{i=1}^{|J^D|} (x^*_{p^D+i-1} + y^*_{p^D+i, s_{d_{1,p^D-1}+i}})$$
$$< d_{1,p^D-1} + |J^D|.$$

First, suppose that $|J^D| = 0$. Then $x^*_{1,p^D-1} = \bar{x}_{1,p^D-1} < \bar{x}_{1,p^D} \leqslant \bar{d}_{1,T} = \bar{d}_{1,p^D-1} = d_{1,p^D-1}$, which establishes our claim. In the sequel, we therefore assume that $|J^D| > 0$. In the proof we use the following observation:

$$\forall_{t \in \{p^D+1, \ldots, \bar{s}\}} \quad \bar{y}_t = 0, \tag{8}$$

where $\bar{s}$ denotes the last period with positive residual demand. For $t \in \{p^D + 1, \ldots, q^D\}$ this is by choice of p^D. Therefore, suppose that $\bar{y}_\tau > 0$ for some $\tau \in \{q^D + 1, \ldots, \bar{s}\}$. Similar as in Part 1, we claim that in this case we can obtain a cheaper solution than (y^*, x^*) by moving an amount $\varepsilon > 0$ from the production in period q^D to period τ. In order to prove our claim, it again suffices to show that the constraints (i)–(iv) that were considered in Part 1 are not satisfied at equality. For most cases the same arguments as in Part 1 can be used. The reader is again referred to [6] for the details.

Since $s_{d_{1,p^D-1}+|J^D|}$ is the last period with positive residual demand, the right-hand side of the constraint under consideration equals $d_{1,t} + |J^D| = d_{1,\bar{s}}$. We have

$$x^*_{1,p^D-1} + \sum_{i=1}^{|J^D|} (x^*_{p^D-1+i} + y^*_{p^D-1+i+1, s_{d_{1,p^D-1}+i}})$$
$$\overset{(\mathrm{I}_1),(\mathrm{I}_2),(8)}{=} \bar{x}_{1,p^D-1+|J^D|} + \sum_{i=1}^{|J^D|} \sum_{\substack{B \in \mathscr{D}: q^B \geqslant p^D-1+i,\\ p^B \leqslant s_{d_{1,p^D-1}+i}}} b^B$$
$$\overset{(*)}{\leqslant} \bar{x}_{1,p^D-1+|J^D|} + \sum_{i=1}^{|J^D|} \sum_{B \in \mathscr{D}: s_{d_{1,p^D-1}+i} \in J^B} b^B$$
$$\overset{(\mathrm{I}_3)}{=} \bar{x}_{1,p^D-1+|J^D|} + \sum_{i=1}^{|J^D|} (1 - \bar{d}_{s_{d_{1,p^D-1}+i}})$$
$$\overset{(\dagger)}{<} d_{1,\bar{s}}.$$

Note that in the current iteration all demand periods in $\{p^D, \ldots, \bar{s}\}$ have positive residual demand. Thus, for each $B \in \mathscr{B}$ with $p^B \leqslant s_{d_{1,p^D-1}+i}$, $i \leqslant |J^D|$, we have $s_{d_{1,p^D-1}+i} \in J^B$. This shows the validity of $(*)$. Moreover, the assumption that $|J^D| < |D|$ yields that $p^D - 1 + |J^D| < q^D$, hence, by definition of q^D and (I_5), we have $\bar{x}_{1,p^D-1+|J^D|} < \bar{x}_{1,q^D} = \bar{d}_{1,\bar{s}}$. From this the validity of $(\dagger)$ immediately follows.

This concludes the proof of $|J^D| = |D|$ and, hence, the proof of Theorem 3. □

As a corollary we can prove Theorem 2 as follows. For arbitrary $\varepsilon > 0$ the cost function $c'_t := c_t + (T-t)\varepsilon$ satisfies the requirements of Theorem 3. Therefore, for every $\varepsilon > 0$ there exists an optimal solution of RDLSP that is an integral extreme point. Since the objective function is continuous in ε, there must be an integer optimal solution of RDLSP for $\varepsilon = 0$. However, we do not necessarily find that for $\varepsilon = 0$ all extreme points of the set of optimal solutions of RDLSP are integral.

References

[1] M.F. Constantino, “A polyhedral approach to production planning models: start-up costs and times, upper and lower

bounds on production", Ph.D. Thesis, Faculté des Sciences, Université Catholique de Louvain, Louvain-la-Neuve, 1995.

[2] T.L. Magnanti and R. Vachani, "A strong cutting plane algorithm for production scheduling with changeover costs", *Oper. Res.* **38** (1990) 456–473.

[3] Y. Pochet and L.A. Wolsey, "Polyhedra for lot-sizing with Wagner–Whitin costs", *Math. Programming* **67** (1994) 297–323.

[4] Y. Pochet and L.A. Wolsey, "Algorithms and reformulations for lot sizing problems", in: W. Cook, L. Lovász and P.D. Seymour (eds.), *Combinatorial Optimization*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 20, 1995, pp. 245–293.

[5] T. Sastry, "Polyhedral structure of the product cycling problem with changeover costs", Ph.D. Thesis, Sloan School of Management, MIT, Cambridge, USA, 1990.

[6] C.A. van Eijl and C.P.M. van Hoesel, "On the discrete lot-sizing and scheduling problem with Wagner–Whitin costs", Memorandum COSOR 95-28 (revised version), Eindhoven University of Technology, Eindhoven, 1995.

[7] S. van Hoesel, "Models and algorithms for single-item lot sizing problems", Ph.D. Thesis, Econometrisch Instituut, Erasmus Universiteit, Rotterdam, 1991.