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Parametric analysis of setup cost in the economic lot-sizing model without speculative motives[☆]

C.P.M. Van Hoesel[a], A.P.M. Wagelmans[b,*]

[a]*Department of Quantitative Economics, Maastricht University, P.O. Box 616, NL-6200 MD, Maastricht, Netherlands*
[b]*Econometric Institute, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, Netherlands*

Abstract

In this paper we consider the important special case of the economic lot-sizing problem in which there are no speculative motives to hold inventory. We analyze the effects of varying all setup costs by the same amount. This is equivalent to studying the set of optimal production periods when the number of such periods changes. We show that this optimal set changes in a very structured way. This fact is interesting in itself and can be used to develop faster algorithms for such problems as the computation of the stability region and the determination of all efficient solutions of a lot-sizing problem. Furthermore, we generalize two related convexity results which have appeared in the literature. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In 1958 Wagner and Whitin published their seminal paper on the "Dynamic Version of the Economic Lot-Size Model", in which they showed how to solve the problem considered by a dynamic programming algorithm. It is well known that the same approach also solves a slightly more general problem to which we will refer as the economic lot-sizing problem (ELS). Recently, considerable improvements have been made with respect to the complexity of solving ELS and some of its special cases (see [1–3]). Similar improvements

can also be made for many extensions of ELS (see [4]).

In this paper we consider the important special case of ELS in which there are no speculative motives to hold inventory, i.e., the marginal cost of producing one unit in some period plus the cost of holding it until some future period is at least the marginal production cost in the latter period. For this model we analyze the effects of varying all setup costs by the same amount. This is equivalent to studying the set of optimal production periods when the number of such periods changes. We will show that this optimal set changes in a very structured way. This fact is interesting in itself and can be used to develop faster algorithms for such problems as the computation of the stability region and the determination of all efficient solutions of a lot-sizing problem. Furthermore, we will generalize

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two related convexity results which have appeared in the literature.

The paper is organized as follows. In Section 2 we state several useful results about the economic lot-sizing problem without speculative motives. In Section 3 we perform a parametric analysis of the problem. We will characterize how the optimal solution changes when all setup costs are reduced by the same amount and we will present a linear time algorithm to calculate the minimal reduction for which the change actually occurs. In Section 4 we discuss applications of the results of Section 3. Finally, Section 5 contains some concluding remarks.

2. The economic lot-sizing problem without speculative motives

In the *economic lot-sizing problem* (ELS) one is asked to satisfy at minimum cost the known demands for a specific commodity in a number of consecutive periods (the *planning horizon*). It is possible to store units of the commodity to satisfy future demands, but backlogging is not allowed. For every period the production costs consist of two components: a cost per unit produced and a fixed setup cost that is incurred whenever production occurs in the period. In addition to the production costs there are holding costs which are linear in the inventory level at the end of the period. Both the inventory at the beginning and at the end of the planning horizon are assumed to be zero.

We will use the following notation:

T: the length of the planning horizon,
d_i: the demand in period $i \in \{1, \ldots, T\}$,
p_i: the unit production cost in period $i \in \{1, \ldots, T\}$,
f_i: the setup cost in period $i \in \{1, \ldots, T\}$,
h_i: the unit holding cost in period $i \in \{1, \ldots, T\}$.

Furthermore, we define $d_{ij} \equiv \sum_{t=i}^{j} d_t$ for all i, j with $1 \leqslant i \leqslant j \leqslant T$.

As shown in [3] an equivalent problem results when all unit holding costs are taken 0, and for all $i \in \{1, \ldots, T\}$ the unit production cost p_i is replaced by c_i, defined as

$$c_i \equiv p_i + \sum_{t=i}^{T} h_t.$$

This reformulation can be carried out in linear time and it changes the objective function value of all feasible solutions by the same amount. From now on we will focus on the reformulated problem.

In this paper, we assume that $c_i \geqslant c_{i+1}$ for all $i \in \{1, \ldots, T-1\}$. Note that if c_i were less than c_j for some $j > i$, then this could be perceived as an incentive to hold inventory at the end of period i (in order to avoid that the higher unit production cost in period j will have to be paid). Under our assumption on the unit production costs this incentive is not present. Therefore, this special case is known as the *economic lot-sizing problem without speculative motives*. Note that in the model originally considered by Wagner and Whitin [5] it is assumed that $h_i \geqslant 0$ and $p_i = 0$ for all $i \in \{1, \ldots, T\}$. Because $c_i = p_i + \sum_{t=i}^{T} h_t$, it is easily seen that this model is an example of a lot-sizing problem without speculative motives.

It is well known that economic lot-sizing problems can be solved using dynamic programming. For problems without speculative motives, the dynamic programming algorithm can be implemented such that it requires only O(T) time (see [1–3]). We will now briefly review such an implementation.

The key observation to obtain a dynamic programming formulation is that it suffices to consider only feasible solutions that have the *zero-inventory property*, i.e., solutions in which the inventory at the beginning of production periods is zero. The latter implies that if i and j are consecutive production periods with $i < j$, then the amount produced in period i equals d_{ij-1}. From now on, we will only consider solutions with this property. Also, note that we may assume that setups only take place in production periods, even if some of the setup costs are zero. Hence, solutions can completely be described by their production periods which coincide with the periods in which the setups occur.

Let the variable $F(i)$, $i \in \{1, \ldots, T\}$, denote the value of the optimal production plan for the instance of ELS with the planning horizon truncated after period i, and define $F(0) \equiv 0$. For $i = 1, \ldots, T$ the value of $F(i)$ can be calculated using the following forward recursion:

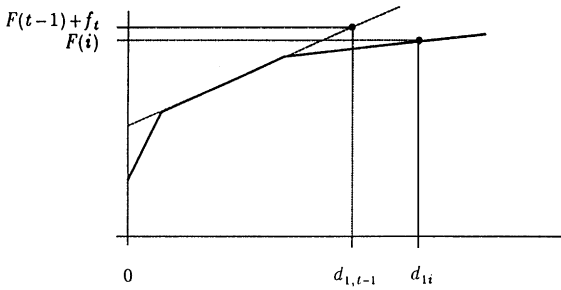$$F(i) := \min_{0 < t \leqslant i} \{F(t-1) + f_t + c_t d_{ti}\}.$$

Fig. 1. Determination of $F(i)$.

To determine $F(i)$ when $F(t-1)$ is already known for all $t \leqslant i$, we can proceed as follows (see Fig. 1): for each $t \leqslant i$ we plot the point $(d_{1,t-1}, F(t-1) + f_t)$ and draw the line with slope c_t that passes through this point. It is easy to verify that $F(i)$ is equal to the value of the concave lower envelope of these lines in coordinate d_{1i} on the horizontal axis. After constructing the line with slope c_i that passes through $(d_{1i}, F(i) + f_{i+1})$, we update the lower envelope and continue with the determination of $F(i+1)$.

The running time of this algorithm depends on the complexity of evaluating the lower envelope in certain points on the horizontal axis and the complexity of updating the concave lower envelope. Because lines are added in order of non-increasing slope, the total computational effort for updating the lower envelope (i.e., over all T iterations) can be done in linear time. (We use a stack to store the breakpoints and corresponding line segments of the lower envelope.) The fact that the points in which the envelope is evaluated have a non-decreasing horizontal coordinate can be used to establish an O(T) bound on the total number of operations required for those evaluations. Hence, the algorithm runs in linear time.

For convenience we will assume from this point on that $d_1 > 0$. Hence, period 1 is the first production period in every feasible solution. Let $i \in \{2, \ldots, T\}$, then $h \in \{1, \ldots, i-1\}$ is called an *optimal predecessor* of i if period h is the last production period before i in some optimal solution in which i is a production period. This means that h is such that $F(i-1) = \{F(h-1) + f_h + c_h d_{h,i-1}\}$. Period h is referred to as an optimal predecessor of

$T+1$ if it is the last production period in some optimal solution.

The following result is a slight generalization of the well-known planning horizon theorem due to Wagner and Whitin. It will be used frequently later on.

Lemma 1. *Let* $1 \leqslant h < i < j < k \leqslant T+1$ *be such that h is an optimal predecessor of k and i is an optimal predecessor of j, then both h and i are optimal predecessors of both j and k.*

Proof. We know that

$$F(j-1) = F(i-1) + f_i + c_i d_{i,j-1}$$
$$\leqslant F(h-1) + f_h + c_h d_{h,j-1} \qquad (1)$$

and

$$F(k-1) = F(h-1) + f_h + c_h d_{h,k-1}$$
$$\leqslant F(i-1) + f_i + c_i d_{i,k-1}. \qquad (2)$$

Combining these inequalities leads to

$$c_h d_{j,k-1} \leqslant c_i d_{j,k-1}.$$

It is easily seen that the lemma holds if $d_{j,k-1} = 0$. Assuming $d_{j,k-1} > 0$, we obtain $c_h \leqslant c_i$. Because $h < i$ we already know that $c_h \geqslant c_i$. Therefore, it must hold that $c_h = c_i$. Substituting this into (1) and (2) leads to

$$F(i-1) + f_i \leqslant F(h-1) + f_h + c_h d_{h,i-1},$$

respectively,

$$F(h-1) + f_h + c_h d_{h,i-1} \leqslant F(i-1) + f_i.$$

Hence, equality must hold in both (1) and (2), which implies the desired result. □

In the next section we perform a parametric analysis of the setup costs of the economic lot-sizing problem without speculative motives.

3. Parametric analysis

In this section we study the parametric problem with setup costs of the form $f_i - \lambda$, $i = 1, \ldots, T$. Here all $f_i \geqslant 0$ are assumed to be non-negative and
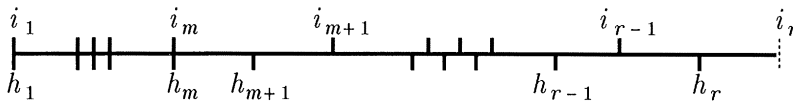
Fig. 2. Structure of optimal solution in Theorem 1.

the domain of the parameter λ is the interval $[0, \Lambda]$, where $\Lambda \leqslant \min_{i=1,\ldots,T}\{f_i\}$. The main issue we will deal with is the following. Suppose we are given an optimal solution for the lot-sizing problem for $\lambda = 0$. Assume that the set of production periods is $\{i_1, \ldots, i_q\}$, where $1 = i_1 < i_2 < \cdots < i_q$ and $q < T$; also define $i_{q+1} \equiv T + 1$. When λ is increased, solutions with more than q setups become relatively more attractive (and solutions with less than q setups become less attractive). We would like to determine the smallest value of $\lambda \in [0, \Lambda]$, if any, such that there exists an optimal solution with at least $q + 1$ setups. Furthermore, we are also interested in that optimal solution itself.

Let λ' denote the parameter value we are looking for. We will use an approach to find this value which is based on a natural decomposition of the problem. To this end we define ELS(t), $t \in \{2, \ldots, q + 1\}$, as the parametric lot-sizing problem with planning horizon consisting of the first $i_t - 1$ periods. Furthermore, we let λ_t, $t \in \{2, \ldots, q + 1\}$, denote the smallest value in $[0, \Lambda]$ for which there exist an optimal solution for ELS(t) with at least t setups; λ_t is defined to be ∞ if there does not exist such a solution for any $\lambda \in [0, \Lambda]$. Clearly, λ' exists and is equal to λ_{q+1} if and only if the latter value is finite.

In the sequel, the values $F(t)$, $t = 1, \ldots, T$, have the same interpretation as in Section 1, i.e., they correspond to $\lambda = 0$. We make the following observations.

Lemma 2. *For all $t \in \{2, \ldots, q + 1\}$ the set $\{i_1, \ldots, i_{t-1}\}$ is an optimal set of production periods for ELS(t) as long as $0 \leqslant \lambda \leqslant \min\{\lambda_t, \Lambda\}$. Moreover, the value of this solution is $F(i_t - 1) - (t - 1)\lambda$.*

Proof. Trivial.

Lemma 3. *The values λ_t, $2 \leqslant t \leqslant q + 1$, are nonincreasing in t.*

Proof. Suppose $2 \leqslant p < r \leqslant q + 1$ and let λ_p be finite. For $\lambda = \lambda_p$ there exists an optimal solution of ELS(p) with at least p setups. Denote the set of production periods in this solution by S; hence, $|S| \geqslant p$. If $\lambda_r > \lambda_p$, then ELS(r) does not have an optimal solution with at least r setups if $\lambda = \lambda_p$. Lemma 2 states that $\{i_1, \ldots, i_p, \ldots, i_{r-1}\}$ is an optimal set of production periods for ELS(r) as long as $0 \leqslant \lambda \leqslant \lambda_r$. However, $S \cup \{i_p, \ldots, i_{r-1}\}$ must also be an optimal solution and $|S \cup \{i_p, \ldots, i_{r-1}\}| \geqslant p + (r - p) = r$. This is a contradiction. Therefore, it must hold that $\lambda_r \leqslant \lambda_p$. $\square$

Lemma 3 will be used in the proof Theorem 1 below. This theorem will enable us to calculate the values λ_t, $2 \leqslant t \leqslant q + 1$, efficiently in order of increasing index. For notational convenience we define $\lambda_1 \equiv \infty$ and we let $\langle t_1, t_2 \rangle$ denote the (possibly empty) set $\{t \in \mathbb{N} \mid t_1 < t < t_2\}$ for every pair of indices t_1 and t_2 with $t_1 < t_2$.

Theorem 1. *Let $r \in \{2, \ldots, q + 1\}$ and suppose $\lambda_r < \lambda_{r-1}$, then ELS($r$) has an optimal solution (Fig. 2) for $\lambda = \lambda_r$ with the following properties*:

● *there are exactly r production periods $h_1 < \cdots < h_r$*
● *there exists an $m \in \{1, \ldots, r - 1\}$ such that*

$i_t = h_t \quad \text{for all } t = 1, \ldots, m,$

$h_t \in \langle i_{t-1}, i_t \rangle \quad \text{for all } t = m + 1, \ldots, r.$

Proof. Consider ELS(r) when $\lambda = \lambda_r$. By definition there exists an optimal solution with at least r production periods. Let $k_1 < \cdots < k_s$ be the production periods in such a solution; hence, $s \geqslant r$. Let n be the largest index such that $k_n \in \{i_1, \ldots, i_{r-1}\}$. Suppose $k_n = i_l$, then both $\{i_1, \ldots, i_{l-1}\}$ and $\{k_1, \ldots, k_{n-1}\}$ are optimal sets of production periods for ELS(l) when $\lambda = \lambda_r$. Because $\lambda_r < \lambda_{r-1} \leqslant \lambda_l$, ELS($l$) does not have an optimal
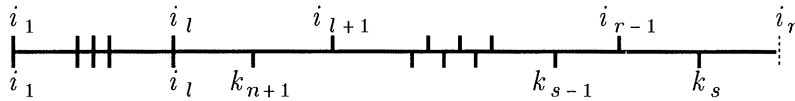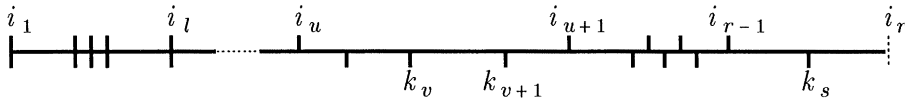
Fig. 3. Solution of the desired form.

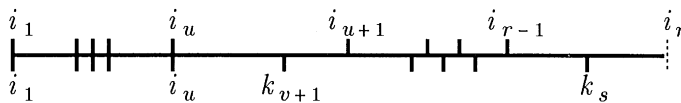Fig. 4. $\langle i_u, i_{u+1} \rangle$ contains several elements of $\{k_{n+1}, \ldots, k_s\}$.

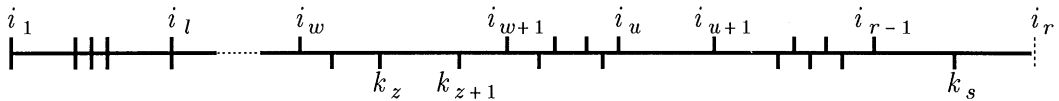Fig. 5. Solution of the desired form.

Fig. 6. $\langle i_u, i_{u+1} \rangle$ does not contain any element of $\{k_{n+1}, \ldots, k_s\}$.

solution with more than $l - 1$ production periods. Hence, $|\{k_1, \ldots, k_{n-1}\}|$ is at most equal to $l - 1 = |\{i_1, \ldots, i_{l-1}\}|$. Now it follows that $\{i_1, \ldots, i_{l-1}\} \cup \{k_n, \ldots, k_s\}$ is an optimal set of production periods for ELS(r) with at least r elements.

If k_{n+1} to k_s are such that every set $\langle i_{l+t-1}, i_{l+t} \rangle$ with $t \in \{1, \ldots, r-l\}$ contains exactly one of them, then the just constructed optimal solution has the desired properties (with $m = l$) (Fig. 3).

Otherwise, let u be the largest index in $\{l, \ldots, r-1\}$ such that $\langle i_u, i_{u+1} \rangle$ does not contain exactly one element of $\{k_{n+1}, \ldots, k_s\}$. First, suppose that $\langle i_u, i_{u+1} \rangle$ contains several of these indices and let k_v and k_{v+1} be the two largest of those (Fig. 4).

Because i_u is an optimal predecessor of i_{u+1} and k_v is an optimal predecessor of k_{v+1}, it follows from Lemma 1 that i_u is an optimal predecessor of k_{v+1}. Hence $\{i_1, \ldots, i_u\} \cup \{k_{v+1}, \ldots, k_s\}$ is also an optimal

set of production periods. Moreover, this solution has the form stated in the theorem (with $m = u$) (Fig. 5).

Now, we are only left with the case that $\langle i_u, i_{u+1} \rangle$ does not contain any element of $\{k_{n+1}, \ldots, k_s\}$. By deducing a contradiction, it will be shown that this case cannot occur. From the fact that $|\{i_1, \ldots, i_l\} \cup \{k_{n+1}, \ldots, k_s\}| \geqslant r$ we obtain $|\{k_{n+1}, \ldots, k_s\}| \geqslant r - l$. Therefore, there must be at least one $t \in \{l, \ldots, r-1\} \backslash \{u\}$ such that $\langle i_t, i_{t+1} \rangle$ contains several elements of $\{k_{n+1}, \ldots, k_s\}$. From the definition of u it follows that indices with this property must be smaller than u. Let $w \in \{l, \ldots, u-1\}$ be the largest index with the property and let k_z and k_{z+1} be the two largest indexed elements in $\langle i_w, i_{w+1} \rangle$ (Fig. 6).

It follows from the definition of u and w that for all $t \in \{w+1, \ldots, r-1\}$ the set $\langle i_t, i_{t+1} \rangle$ contains at most one element of $\{k_{z+2}, \ldots, k_s\}$. Because

$\langle i_u, i_{u+1} \rangle$ does not contain any element of the latter set, it is now easy to show that

$$|\{i_1, \ldots, i_l\} \cup \{k_{n+1}, \ldots, k_z\}|$$

$$= |\{i_1, \ldots, i_l\} \cup \{k_{n+1}, \ldots, k_s\}| - |\{k_{z+1}, \ldots, k_s\}|$$

$$\geqslant r - (r - w - 1) = w + 1.$$

Furthermore, it follows from Lemma 1 that k_z is an optimal predecessor of i_{w+1}. Hence, $\{i_1, \ldots, i_l\} \cup \{k_{n+1}, \ldots, k_z\}$ is an optimal set of production periods for ELS($w + 1$) when $\lambda = \lambda_r$. However, because $\lambda_r < \lambda_{r-1} \leqslant \lambda_{w+1}$, ELS($w + 1$) does not have an optimal solution with more than w setups for λ_r. Hence, we have obtained a contradiction. This completes the proof. $\square$

Theorem 1 is basically a characterization of how the set of optimal production periods changes — or to be more precise, may be assumed to change — when λ becomes equal to λ'. Let r be the smallest index such that $\lambda_r = \lambda'$, then there exists an optimal solution with exactly $q + 1$ setups of which the production periods before i_r are as described in the theorem and the other production periods are i_r to i_q. This characterization resembles a result given by Murphy and Soyster [6], who consider the lot-sizing problem in which the setup and unit production costs are non-increasing over time, and the holding costs in each period are concave and non-decreasing functions of the inventory level at the end of that period. They show that when all setup costs are decreased *proportionally* (instead of by the same amount), then the number of production periods is non-decreasing and the kth production period in the perturbed problem instance occurs not later than the kth production period in the original instance.

We now turn to the issue of determining λ' and a corresponding optimal solution with $q + 1$ setups efficiently. As noted before, we will determine the values λ_t, $2 \leqslant t \leqslant q + 1$, in order of increasing index. To explain our method we need some additional notation. For every pair of indices t_1 and t_2 with $t_1 < t_2$ define $\langle t_1, t_2] \equiv \{t \in \mathbb{N} \mid t_1 < t \leqslant t_2\}$, i.e., $\langle t_1, t_2] = \langle t_1, t_2 \rangle \cup \{t_2\}$. Furthermore, $G(j)$ is defined for $j \in \{2, \ldots, T\}$ as follows:

● if $j = i_r$: $G(j) \equiv F(i_r - 1) + f_{i_r}$,

● if $j \in \langle i_{r-1}, i_r \rangle$: $G(j) \equiv$ the optimal value when $\lambda = 0$ of the lot-sizing problem with planning horizon consisting of the first $i_r - 1$ periods under the restriction that exactly one setup occurs in $\langle i_t, i_{t+1}]$ for all $t \in \{1, \ldots, r - 2\}$, and j is the only production period in $\langle i_{r-1}, i_r \rangle$.

The reason why these values are introduced is the following. Let $r \in \{2, \ldots, q + 1\}$ and suppose $\lambda_r < \lambda_{r-1}$. Consider a fixed $j \in \langle i_{r-1}, i_r \rangle$ and note that the restriction in the definition of $G(j)$ makes the corresponding optimal solution a candidate for the solution described in Theorem 1. Because this solution has r setups, its value equals $G(j) - r\lambda_r$ when $\lambda = \lambda_r$. Clearly, the optimal solution of Theorem 1 is the best one among all candidates, i.e., its value is $\min_{j \in \langle i_{r-1}, i_r \rangle} \{G(j)\} - r\lambda_r$. Obviously, this value equals $F(i_r - 1) - (r - 1)\lambda_r$ (cf. Lemma 2), and therefore,

$$\lambda_r = \min_{j \in \langle i_{r-1}, i_r \rangle} \{G(j)\} - F(i_r - 1). \tag{3}$$

Note that (3) holds under the assumption that $\lambda_r < \lambda_{r-1}$. Because $\lambda_r \leqslant \lambda_{r-1}$, λ_r equals $\min\{\lambda_{r-1}, \min_{j \in \langle i_{r-1}, i_r \rangle} \{G(j)\} - F(i_r - 1)\}$, unless this value is greater than Λ. In the latter case λ_r is set equal to ∞.

We will now show how the values $G(j)$ can be calculated for all $j \in \langle i_{r-1}, i_r \rangle, r \in \{2, \ldots, q + 1\}$, given the values $G(h)$ for all $h \in \langle i_{r-2}, i_{r-1}]$; where $i_0 \equiv 0$. Note that the latter values are defined with respect to the planning horizon with total demand equal to $d_{1, i_{r-1} - 1}$. Therefore, the following recursion holds:

$$G(j) := \min_{h \in \langle i_{r-2}, i_{r-1}]} \{G(h) + c_h d_{i_{r-1}, j-1}\} + f_j$$

$$+ c_j d_{j, i_r - 1} \quad \text{for } j \in \langle i_{r-1}, i_r \rangle. \tag{4}$$

The minimization in (4) determines an optimal predecessor of j in the restricted problem corresponding to $G(j)$. Because the last two terms do not depend on h, we are mainly concerned with calculating the values $\min_{h \in \langle i_{r-2}, i_{r-1}]} \{G(h) + c_h d_{i_{r-1}, j-1}\}$ for all $j \in \langle i_{r-1}, i_r \rangle$. To this end, we construct the lower envelope of the lines with constant term $G(h)$ and slope c_h for $h \in \langle i_{r-2}, i_{r-1}]$. For a fixed $j \in \langle i_{r-1}, i_r \rangle$ the value of interest is found by

evaluating the lower envelope in coordinate $d_{i_{r-1}, j-1}$ on the horizontal axis. Using similar arguments as in Section 1, one can easily show that determining $\min_{h \in \langle i_{r-2}, i_{r-1} \rangle} \{G(h) + c_h d_{i_{r-1}, j-1}\}$ in this way for all $j \in \langle i_{r-1}, i_r \rangle$ takes a computational effort that is bounded by a constant times the sum of the cardinalities of the sets $\langle i_{r-2}, i_{r-1} \rangle$ and $\langle i_{r-1}, i_r \rangle$. Subsequently, the values $G(j)$ are easily obtained for all $j \in \langle i_{r-1}, i_r]$. One can now determine λ_r and proceed with the analogous calculation of $G(k)$ for all $k \in \langle i_r, i_{r+1} \rangle$. The complexity of this algorithm to determine λ_{q+1}, and thus λ', is easily seen to be O(T). Note that a solution with $q + 1$ setups that is optimal for $\lambda = \lambda'$ can be constructed in linear time if we have stored an optimal predecessor of j when calculating $G(j)$. To summarize, we have the following result.

Theorem 2. *It takes linear time to calculate λ' (or to find out that it does not exist) and to determine a solution with exactly $q + 1$ production periods that is optimal for this value.*

We have only looked at the parametric problem in which all setup costs are reduced when the parameter increases. It is left to the reader to verify that similar results as presented in this section hold for the parametric problem in which all setup costs increase by the same amount when the parameter increases. Therefore, we state the following theorem without proof.

Theorem 3. *Consider an economic lot-sizing problem without speculative motives that has an optimal solution with $q > 1$ production periods. Let λ'' be the smallest amount such that there exists an optimal solution with less than q production periods when all setup costs are increased by λ''. The value of λ'' and a corresponding optimal solution with exactly $q - 1$ setups can be determined in linear time.*

4. Improved algorithms and theoretical results

In this section we show that the results obtained in Section 3 are not only interesting by themselves,

but that they also lead to improved algorithms for several problems which have been discussed before in the literature. Furthermore, we are able to generalize two known convexity results.

4.1. Computing stability regions

Richter [7,8] considers the economic lot-sizing model with stationary cost coefficients, i.e., $f_i = f \geqslant 0$, $h_i = h \geqslant 0$ and $p_i = p$ for all $i \in \{1, \ldots, T\}$. Without loss of generality, we may assume $p = 0$ and therefore only the values of f and h are relevant. It is easily seen that not the absolute value of these coefficients, but rather their ratio determines the optimal solution. Hence, the non-negative quadrant of the (f, h)-space can be partitioned into convex cones, each of which corresponds to another optimal solution. Moreover, there are at most T of these cones, each corresponding to another number of setups in the optimal solution. For fixed f_0 and h_0 and a given optimal solution Richter determines the corresponding convex cone ("stability region") using an algorithm that runs in at least O(T^2) time. Van Hoesel and Wagelmans [9] point out that this time bound can actually be achieved. However, Theorems 2 and 3 imply an even stronger result. To use those theorems we fix the unit holding cost to h_0 and consider the two parametric problems that result when λ is subtracted from f_0, respectively, added to f_0. Both λ' and λ'', defined as before, can be calculated in linear time. It is easily seen that the given solution is optimal for all pairs (f, h) that satisfy $(f_0 - \lambda')/h_0 \leqslant f/h \leqslant (f_0 + \lambda'')/h_0$, and not for any other pair. Hence, computing the stability region can be done in linear time, which is as fast as solving the problem.

We also note that if the setup cost is stationary, whereas the marginal production costs and holding costs are non-stationary, then the setup cost stability region (interval) can still be computed in linear time, provided that there are no speculative motives. This constitutes a considerable improvement over the O(T^3) algorithm of Chand and Vörös [10], who only allow the holding costs to be non-stationary. However, these authors also study the model in which backlogging is allowed. Clearly, our results are not applicable to that model.

4.2. Computing the value function and efficient solutions

Zangwill [11] studies the implications of setup cost reduction in the economic lot-sizing model by performing a parametric analysis (see also [12,13]). His main motivation is to analyze the concepts of the Zero Inventory philosophy, which states that the inventory levels should be as small as possible and that this can be accomplished by reducing the setup costs. Zangwill shows that reducing all setup costs by the same amount may sometimes increase total holding costs. However, if the setup costs and unit production costs are stationary ($f_i = f$ and $p_i = p$ for all $i \in \{1, \ldots, T\}$), then setup cost reduction leads to reduction of both the total holding costs and the number of periods with positive inventory.

Zangwill's results are partly based on the analysis of the *value function*, i.e., the function that gives the optimal value of the lot-sizing problem for every $\lambda \in [0, \Lambda]$. It is easily seen that the value function is piecewise linear, decreasing and concave. Moreover, the function has at most T linear segments. To construct this function Zangwill proposes an algorithm that runs in $O(T^3)$. Instead of this specialized algorithm one may use a well-known general method that is often attributed to Eisner and Severance [14]. This method constructs the value function by solving at most $2T + 1$ non-parametric lot-sizing problems. If the Wagner–Whitin algorithm is used to solve the latter problems again an $O(T^3)$ time bound results. However, we may also use the linear time algorithm, because only lot-sizing problems without speculative motives are considered. Hence, the value function can be constructed in $O(T^2)$ time.

Theorem 2 implies yet another approach to construct the value function. We may apply the procedure given in Section 2 repeatedly. Starting with an optimal solution for $\lambda^0 \equiv 0$, we first find λ', the largest value of λ for which the given solution is optimal. At the same time we find a solution that is optimal for λ' and that has one setup less than the original optimal solution. We now proceed by letting λ' play the role of λ^0. Clearly, we will find the complete value function after at most $T - 1$ applications of our procedure. Hence, this approach also takes $O(T^2)$ time, and from a complex-

ity point of view it does not perform better than the Eisner–Severance method. However, in the following application this approach is particularly useful.

Richter [15] analyzes the stationary cost model with respect to the criteria total costs and total inventory. The goal is to find all *efficient solutions*, i.e., all solutions for which there does not exist another solution that is better on one criterion and not worse on the other. Assume that the there exists an optimal solution (w.r.t. total costs) that has $q < T$ production periods. One can show that the total inventory is non-increasing in the number of setups; for instance, this follows from the result by Zangwill [11] mentioned earlier and also from Theorem 4 in the next subsection. Hence, to find all efficient solutions it suffices to determine for all $k \in \{q, \ldots, T\}$ the optimal value of the problem in which the number of setups is restricted to be exactly k. The latter can be done by calculating the value function of the parametric problem in the way indicated above (where Λ equals the setup cost). This approach has a lower running time than the one used by Richter, which is based on the Wagner–Whitin algorithm and runs in $O(T^3)$ time or worse (no complexity analysis is given). We should also mention that the Eisner–Severance method can not be used, because it does not necessarily determine optimal solutions for all $k \in \{q, \ldots, T\}$. In particular the latter may happen if for some $k \in \{q, \ldots, T\}$ the corresponding solution is only optimal for one value of $\lambda \in [0, \Lambda]$.

4.3. A convexity result

Consider a lot-sizing problem without speculative motives and suppose that there exists an optimal solution with $q > 1$ setups. For $k \in \{1, \ldots, T\}$ we let $TC(k)$ denote the optimal value of the problem in which the number of setups is restricted to be exactly k.

Theorem 4. *The function* TC *is non-increasing on* $\{1, \ldots, q\}$ *and non-decreasing on* $\{q + 1, \ldots, T\}$. *Furthermore,* TC *is convex on* $\{1, \ldots, T\}$.

Proof. We will provide the proof only for the range $\{1, \ldots, q\}$. A similar proof holds for the other range and is left to the reader.

Consider the parametric problem in which the setup cost in period i is equal to $f_i + \Lambda - \lambda$, where $\lambda \in [0, \Lambda]$ and $\Lambda = c_1 d_{1T}$. Hence, for $\lambda = \Lambda$ there exists an optimal solution with q setups and for $\lambda = 0$ it is optimal to produce only in period 1. It follows that there exist values $0 = \lambda^0 \leqslant \lambda^1 \leqslant \cdots \leqslant \lambda^q = \Lambda$ such that for every $k \in \{1, \ldots, q\}$ there are k setups in an optimal solution if and only if $\lambda \in [\lambda^{k-1}, \lambda^k]$. The value of an optimal solution with k setups is equal to $\mathrm{TC}(k) + k\Lambda - k\lambda$. For $\lambda = \lambda^k, 1 \leqslant k < q$, optimal solutions with k and $k + 1$ setups exist. Therefore,

$$\mathrm{TC}(k) + k\Lambda - k\lambda^k = \mathrm{TC}(k + 1) + (k + 1)\Lambda$$
$$- (k + 1)\lambda^k$$

or equivalently

$$\mathrm{TC}(k) - \mathrm{TC}(k + 1) = \Lambda - \lambda^k. \tag{5}$$

Because the right-hand side of this equality is non-negative, it follows that TC is non-increasing on $\{1, \ldots, q\}$. Clearly, for $1 < k < q$ it also holds that

$$\mathrm{TC}(k - 1) - \mathrm{TC}(k) = \Lambda - \lambda^{k-1}.$$

Combining this with (5) and $\lambda^{k-1} \leqslant \lambda^k$, we obtain

$$\mathrm{TC}(k) - \mathrm{TC}(k - 1) \leqslant \mathrm{TC}(k + 1) - \mathrm{TC}(k)$$

and this means that TC is convex on $\{1, \ldots, q\}$. $\quad\square$

Remark. Note that the problem reformulation that we have carried out (eliminating the holding costs and replacing p_i by c_i) does not affect this result, because it has caused the value of all feasible solutions to change by the same amount. For convenience, we assume that $\mathrm{TC}(k)$ equals the solution value w.r.t. the original objective function.

An obvious application of Theorem 4 concerns the problem in which the number of setups is restricted to be at most n. The theorem states that if the unrestricted problem has an optimal solution with $q > n$ production periods, then there exists an optimal solution of the restricted problem with exactly n setups and it follows from the proof of Theorem 4 that this solution can be determined in $\mathrm{O}(nT)$ time.

Theorem 4 generalizes a result by Tunçel and Jackson [16], who use a completely different ap-

proach to show this result for the special case in which $p_i = p$ for all $i \in \{1, \ldots, T\}$. It also generalizes a result by Chand and Sethi [17], who also consider the special case of stationary marginal production costs. They define $\mathrm{HC}(k)$ to be the minimum holding cost if the number of setups is restricted to $k \in \{1, \ldots, T\}$ and show that this function is non-increasing and convex. To see that this is a special case of Theorem 4, it suffices to assume $f_i = p_i = 0$ for all $i \in \{1, \ldots, T\}$ and to note that in that case $q = T$ and $\mathrm{TC}(k) = \mathrm{HC}(k)$ for all $k \in \{1, \ldots, T\}$.

5. Concluding remarks

By carrying out a parametric analysis of the setup costs in the economic lot-sizing problem, we have obtained new results about the structure of optimal solution for given number of setups, we have been able to design fast algorithms for several related problems and we have obtained additional theoretical results which may be useful. Our analysis and the algorithms which we have proposed differ significantly from existing approaches. We think that the characterization given in Theorem 1 and the algorithm it suggests are particularly interesting. An interesting topic for future research is the question whether similar results hold if we allow backlogging.

Acknowledgements

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