

Enumeration of Circuits and Minimal Forbidden Sets

Frederik Stork^a Marc Uetz^b

^a*ILOG Deutschland GmbH, Germany (fstork@ilog.de)*

^b*Universiteit Maastricht, The Netherlands (m.uetz@ke.unimaas.nl)*

Abstract

In resource-constrained scheduling, it is sometimes important to know all inclusion-minimal subsets of jobs that must not be scheduled simultaneously. These so-called minimal forbidden sets are given implicitly by a linear inequality system, and can be interpreted more generally as the circuits of a particular independence system. We present several complexity results related to computation, enumeration, and counting of the circuits of an independence system. On this account, we also propose a backtracking algorithm that enumerates all minimal forbidden sets for resource constrained scheduling problems.

Key words: Independence system, Circuit, Enumeration, Minimal forbidden set

1 Introduction

Given a finite ground set V , an independence system is defined as a family \mathcal{I} of subsets of V with two properties. First, $\emptyset \in \mathcal{I}$, and second, any subset of any member of \mathcal{I} also belongs to \mathcal{I} . The sets in \mathcal{I} are called independent sets, and the inclusion-maximal independent sets are the *bases* \mathcal{B} of \mathcal{I} . The sets not in \mathcal{I} are called dependent sets, and inclusion-minimal dependent sets are the *circuits* \mathcal{C} of \mathcal{I} ; see also [6]. Given a membership oracle for such an independence system \mathcal{I} , we are interested in the problem to enumerate all circuits of \mathcal{I} . It is obvious that the output size may be exponential in terms of the size of V . The complexity is thus measured in terms of the size of both, in- and output. Given that $|V| = n$, and given a (poly(n)) membership oracle for some collection of subsets $\mathcal{S} \subseteq 2^V$, we use the following definitions; see [4].

Definition 1 *The enumeration problem for \mathcal{S} is solvable in polynomial total time if there exists an algorithm and a polynomial $p(\cdot, \cdot)$, such that the algorithm correctly outputs all members of \mathcal{S} in $p(n, |\mathcal{S}|)$ time.*

Given a sub-collection $\mathcal{X} \subseteq \mathcal{S}$, the *increments problem* is the problem: either decide that $\mathcal{X} = \mathcal{S}$, otherwise output a new element in $\mathcal{S} \setminus \mathcal{X}$.

Definition 2 *The enumeration problem for \mathcal{S} is solvable in incremental polynomial time if there exists an algorithm and a polynomial $p(\cdot, \cdot)$, such that the algorithm correctly solves the increments problem in $p(n, |\mathcal{X}|)$ time, for any $\mathcal{X} \subseteq \mathcal{S}$.*

If the enumeration problem for \mathcal{S} is solvable in incremental polynomial time, it is also solvable in polynomial total time: Starting with $\mathcal{X} = \emptyset$, the iterative solution of the increments problem yields the complete collection \mathcal{S} , in time polynomial in n and $|\mathcal{S}|$. The reverse, however, need not be true.

2 The general case

The following theorem is well known; it is proved using a reduction from the NP-complete decision problem SATISFIABILITY [3].

Theorem 3 ([5]) *Unless $P=NP$, there does not exist a polynomial total time algorithm that enumerates the bases \mathcal{B} of any independence system \mathcal{I} .*

Likewise, using a simple duality argument, we can show the following.

Theorem 4 *Unless $P=NP$, there does not exist a polynomial total time algorithm that enumerates the circuits \mathcal{C} of any independence system \mathcal{I} .*

The proof uses the fact that the circuits of \mathcal{I} are the bases of the following, dual independence system: $\mathcal{I}^D = \{W \subseteq V \mid V \setminus W \notin \mathcal{I}\}$. These two theorems say that, unless $P=NP$, an algorithm cannot exist which solves the problem for *any* independence system \mathcal{I} . For particular realizations of the membership oracle of \mathcal{I} however, efficient enumeration algorithms may well exist.

3 Scheduling and linear inequality systems

In resource-constrained scheduling, the input consists of a set of partially ordered jobs (V, \prec) (the partial order representing the precedence constraints) and resource constraints. The latter are given through a number of resource types k with availabilities b_k , and resource requirements a_{kj} of these resource types for all jobs $j \in V$. If a subset S of jobs consumes more of a resource type than available, the respective jobs in S may not be processed in parallel. The subsets of jobs that may be processed in parallel define an inde-

pendence system. The circuits of this independence system are either pairs of (precedence-)related jobs, $\{i, j\}$ with $i \prec j$, or the so-called *minimal forbidden sets* (minimal anti-chains of (V, \prec) which may not be processed in parallel). For several algorithmic purposes, e.g. in stochastic scheduling [8], a complete list of the minimal forbidden sets is required. This amounts to the computation of the circuits of the corresponding independence system. The membership oracle of this system is a linear inequality system $Ax \leq b$, where A, b contains one row in for each resource type k , and one row for each edge in the comparability graph of the partial order (V, \prec) . The circuits are the minimally infeasible $\{0, 1\}$ -vectors for $Ax \leq b$; they are the incidence vectors of either pairs of (precedence-)related jobs, or minimal forbidden sets.

Inspired by [2], and using a reduction from the NP-complete decision problem INDEPENDENT SET in graphs [3], we show:

Theorem 5 *Unless $P=NP$, there does not exist a polynomial total time algorithm that enumerates the minimally infeasible $\{0, 1\}$ -vectors of an arbitrary linear inequality system $Ax \leq b$.*

In fact, in [2] it is shown that the decision version of the corresponding increments problem is NP-complete. What is interesting is the fact that the ‘dual’ problem is apparently much easier: It follows from [2] that the increments problem for the maximally feasible $\{0, 1\}$ -vectors of an arbitrary linear inequality system $Ax \leq b$ can be solved in quasi-polynomial time, hence there is also a quasi-polynomial total time algorithm for this problem. (Such a result is not likely for the problem considered here, because then all NP-hard problems could be solved in quasi-polynomial time.)

In terms of scheduling, Theorem 5 immediately yields:

Corollary 6 *Unless $P=NP$, there does not exist a polynomial total time algorithm that enumerates the minimal forbidden sets for any instance of the resource-constrained project scheduling problem.*

It turns out that even the computation of the *number* of minimal forbidden sets is a hard problem, because we can show:

Theorem 7 *The problem to compute the number of minimally infeasible (maximally feasible) $\{0, 1\}$ -vectors of an arbitrary linear inequality system $Ax \leq b$ is #P-complete.*

The proof uses a reduction from the problem to compute a maximum cardinality anti-chain of a partial order. While this problem is polynomially solvable, the associated counting problem is known to be #P-complete [7].

Nevertheless, for practical purposes we implemented a simple backtracking al-

gorithm that lists the minimal forbidden sets for any instance of the resource-constrained project scheduling problem. In general, this algorithm can have an exponential running time in terms of in- and output of the problem. Yet, empirically it improves considerably upon a divide-and-conquer algorithm previously suggested in [5,1]; see [9]. Moreover, we can show that the algorithm is efficient for an important special case.

Proposition 8 *There exists an incremental polynomial time (hence also a polynomial total time algorithm) that enumerates the minimal forbidden sets for any instance of the resource-constrained project scheduling problem, given that the number of resource types is 1.*

We refer to [9] for several other results related to enumeration and computation of minimal forbidden sets, as well as detailed computational results.

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