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The lexicographic equal-loss solution

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We introduce a new solution for Nash's bargaining problem, called the *lexicographic equal-loss solution*. This solution is a lexicographic extension of the equal-loss solution, which equalizes across agents the losses from the ideal point, to satisfy Pareto optimality. An axiomatic characterization is presented by using the following five axioms: Pareto optimality, anonymity, translation invariance, weak monotonicity, and independence of alternatives other than the ideal point.

Key words: Bargaining problem; lexicographic equal-loss solution; axiomatic characterization.

1. Introduction

Suppose a bundle of goods is to be divided or redivided between a number of agents. Among the many criteria according to which such a division might be chosen, consider the following pair of more or less complementary criteria: either the utility *gains* of agents relative to their utility levels of initial holdings are important, or their utility *losses* with respect to their maximally attainable utilities.

One appropriate framework for studying problems like the division problem above, is axiomatic bargaining theory, which started with the seminal paper by Nash (1950).[1] Solutions can be judged according to the gains or losses criteria. Typical examples of solutions for which the gains criterion is central, are the egalitarian and lexicographic egalitarian solutions (for formal definitions of these solutions, see the next section); for which either the gains or the losses criterion can be relevant—the Kalai-Smorodinsky (1975) solution.

In this paper we propose and axiomatically characterize a solution for which the losses criterion is taken as a lead: the so-called *lexicographic equal-loss solution*. Ac-

[1] See Thomson (forthcoming) for an extensive review of the literature on axiomatic bargaining theory.

cording to this solution, an outcome in a bargaining problem is determined as follows. First, one takes the maximal feasible outcome in which the players suffer equal losses from their utility levels at the *ideal point* (each coordinate of which is the maximally attainable utility level of an agent guaranteeing to the others their initial utility levels). If this outcome is not (strongly) Pareto optimal, then a lexicographic procedure is used to arrive at a Pareto optimal outcome. So the ideal point plays a central role; the disagreement point (e.g. the point representing the utility levels of initial holdings in a division problem) only matters insofar as it determines the ideal point. The same idea is embodied in two of the axioms used to characterize the lexicographic equal-loss solution, namely *weak monotonicity* and *independence of alternatives other than the ideal point*. The remaining three axioms in our characterization are *Pareto optimality*, *anonymity*, and *translation invariance*.

The axioms also show that the lexicographic equal-loss solution offers a compromise between monotonicity and Pareto optimality. Recall that Luce and Raiffa (1957) have already shown that (strong) monotonicity is inconsistent with Pareto optimality.

The next section discusses definitions and axioms, and Section 3 contains the characterization result and its proof. Section 4 concludes.

2. Preliminaries

An n-person bargaining problem, or simply a *problem*, is a pair (S,d), where S is a subset of $\boldsymbol{R}^n$ and d is a point in S, such that

(1) S is convex and closed,
(2) $a_i(S,d) \equiv \max\{x_i \mid x \equiv (x_1,\dots,x_n) \in S, x \geqq d\}$[2] exists for all i,
(3) S is *comprehensive*, i.e. for all $x \in S$ and for all $y \in \boldsymbol{R}^n$, if $y \leqq x$, then $y \in S$,
(4) there exists $x \in S$ with $x > d$.

We denote by Σ the class of all n-person problems. We interpret a problem $(S,d) \in \Sigma$ as follows: the n agents can achieve any point of the *feasible set* S if they unanimously agree on it; otherwise, they end up at the *disagreement point* d. Points of S are called *feasible outcomes*. The coordinates of an $x \in S$ may be the utility levels attained by the n agents through the choice of some joint action. Closedness of S is required for mathematical convenience; convexity stems from allowing lotteries in an underlying bargaining situation. Property (2) is a boundedness condition, and comprehensiveness may be justified by the assumption of free disposability of utility. Condition (4) is a nondegenerateness requirement, and may be interpreted as providing the agents with an incentive to reach some agreement.

A *solution* is a function $F: \Sigma \to \boldsymbol{R}^n$ such that for all $(S,d) \in \Sigma$, $F(S,d) \in S$. $F(S,d)$, the value taken by the solution F when applied to the problem (S,d), is

[2] Vector inequalities: given $x, y \in \boldsymbol{R}^n$, $x \geqq y$, $x \geq y$, $x > y$.

called the *solution outcome of* (S, d). Nash (1950) proposed to handle bargaining problems by looking for solutions satisfying appealing axioms, and we will adopt his approach here.

In what follows, we will be interested in the following axioms. First, we introduce some notation. For $(S, d) \in \Sigma$, let $\mathrm{PO}(S) \equiv \{x \in S \mid$ for all $x' \in \boldsymbol{R}^n$, $x' \geq x$ implies $x' \notin S\}$ be the set of *Pareto optimal points of* S. Similarly, let $\mathrm{WPO}(S) \equiv \{x \in S \mid$ for all $x' \in \boldsymbol{R}^n$, $x' > x$ implies $x' \notin S\}$ be the set of *weakly Pareto optimal points of* S.

Pareto optimality (PO). For all $(S, d) \in \Sigma$, $F(S, d) \in \mathrm{PO}(S)$.

Let $N \equiv \{1, \ldots, n\}$ denote the set of agents, and let $\pi : N \to N$ be a permutation of N. For $x \equiv (x_1, \ldots, x_n) \in \boldsymbol{R}^n$, let $\pi x \equiv (x_{\pi(1)}, \ldots, x_{\pi(n)})$, and for $S \subset \boldsymbol{R}^n$, let $\pi S \equiv \{\pi x \mid x \in S\}$.

Anonymity (AN). For all $(S, d) \in \Sigma$ and all permutations π of $N, F(\pi S, \pi d) = \pi F(S, d)$.

Given $S \subset \boldsymbol{R}^n$ and $t \in \boldsymbol{R}^n$, let $S + t \equiv \{x + t \mid x \in S\}$.

Translation invariance (TINV). For all $(S, d) \in \Sigma$ and for all $t \in \boldsymbol{R}^n$, $F(S + t, d + t) = F(S, d) + t$.

PO requires that the solution outcome should exhaust all gains from cooperation. AN requires that the names of the agents do not affect the solution outcome. TINV requires that the choice of zeros for utility functions does not matter in the determination of the solution outcome.

The following two axioms are less well known, but have been discussed before in the literature. For $x \in \boldsymbol{R}^n$ and $i \in N$, let x_{-i} be the $(n-1)$-dimensional vector obtained after deleting the ith component of x. Also, for $(S, d) \in \Sigma$, let $S_{d,-i} \equiv$ the closure of $\{x_{-i} \mid x \in S, x \leqq a(S, d)\}$.

Weak monotonicity (WMON). For all $(S, d), (S', d') \in \Sigma$, if $S \subseteq S'$, $d = d'$ and $S_{d,-i} = S'_{d',-i}$ for all i, then $F(S', d') \geqq F(S, d)$.

Independence of alternatives other than the ideal point (IAIP). For all (S, d), $(S', d') \in \Sigma$, if $S \supseteq S'$, $a(S, d) = a(S', d')$ and $F(S, d) \in S'$, then $F(S', d') = F(S, d)$.

WMON, introduced for two-person problems by Kalai and Smorodinsky (1975), requires that an expansion of the feasible set which does not affect the ideal point should not hurt any agent. However, a straightforward extension of this axiom may be incompatible with Pareto optimality for more than two-person bargaining problems, as shown by Roth (1979). Our version of the axiom, introduced by Imai (1983), is a modification of the original axiom to be compatible with Pareto optimality. IAIP introduced by Roth (1977), requires that if the feasible set contracts and the disagreement point changes without affecting the ideal point, and the solution outcome for the original problem is still feasible for the smaller problem, then the solution outcome for the smaller problem should be the same as that for the original problem. This requirement could be regarded as 'dual' to Nash's (1950) *independence of irrelevant alternatives*, which requires that if the feasible set contracts without affecting the disagreement point, and the solution outcome for the original problem is still feasible for the smaller problem, then the solution outcome for the smaller problem should be the same as that for the original problem.

Now we introduce the lexicographic equal-loss solution. For this definition, and in what follows, we need an additional notation. For $\emptyset \neq M \subseteq N$, we denote by e_M the n-dimensional vector with ith coordinate 1 if $i \in M$ and 0 otherwise.

Definition. Let $>^{\mathrm{l}}$ denote the *lexicographic ordering* on $\boldsymbol{R}^n$, i.e. $x >^{\mathrm{l}} y$ $(x, y \in \boldsymbol{R}^n)$ if there is an $i \in N$ with $x_i > y_i$ and $x_j = y_j$ for all $j < i$. Let $\alpha : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be such that for each $x \in \boldsymbol{R}^n$ there is a permutation π of N with $\alpha(x) = \pi(x)$ and $\alpha_1(x) \leqq \alpha_2(x) \leqq \cdots \leqq \alpha_n(x)$. Then the *lexicographic maximin ordering* $>^{\mathrm{lm}}$ on $\boldsymbol{R}^n$ is defined by $x >^{\mathrm{lm}} y (x, y \in \boldsymbol{R}^n)$ if $\alpha(x) >^{\mathrm{l}} \alpha(y)$. The *lexicographic equal-loss solution* $L^* : \Sigma \to \boldsymbol{R}^n$ assigns to each problem $(S, d) \in \Sigma$ the unique point of S in the following way:

(1) let $t \equiv a(S, d)$ and $S^* \equiv S - t$,

(2) find a maximal element x^* of S^* with respect to $>^{\mathrm{lm}}$,

(3) $L^*(S, d) \equiv x^* + t$.

(The *equal-loss solution* $E^* : \Sigma \to \boldsymbol{R}^n$ assigns to each problem $(S, d) \in \Sigma$ the maximal point x of S such that $a_i(S, d) - x_i = a_j(S, d) - x_j$ for all $i, j \in N$.)

It can be shown that L^* is well defined, in much the same way as this is done for the so-called lexicographic egalitarian solution.[3] L^* can be regarded as 'dual' to this solution, just as E^* corresponds to the well-known egalitarian solution. The egalitarian solution and its lexicographic version have been studied extensively, whereas the equal-loss solution has been characterized very recently by Chun (1988).[4] For two-person bargaining problems, E^* satisfies Pareto optimality and consequently coincides with its lexicographic version. However, for more than two-person bargaining problems, E^* suffers the following two limitations: it satisfies only weak Pareto optimality and, as pointed out by Thomson (forthcoming), it does not satisfy individual rationality.[5] Here we propose its lexicographic version, which satisfies Pareto optimality for all bargaining problems. Also we will discuss, in the con-

[3] If we set $t \equiv d$ in the above definition, then the resulting solution is the *lexicographic egalitarian solution*. The *egalitarian solution* $E : \Sigma \to \boldsymbol{R}^n$ assigns to each problem $(S, d) \in \Sigma$ the maximal point x of S such that $x_i - d_i = x_j - d_j$ for all $i, j \in N$. The egalitarian solution has been characterized by Kalai (1977), and its lexicographic version by Lensberg (1982), Chun (1989), and Chun and Peters (1988, 1989).

[4] E^* is a variant of the family of solutions proposed by Yu (1973). Actually, Yu's 'solution' is not a solution in our sense since it may result in multiple solution outcomes.

[5] *Individual rationality*: for all $(S, d) \in \Sigma$, $F(S, d) \geqq d$.

cluding section, how the lexicographic equal-loss solution can be modified to satisfy both Pareto optimality and individual rationality.

There is an interesting procedure to find $L^*(S,d)$ $((S,d) \in \Sigma)$. First, *decrease* the utilities of the n agents in $N^1 \equiv N$ equally from $a(S,d)$, along $a(S,d) - \zeta^1 e_{N^1}$ $(\zeta^1 \geqq 0)$, until a boundary point is reached, say z^1. If $z^1 \in \mathrm{PO}(S)$, then set $z \equiv z^1$. Otherwise, let $N^2 \subset N$ be the largest possible subset of agents whose utilities can be equally *increased* in a non-negative direction starting from z^1, i.e. go along the direction $z^1 + \zeta^2 e_{N^2}$ $(\zeta^2 > 0)$. Let z^2 be the maximal point in this direction and still in S; if $z^2 \in \mathrm{PO}(S)$, then $z \equiv z^2$, otherwise we continue along the direction $z^2 + \zeta^3 e_{N^3}$ $(\zeta^3 > 0)$, where $N^3 \subset N^2$ is the largest possible subset of agents for which an *increase* along $z^2 + \zeta^3 e_{N^3}$ is still possible, etc. In this way we end up, after a finite number of steps, at a point $z \in \mathrm{PO}(S)$. It is not hard to show that $z = L^*(S,d)$; one may adapt Lemma 3 in Imai (1983) to our context. This procedure to find $L^*(S,d)$ illustrates our expression *lexicographic equal-loss solution*. The *equal-loss solution* assigns to each problem $(S,d) \in \Sigma$ the point z^1 above.

3. Main result

In this section we show that the lexicographic equal-loss solution, L^*, is the unique solution satisfying the five axioms introduced above.[6]

Theorem. *The lexicographic equal-loss solution L^* is the unique solution satisfying Pareto optimality, anonymity, translation invariance, weak monotonicity, and independence of alternatives other than the ideal point.*

It is straightforward to verify that L^* satisfies PO, AN, and TINV. The fact that it satisfies IAIP and WMON is proved in the following lemmas.

Lemma 1. *The lexicographic equal-loss solution satisfies independence of alternatives other than the ideal point.*

Proof. Let $(S,d), (S',d') \in \Sigma$ be two problems satisfying the hypotheses of IAIP. Also, let $\{z^t\} \subset S$ be the sequence as defined in the process of finding $L^*(S,d) \equiv z^T$. Since $z^T \in S'$, $z^t \leqq z^T$ for all t, and S' is comprehensive, we have $z^t \in S'$ for all t. Now we construct the sequence $\{\bar{z}^t\} \subset S'$ to find $L^*(S',d')$. Since $S' \subseteq S$, $a(S',d') = a(S,d)$, and $z^t \in S'$ for all t, $\bar{z}^t = z^t$ for all t. Therefore, we conclude that $L^*(S',d') = z^T = L^*(S,d)$. □

[6] Although we characterize the lexicographic equal-loss solution, whereas Imai (1983) characterizes the lexicographic version of the Kalai–Smorodinsky (1975) solution, some parts of our proofs are similar to those of Imai.

Lemma 2. *The lexicographic equal-loss solution satisfies weak monotonicity.*

Proof. Let $(S,d),(\bar{S},\bar{d})\in\Sigma$ be such that $S\subseteq\bar{S}$, $d=\bar{d}$ and $S_{d,-i}=\bar{S}_{d,-i}$ for all i. Note that $S_{d,-i}=\bar{S}_{d,-i}$ for all i implies that $a(S,d)=a(\bar{S},d)$. Since L^* satisfies TINV, we may assume that $a(S,d)=e_N$. The proof is done by the help of two claims, which require the following additional notation. For $y\in S$, let $N(S,y)\subseteq N$ be defined by $N(S,y)\equiv\{i\in N\mid y+\zeta e_{\{i\}}\in S$ for some $\zeta>0\}$. $N(S,y)$ denotes the largest subset of players of N, whose utilities could be increased equally from y in S. Let ζ^* be the minimal number such that for all $\zeta>\zeta^*$, $y+\zeta e_{N(S,y)}\notin S$. Finally, let $z(S,y)\equiv y+\zeta^* e_{N(S,y)}$.

Claim 1. For all $y\in S$, if $N(S,y)\neq\emptyset$, then $N(S,y)=N(\bar{S},y)$.

Proof. Since $S\subseteq\bar{S}$, it is clear that $N(S,y)\subseteq N(\bar{S},y)$. We will show that $N(\bar{S},y)\subseteq N(S,y)$. Suppose, by way of contradiction, that there exists $j\in N(\bar{S},y)\setminus N(S,y)$. Let $z\equiv z(S,y)$ and $\bar{z}\equiv z(\bar{S},y)$. Clearly, $z\leqq\bar{z}$. Now pick $k\in N(S,y)$. Since $S_{d,-i}=\bar{S}_{d,-i}$ for all i, there exists $x\in S$ such that $x_{-k}=\bar{z}_{-k}$. By the convexity of S, for all $\lambda\in[0,1]$, $\lambda x+(1-\lambda)z\equiv x^\lambda\in S$. Since $x_{-k}=\bar{z}_{-k}\geqq y_{-k}$, $z_{-k}\geqq y_{-k}$ and $z_k>y_k$, there exists $\lambda\in(0,1]$ such that $x^\lambda\geqq y$. Since $\bar{z}_j>y_j=z_j$, $x_j^\lambda>y_j$. Altogether, we obtain $x_j^\lambda>y_j$, $x^\lambda\geqq y$ and $x\in S$, which implies that $j\in N(S,y)$, a contradiction.

Claim 2. Let $T>1$ be the final step in finding $L^*(S,d)$. Also, let $\{z^t\}$ and $\{\bar{z}^t\}$ be the two sequences as defined in the process of finding $L^*(S,d)$ and $L^*(\bar{S},d)$ respectively. Then, for all $t=1,\dots,T-1$, $z^t=\bar{z}^t$.

Proof. First, we will consider the case when $t=1$. Since $S\subseteq\bar{S}$ and $a(S,d)=a(\bar{S},d)$, it is clear that $z^1\leqq\bar{z}^1$. We need to show that $\bar{z}^1\leqq z^1$. Suppose, by way of contradiction, that there exists $j\in N$ such that $\bar{z}_j^1>z_j^1$. Since $\bar{z}^1$ and z^1 are points with equal coordinates, it follows that $\bar{z}^1>z^1$. Since T is the final step, $z^1\in\mathrm{WPO}(S)\setminus\mathrm{PO}(S)$. Therefore, there exists $x\in S$ such that $x\geq z^1$. Let $k\in N$ be such that $x_k>z_k^1$. On the other hand, since $S_{d,-i}=\bar{S}_{d,-i}$ for all i, there exists $y\in S$ such that $y_{-k}=\bar{z}_{-k}^1$. By the convexity of S, for all $\lambda\in[0,1]$, $\lambda x+(1-\lambda)y\equiv y^\lambda\in S$. Since $x\geq z^1$, $x_k>z_k^1$ and $y_{-k}=\bar{z}_{-k}^1>z_{-k}^1$, there exists $\lambda\in(0,1)$ such that $y^\lambda>z^1$. This is a contradiction to $z^1\in\mathrm{WPO}(S)$.

The proofs for $t=2,\dots,T-1$ are analogous, using Claim 1, and thus are omitted.

Finally, by combining the results of Claims 1 and 2, it follows that $z^T=L^*(S,d)\leqq\bar{z}^T\leqq L^*(\bar{S},d)$. Therefore, L^* satisfies WMON. □

For the proof of the theorem, we need some additional notation. Given $(S,d)\in\Sigma$, $\mathrm{Int}(S)$ is the *interior of* S. And given $p\in\boldsymbol{R}^n$ and $z\in\boldsymbol{R}^n$, $H(p,pz)\equiv\{x\in\boldsymbol{R}^n\mid px\leqq pz\}$. Here, juxtaposition of vectors denotes inner product.

We will try to sketch the main idea behind the proof before diving into its mathematics. The proof uses the procedure for finding $L^*(S,d)$ ($(S,d)\in\Sigma$) described at the end of Section 2. Note that we need to figure out $z^1,\ldots,z^T$ to obtain $L^*(S,d)=z^T$. First, by translation invariance, we may assume that the ideal point has all coordinates equal. The main step of the proof lies in the construction of sequences of problems, whose solution outcome is z^t ($t=1,\ldots,T$). The first problem of the sequence is symmetric, whence its solution outcome is determined to be z^1 by Pareto optimality and anonymity. In the induction argument, using weak monotonicity and independence of alternatives other than the ideal point, we obtain that the solution outcome for step t ($t=2,\ldots,T$) should be greater than or equal to the solution outcome for step $t-1$, z^{t-1}. By Pareto optimality, we can conclude that it is equal to z^t.

Proof of theorem. Let F be a solution satisfying the five axioms. Also, let $(\bar{S},\bar{d})\in\Sigma$ be given. By TINV, we may assume that $a(\bar{S},\bar{d})=e_N$. Let $S\equiv\{x\in\bar{S}\mid x\leqq a(\bar{S},\bar{d})\}$ and $d'\in\operatorname{Int}(S)$ be such that $d'_i=d'_j\equiv 1-\delta$ for all $i,j\in N$ and $a(S,d')=e_N$. Equivalently, we may take, by TINV, $d=0$ and $a(S,d)=\delta e_N$. Note that $\delta>0$. Now let $\{z^t\}_{t=1}^T$ and $\{N^t\}_{t=1}^T$ be the sequences as defined in the process of finding $L^*(S,d)$. We will show that $F(S,d)=z^T$. Then, by IAIP we have $F(S,\bar{d})=z^T$, and by WMON, we have $F(\bar{S},\bar{d})=z^T=L^*(\bar{S},\bar{d})$, which then concludes the proof.

Now we construct auxiliary problems. Let $M^t=N\backslash N^t$ and $p^t\equiv e_{M^t}$ for $t=1,\ldots,T$ (where $M^1=\emptyset$ and $p^1=0$). Define

$$S^{1,t}\equiv H(e_N,\ \textstyle\sum z_i^t)\cap\left(\bigcap_{k=1}^{t}H(p^k,p^kz^k)\right)\cap(\delta e_N-\boldsymbol{R}_+^n),\quad \text{for } t=1,\ldots,T,$$

$$S^{2,t}\equiv S^{1,t}\cap H(p^{t+1},p^{t+1}z^{t+1}),\quad \text{for } t=1,\ldots,T-1,$$

$$S^{3,t}\equiv H(e_N,\ \textstyle\sum z_i^t)\cap S,\quad \text{for } t=1,\ldots,T, \text{ and}$$

$$S^{4,t}\equiv S^{1,t}\cap S,\quad \text{for } t=1,\ldots,T.$$

Claim 1. $z^t\in S^{r,t}$ and $d\in\operatorname{Int}(S^{r,t})$ for all r and t.

Proof. By definition of p^t, $p^tz^t=p^tz^{t+s}$ for all $t=1,\ldots,T$ and for all $s=1,\ldots,T-t$. Also, note that by definition of the sequence $\{z^t\}$, $z^t\leq z^{t+1}$ for all $t=1,\ldots,T-1$. Now it follows immediately that $z^t\in S^{r,t}$ for all r and t. Since $d<z^1$ and S is comprehensive, $d\in\operatorname{Int}(S^{r,t})$ for all r and t. This proves the claim.

Claim 2. $a(S^{r,t},d)=\delta e_N$ for all r and t.

Proof. For all $i\in N$, let y^i be such that $y^i_i=\delta$ and $y^i_j=0$ for all $j\neq i$. Since $a(S,d)=\delta e_N$ and S is comprehensive, $y^i\in S$ for all i. It is enough to show that all y^i's belong to the half-spaces defined above. For $t=1$, $p^1y^i=0$ and trivially $y^i\in H(p^1,p^1z^1)$ for all i.

Before we consider the case when $t>1$, we first need to establish the following fact. Let m be such that $m=n$ if $T=1$ and $m=|M^2|$ otherwise. We will show that $z^1 \geqq (1/m)\delta e_N$. Since $z^1 \in \mathrm{WPO}(S)$, there exists $p \in \boldsymbol{R}^n_+$ such that for all $z \in S$, $pz \leqq pz^1$. Since $y^i \in S$, $py^i \leqq pz^1$ for all i. Furthermore, if $i \in N^2$, $z^1 + \zeta e_{\{i\}} \in S$ for some $\zeta > 0$, and consequently $p_i = 0$, hence $py^i = 0$. Therefore,

$$\sum_{i \in N} py^i = \sum_{i \in N^2} py^i + \sum_{i \in M^2} py^i = \sum_{i \in M^2} py^i \leqq \sum_{i \in M^2} pz^1.$$

Since $|M^2| \leqq m$, $p \sum y^i \leqq mpz^1$. Equivalently, $(1/m)p \sum y^i \leqq pz^1$. Since $\sum y^i = \delta e_N$, $p((1/m)\delta e_N) \leqq pz^1$. Using the fact that both $(1/m)\delta e_N$ and z^1 are points with equal coordinates, we obtain $(1/m)\delta e_N \leqq z^1$.

Now we go back to the case when $t>1$. Note that if $T>1$, then $m \leqq |M^t|$ for all $t=2, \ldots, T$. Since $(1/m)\delta e_N \leqq z^1 \leqq z^t$,

$$p^t y^i \leqq \delta \leqq \frac{|M^t|}{m} \delta = p^t \left(\frac{1}{m} \delta e_N \right) \leqq p^t z^1 \leqq p^t z^t,$$

for all $t=2, \ldots, T$. Therefore, $y^i \in H(p^t, p^t z^t)$ for all i and for all $t=2, \ldots, T$.

Also, $e_N y^i = \delta = e_N((1/n)\sum y^i) \leqq e_N z^1$. Therefore, $y^i \in H(e_N, \sum z^t_i)$ for all i and t. Altogether, we obtain the desired conclusion.

Claim 3. $S^{1,t+1}_{d,-i} = S^{2,t}_{d,-i}$ and $S^{3,t+1}_{d,-i} = S^{3,t}_{d,-i}$ for all $i=1, \ldots, n$ and for all $t=1, \ldots, T-1$.

Proof. It is clear that $S^{1,t+1}_{d,-i} \supseteq S^{2,t}_{d,-i}$ for all $i=1, \ldots, n$ and for all $t=1, \ldots, T-1$. For the other inclusion relation, let $i \in N$ and $w \in S^{1,t+1}_{d,-i}$ be given. Then there exists $x \in S^{1,t+1}$ such that $x_{-i} = w$. If $e_N x \leqq e_N z^t$, then we are done. Otherwise, let y be such that $y \equiv x - (\sum x_j - \sum z^t_j) e_{\{i\}}$. By the comprehensiveness of $S^{1,t+1}$, $y \in S^{1,t+1}$. Since $e_N y = \sum z^t_j = e_N z^t$ and $y_{-i} = x_{-i} = w$, $w \in S^{2,t}_{d,-i}$. Similarly, we can show that $S^{3,t+1}_{d,-i} = S^{3,t}_{d,-i}$ for all $i=1, \ldots, n$ and for all $t=1, \ldots, T-1$.

Claim 4. $F(S^{r,1}, d) = z^1$ for all r.

Proof. Note that $S^{1,1} \equiv H(e_N, \sum z^1_i) \cap (\delta e_N - \boldsymbol{R}^n_+)$. Therefore, by PO and AN, $F(S^{1,1}, d) = z^1$. By IAIP and Claim 2, $F(S^{2,1}, d) = F(S^{3,1}, d) = F(S^{4,1}, d) = z^1$, as desired.

Claim 5. $F(S^{r,t}, d) = z^t$ for all r and t.

Proof. We use induction on t, based on Claim 4. Suppose, as an induction hypothesis, that the conclusion of Claim 5 holds for all $t=1, \ldots, h-1$. Now we consider the case when $t=h$. We will use Claims 2 and 3 several times, without explicit mentioning. By WMON applied between $(S^{2,h-1}, d)$ and $(S^{1,h}, d)$, $F(S^{1,h}, d) \geq F(S^{2,h-1}, d) = z^{h-1}$. Therefore, by PO and AN, $F(S^{1,h}, d) = z^h$. By IAIP applied between $(S^{1,h}, d)$ and $(S^{4,h}, d)$, $F(S^{4,h}, d) = F(S^{1,h}, d) = z^h$. By WMON applied between $(S^{3,h-1},d)$ and $(S^{3,h},d)$, $F(S^{3,h},d) \geqq F(S^{3,h-1},d) = z^{h-1}$. Note that $z \in S^{3,h}$ and $z \geqq z^{h-1}$ implies that $z_i = z_i^{h-1}$ for all $i \in M^h$. Then $p^t z = p^t z^h = p^t z^t$ for all $t = 1, \ldots, h$ and, consequently, $z \in S^{4,h}$. Since $F(S^{3,h},d) \geqq z^{h-1}$, $F(S^{3,h},d) \in S^{4,h}$. Therefore, by IAIP applied between $(S^{3,h},d)$ and $(S^{4,h},d)$, $F(S^{3,h},d) = F(S^{4,h},d) = z^h$. Finally, by IAIP applied between $(S^{1,h},d)$ and $(S^{2,h},d)$, $F(S^{2,h},d) = z^h$ (this final step is not applicable when $h = T$). This completes the proof for Claim 5.

Claim 6. $F(S,d) = L^*(S,d) \equiv z^T$.

Proof. From a proof similar to that of Claim 3, we can show that $S^{3,T}_{d,-i} = S_{d,-i}$ for all i. Therefore, by applying WMON between $(S^{3,T},d)$ and (S,d), $F(S,d) \geqq F(S^{3,T},d) = z^T$, where the equality follows from Claim 5. Since $z^T \in \mathrm{PO}(S)$, $F(S,d) = z^T$, as desired. Q.E.D.

4. Concluding remarks

We have introduced and characterized the lexicographic equal-loss solution for n-person bargaining problems, using five axioms. Unfortunately, $L^*(S,d)$ does not satisfy individual rationality for more than two-person problems. One possible modification $\bar{L}^*$ of L^*, which satisfies individual rationality, can be defined in the following way. Given $(S,d) \in \Sigma$, let $\bar{S}$ be the comprehensive hull of the individually rational points of (S,d), i.e. the smallest comprehensive set containing $\{x \in S \mid x \geqq d\}$. Then, take $\bar{L}^*(S,d) = L^*(\bar{S},d)$. It is not hard to verify that $\bar{L}^*$ satisfies both Pareto optimality and individual rationality for all bargaining problems.[7] However, its axiomatic characterization remains an open question.

Another drawback of the solution L^* may be that it is not independent of the von Neumann–Morgenstern utility representations chosen (if any); in other words, the solution involves an implicit utility comparison between the agents.[8]

In this paper we mentioned three other solutions, which are also not independent of utility representations, namely the egalitarian, the lexicographic egalitarian, and the equal-loss solutions. Table 1 summarizes the axiomatic properties of these solutions together with the lexicographic equal-loss solution, and indicates where their characterization results can be found.

In our axioms, as well as in the definition of the lexicographic equal-loss solution, the disagreement point plays only a modest role. Hence it is not difficult—as was pointed out to us by a referee—to adapt the model for n-person social choice problems with cardinal utility, i.e. to the present model without a disagreement point.

[7] We are grateful to Walter Bossert for pointing out our earlier mistake and suggesting this solution.

[8] In fact, Roth (1977, 1979, p. 108) showed that a solution satisfying PO, AN and IAIP should involve interpersonal utility comparisons.

Table 1
Axiomatic properties of four bargaining solutions (which are not independent of utility representations). *Weak Pareto optimality* requires that the solution outcome be in the set of weakly Pareto optimal points of a feasible set. *Continuity* requires that a small change in the feasible set results in a small change in the solution outcome. The other axioms are discussed in the paper

	Egalitarian	Lexicographic egalitarian	Equal-loss	Lexicographic equal-loss
Weak Pareto optimality	yes	yes	yes	yes
Pareto optimality	no	yes	no	yes
Continuity	yes	no	yes	no
Anonymity	yes	yes	yes	yes
Translation invariance	yes	yes	yes	yes
Weak monotonicity	yes	yes	yes	yes
Independence of alternatives other than the ideal point	no	no	yes	yes
Independence of irrelevant alternatives	yes	yes	no	no
Characterization results	Kalai (1977)	Chun and Peters (1988) (see also Imai, 1983)	Chun (1988)	Current paper

Instead of the ideal point $a(S,d)$ one could take the global ideal point defined by $a_i(S) \equiv \sup\{x_i \mid x \in S\}$. (Of course, $a_i(S)$ is assumed to be finite.) Further details are omitted.

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