ELSEVIER

Income inequality, quasi-concavity, and gradual population shifts[☆]

Kristof Bosmans

Center for Economic Studies, Katholieke Universiteit Leuven, Naamsestraat 69, B-3000 Leuven, Belgium

Abstract

An income distribution is a *mixture* of two given income distributions if the relative frequency it associates with each income level is a convex combination of the relative frequencies associated with it by the given two income distributions—e.g., the income distribution of a country is obtained as a mixture of the income distributions of its regions. In this article, it is established that all inequality measures commonly considered in the literature—the class of decomposable inequality measures and the class of normative inequality measures based on a social welfare function of the rank-dependent expected utility form—satisfy quasi-concavity properties, which imply, loosely speaking, that mixing income distributions increases inequality. These quasi-concavity properties are then shown to greatly reduce the possible patterns describing the evolution of inequality in the overall income distribution (a mixture) during a process in which population gradually shifts from one of its constituent income distributions to another over time.
© 2006 Elsevier B.V. All rights reserved.

1. Introduction

In the analysis of income inequality, it is often useful to view the income distribution of interest as being composed of several constituent income distributions, e.g., the income distributions

corresponding to different regions, sectors, or genders. The question of how inequality in the overall income distribution is affected if the constituent income distributions change, has received considerable attention in the form of decomposability analysis.[1] By contrast, the complementary question of how overall inequality changes if the population shares corresponding to the constituent income distributions change, has not been studied much. Nevertheless, the latter question is interesting both from the empirical and the theoretical perspective.

There are several empirical phenomena that involve a shift of the population from one constituent income distribution to another. Take as an example the phenomenon of demographic ageing. In this case, the overall income distribution changes over time because population gradually shifts from the income distribution of working consumers to the income distribution of retired consumers. Another (particularly natural) example is that of a country with two regions that have different population growth rates: here population shifts from the income distribution corresponding to the region with the lower growth rate to that corresponding to the region with the higher growth rate. As a final example, consider the development process studied by Kuznets (1955) which involves a gradual population shift from the income distribution of the agricultural sector to that of the industrial sector.

Besides being of empirical interest, the gradual population shift process is relevant theoretically. In order to see this, assume that the overall income distribution is constituted of two perfectly equal income distributions: one in which everyone has income 10 and another in which everyone has income 50. Now, suppose that we start off with the entire population in the former income distribution, and that population gradually shifts to the latter over time. The income distribution will take, among others, the following three forms at various stages of this simple process:

$$A = \begin{cases} 90\% \text{ has } 10 \\ 10\% \text{ has } 50 \end{cases}, \qquad B = \begin{cases} 50\% \text{ has } 10 \\ 50\% \text{ has } 50 \end{cases}, \qquad C = \begin{cases} 10\% \text{ has } 10 \\ 90\% \text{ has } 50 \end{cases}.$$

Thinking about how inequality evolves as the income distribution changes from A to B and from B to C obviously means thinking about how inequality judgements are influenced by the relative population sizes of the "rich" and "poor." For this reason, this simple case of the gradual population shift process has been considered of importance for the theoretical question of how inequality comparisons ought to be made in the first place. It has been studied in this way by Fields (1987, 1993), among others.

The key to tackling the question of how inequality evolves during a gradual population shift lies in the behaviour of inequality measures with respect to mixing income distributions. Let us first explain what we mean by mixing income distributions. Assuming that income distributions are defined in terms of relative frequencies, each income distribution can be defined as a mixture, i.e., a convex combination, of its constituent income distributions. As an illustration, consider a country with two regions: "region P" and "region Q," representing population shares of α and $1 - \alpha$, respectively. Indeed, if p_x and q_x are the proportions of the population with income x in regions P and Q, respectively, then the proportion of the population with income x in the country is equal to $\alpha p_x + (1 - \alpha)q_x$. Now, during a gradual population shift process, the income distribution at each stage is a mixture of the income distribution at any earlier stage and the income distribution at any later stage—as an illustration, note that income distribution B in the example above of the simple case of the process, is a fifty-fifty mixture of income distributions A and C. In order to describe the evolution of inequality during a gradual population shift process, the important question is whether income inequality in a mixture is greater than, smaller than, or equal to, income inequality in each

---

[1]See, e.g., the overview of the literature on inequality measurement by Cowell (2000).

of its constituent income distributions. Moreover, can a general answer even be given to this question, or does the answer depend on the specifics of the constituent income distributions and on the particular inequality measure that is used?

In this article, we show that a general answer can indeed be given to the question of how inequality measures behave with respect to mixing income distributions. It is demonstrated that virtually all inequality measures that are studied in the literature on inequality measurement—viz., the class of decomposable inequality measures and the class of normative inequality measures based on the general social welfare function of the rank-dependent expected utility form—satisfy quasi-concavity properties, which say, loosely speaking, that mixing income distributions tends to increase inequality. For instance, the properties imply the following for the case where inequality is equal in the two constituent income distributions that are mixed: inequality in the mixture is at least as great as that in each of its constituent income distributions, and if the mean incomes of the constituent income distributions are not equal, then inequality in the mixture is *strictly* greater than that in each of its constituent income distributions. We emphasise that while all well known inequality measures satisfy these quasi-concavity properties, the properties are not implied by the fundamental Lorenz type axioms on their own. With respect to the problem of how inequality evolves during a gradual population shift process, the quasi-concavity properties are shown to reduce the possible patterns describing the evolution of inequality to only three: (i) an increasing pattern in which inequality increases during the entire process, (ii) a decreasing pattern in which inequality decreases during the entire process, and (iii) an inverted-U pattern in which inequality increases in the first stages of the process and decreases afterwards. This result generalises some results of Kakwani (1988) and Anand and Kanbur (1993) in the same context.

The article is structured as follows. Section 2 deals with notation and basic concepts. In Section 3, we show axiomatically that the quasi-concavity properties are satisfied by all inequality quasi-orderings satisfying the transfer principle, a weak invariance axiom, and decomposability. Instead of focusing exclusively on relative inequality concepts, as is common in the literature, we consider the weak invariance axiom of Bossert and Pfingsten (1990) that allows for relative and absolute inequality concepts as well as intermediate ones. While the result of Section 3 applies to, among others, the inequality measures based on a social welfare function of the expected utility form, it does not apply to its rank-based alternatives, the generalised Gini indices, as these are not decomposable. Therefore, we consider in Section 4 the class of inequality measures (absolute, relative as well as intermediate cases) based on a social welfare function of the rank-dependent expected utility form, which generalises both the class of expected utility inequality measures and the class of generalised Gini indices. Benefiting from functional representability of the given inequality orderings, it is shown that the quasi-concavity properties are also satisfied by all members of this general class of normative inequality measures. In Section 5 we spell out the implications of the results of Sections 3 and 4 for the question of how inequality evolves during a gradual population shift process. Section 6 concludes. All the proofs are contained in an Appendix A.

2. Preliminaries

2.1. Notation and basic axioms

An income distribution is an ordered pair (p, x) with $p = (p_1, p_2, ..., p_n) \in (0, 1]^n$ a vector (of finite length) of relative frequencies such that $p_1 + p_2 + \cdots + p_n = 1$, and with $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_{++}$ the

corresponding vector of income levels. So, for all $i=1, 2, ..., n$, the proportion of the population with income x_i is equal to p_i, which we sometimes write as p_{x_i}. The components of x are arranged such that $0<x_1<x_2<\cdots<x_n$. The set $\mathcal{P}$ collects all income distributions. For all $(p,x)\in\mathcal{P}$, the set $\{x_1, x_2, ..., x_n\}$ is referred to as the support of the income distribution (p, x), and the mean income $p_1x_1+p_2x_2+\cdots+p_nx_n$ is denoted by $\mu(p, x)$. An income distribution $(p,x)\in\mathcal{P}$ is said to be perfectly equal if there exists an income level e such that $p_e=1$, and is said to be unequal otherwise. Inequality comparisons of income distributions are captured by a binary relation $\preceq$ ("is at most as unequal as") on $\mathcal{P}$. The relation's asymmetric and symmetric factors are denoted by $\prec$ ("is less unequal than") and $\sim$ ("is equally unequal as"), respectively. We assume that the relation $\preceq$ is a quasi-ordering, i.e., is reflexive and transitive. A quasi-ordering that is complete is an ordering. An inequality measure is defined as a function $I : \mathcal{P} \rightarrow \mathbb{R}$ that represents an inequality ordering.

Throughout this article, we are often required to view the overall income distribution as a mixture, i.e., a convex combination, of its constituent income distributions. Suppose $(r,z)\in\mathcal{P}$ is the overall income distribution, constituted of the income distributions $(p,x)\in\mathcal{P}$ and $(q,y)\in\mathcal{P}$ with population shares $\alpha\in(0, 1)$ and $(1-\alpha)\in(0, 1)$, respectively. Then, the support of (r, z) is the union of the supports of (p, x) and (q, y), and, for each element z_i in the support of (r, z), we have

$$r_i = \begin{cases} \alpha p_{z_i} & \text{if } z_i \text{ occurs in } x \text{ and not in } y; \\ (1-\alpha)q_{z_i} & \text{if } z_i \text{ occurs in } y \text{ and not in } x; \\ \alpha p_{z_i} + (1-\alpha)q_{z_i} & \text{if } z_i \text{ occurs in } x \text{ and in } y. \end{cases}$$

This mixture (r, z) of (p, x) and (q, y) is denoted by $\alpha(p, x)+(1-\alpha)(q, y)$.

We now consider three basic axioms. To define the well known transfer principle, we require the concept of the mean preserving spread. Consider an arbitrary $(p,x)\in\mathcal{P}$ and let $0<z_1<z_2\leq z_3<z_4$ be four arbitrary income levels with z_2 and z_3 belonging to the support of (p, x). The income distribution (q, y) is said to be obtained from (p, x) by a mean preserving spread if there exists a scalar $\delta>0$ such that

$$q_{z_1} = p_{z_1} + \delta>0, \quad q_{z_2} = p_{z_2}-\delta\geq0, \quad q_{z_3} = p_{z_3}-\delta\geq0, \quad q_{z_4} = p_{z_4} + \delta>0,$$

(if z_1, respectively z_4, does not belong to the support of (p, x), then we set p_{z_1}, respectively p_{z_4}, equal to 0), $q_{x_i}=p_{x_i}$ for all other elements x_i in the support of (p, x), and $\mu(p, x)=\mu(q, y)$. In other words, whenever (q, y) is obtained from (p, x) by a mean preserving spread, this means that (q, y) is obtained from (p, x) by a series of poorer-to-richer transfers. The transfer principle demands that such transfers increase inequality.

Axiom 1 *(TP).* For all $(p,x)\in\mathcal{P}$, we have that if (q, y) is obtained from (p, x) by a mean preserving spread, then $(p, x) \prec (q, y)$.

The second axiom we consider is an invariance condition, i.e., it defines a transformation, by which all incomes are changed in the same direction, that leaves inequality invariant. For all transformations $f : \mathbb{R}_{++}\rightarrow\mathbb{R}$ and all $(p,x)\in\mathcal{P}$, we denote the transformed vector of income levels $(f(x_1), f(x_2), ..., f(x_n))$ by $f(x)$. So, for instance, $(p, \tau x)$ denotes the income distribution obtained from (p, x) by multiplying each individual's income by τ. The β-invariance axiom is a general, linear, invariance condition first proposed by Bossert and Pfingsten (1990).

Axiom 2 *(βINV).* There exists a scalar $\beta \in [0, 1]$ such that the following holds. For all $(p, x) \in \mathcal{P}$ and all scalars λ such that $(p, x + \lambda(\beta x + 1 - \beta)) \in \mathcal{P}$, we have $(p, x) \sim (p, x + \lambda(\beta x + 1 - \beta))$.

The axiom βINV encompasses both the popular relative $(\beta = 1)$ case, which says that multiplication of all incomes by the same scalar leaves inequality invariant, and the absolute $(\beta = 0)$ case, which says that addition to all incomes of the same scalar leaves inequality invariant. Inequality relations satisfying βINV for $\beta \in (0, 1)$ are referred to as intermediate inequality relations. In line with the literature, we consider TP and βINV to be the two fundamental axioms. Accordingly, both axioms are satisfied by all concepts of inequality comparisons considered in this article.

Decomposability, finally, is a popular axiom, but is usually interpreted as being less compelling than TP and βINV.[2] Roughly speaking, decomposability says that each transformation of the overall income distribution that changes only one of its constituent income distributions and leaves population shares and mean income unaffected, should affect inequality in the overall income distribution in the same direction as it affects inequality in the given constituent income distribution.

Axiom 3 *(DEC).* For all $(p, x), (q, y), (r, z) \in \mathcal{P}$ with $\mu(p, x) = \mu(q, y)$, we have

$$(p, x) \preceq (q, y) \Leftrightarrow \alpha(p, x) + (1 - \alpha)(r, z) \preceq \alpha(q, y) + (1 - \alpha)(r, z)$$

for all $\alpha \in (0, 1)$.

2.2. Properties concerning mixtures

The main focus of this article are properties that describe how inequality relations behave with respect to mixing income distributions, i.e., how a mixture compares in terms of inequality to its constituent income distributions. We will not impose these properties as a priori desirable properties on inequality relations – which is the reason why we do not refer to them as "axioms" – but are interested, instead, in how broadly they are satisfied and in what their implications are.

Quasi-concavity and strict quasi-concavity describe a positive inequality attitude to mixing income distributions. Loosely speaking, the properties say that mixing tends to increase inequality. To give an example, (strict) quasi-concavity implies that a mixture of two equally unequal income distributions is at least as unequal as (is more unequal than) the given two income distributions.

Property 1 *(QC).* For all $(p, x), (q, y) \in \mathcal{P}$, we have

$$(p, x) \preceq (q, y) \Rightarrow (p, x) \preceq \alpha(p, x) + (1 - \alpha)(q, y) \qquad \text{for all } \alpha \in (0, 1). \tag{1}$$

Property 2 *(SQC).* For all $(p, x), (q, y) \in \mathcal{P}$ with $(p, x) \neq (q, y)$, we have

$$(p, x) \preceq (q, y) \Rightarrow (p, x) \prec \alpha(p, x) + (1 - \alpha)(q, y) \qquad \text{for all } \alpha \in (0, 1). \tag{2}$$

---

[2]For a critique of decomposability, see Sen and Foster (1997, pp. 149–163). The axiom they refer to as subgroup consistency is similar to our definition of the concept.

More relevant than SQC, however, will turn out to be the following conditional strict quasi-concavity property, which requires Eq. (2) to be satisfied only if the means of the two constituent income distributions are not equal.

Property 3 *(CSQC).* For all $(p, x), (q, y) \in \mathcal{P}$ with $\mu(p, x) \neq \mu(q, y)$, Eq. (2) holds.

Note that SQC implies both QC and CSQC, while the latter two properties are independent. Quasi-convexity and strict quasi-convexity describe negative inequality attitudes to mixing income distributions and, thus, are the natural counterparts of QC and SQC. Loosely speaking, these properties say that mixing tends to decrease inequality. For instance, (strict) quasi-convexity implies that a mixture of two equally unequal income distributions is at most as unequal as (is less unequal than) each of the given two income distributions.

Property 4 *(QV).* For all $(p, x), (q, y) \in \mathcal{P}$ with neither (p, x) nor (q, y) perfectly equal, we have

$$(p, x) \preceq (q, y) \Rightarrow \alpha(p, x) + (1-\alpha)(q, y) \preceq (q, y) \qquad \text{for all } \alpha \in (0, 1). \tag{3}$$

Property 5 *(SQV).* For all $(p, x), (q, y) \in \mathcal{P}$ with $(p, x) \neq (q, y)$ and neither (p, x) nor (q, y) perfectly equal, we have

$$(p, x) \preceq (q, y) \Rightarrow \alpha(p, x) + (1-\alpha)(q, y) \prec (q, y) \qquad \text{for all } \alpha \in (0, 1). \tag{4}$$

The reason why the perfectly equal income distributions are excluded from the set of income distributions over which Eqs. (3) and (4) are required to hold, is that the properties QV and SQV would otherwise be incompatible with the commonsense requirement that each unequal income distribution is strictly more unequal than each perfectly equal one.[3]

Using the minimal framework of inequality quasi-orderings, we demonstrate in Section 3 that the three axioms TP, βINV, and DEC are sufficient for the properties QC and CSQC to be satisfied. In Section 4, similar results are shown to hold for the members of an important class of inequality orderings consistent with TP and βINV but not (necessarily) with DEC. We remark that although QC and CSQC turn out to be satisfied very generally, this does not necessarily imply that these are *desirable* properties for inequality relations—indeed, in Section 5 we discuss a critique of some of the implications of these properties that has been put forward in the literature.

3. Inequality quasi-orderings

In this section, we examine the implications of the three basic axioms, TP, βINV, and DEC, for the behaviour of inequality quasi-orderings with respect to mixing income distributions.

The two fundamental axioms TP and βINV are sufficient to rule out the two quasi-convexity properties, but are not sufficient to imply any of the three quasi-concavity properties. To see this, consider the following lemma.

---

[3] This can be seen by letting (p, x) and (q, y) in Eqs. (3) and (4) both be perfectly equal income distributions (with $(p, x) \neq (q, y)$). Note, furthermore, that the "commonsense requirement" is implied by TP and βINV jointly.

Lemma 1. *Let $\preceq$ be an inequality quasi-ordering that satisfies TP and βINV. Then, for all $(p, x) \in \mathcal{P}$ and all scalars λ such that $(p, x + \lambda(\beta x + 1 - \beta)) \in \mathcal{P}$ and $(p, x) \neq (p, x + \lambda(\beta x + 1 - \beta))$, we have*

$$(p, x) \prec \alpha(p, x) + (1 - \alpha)(p, x + \lambda(\beta x + 1 - \beta)) \qquad \text{for all } \alpha \in (0, 1).$$

Note that βINV requires, moreover, that $(p, x) \sim (p, x + \lambda(\beta x + 1 - \beta))$ in Lemma 1. By letting $(q, y) = (p, x + \lambda(\beta x + 1 - \beta))$ in conditions (3) and (4), it thus follows from the lemma that TP and βINV imply violations of these conditions and, hence, of QV and SQV. In a similar way, it is established that TP and βINV imply Eqs. (1) and (2) in all cases where $(q, y) = (p, x + \lambda(\beta x + 1 - \beta))$. Although the latter reveals that TP and βINV imply instances of QC, SQC, and CSQC, the two axioms are not sufficient for any of these three quasi-concavity properties to be satisfied in general. The following example provides an illustration of this point.

Example 1. Consider the inequality measure $I : \mathcal{P} \to \mathbb{R} : (p, x) \mapsto \kappa I_{\text{GE}}^{10}(p, x) + I_{\text{GE}}^{-9}(p, x)$ where $\kappa > 0$ is a scalar and where I_{GE}^{θ} is the generalised entropy inequality measure, given by

$$I_{\text{GE}}^{\theta} : \mathcal{P} \to \mathbb{R} : (p, x) \mapsto \frac{1}{\theta^2 - \theta} \sum_{i=1}^{n} p_i \left[\left(\frac{x_i}{\mu(p, x)} \right)^{\theta} - 1 \right],$$

with θ a scalar. Since the generalized entropy inequality measure satisfies TP and βINV (for $\beta = 1$), I satisfies these axioms as well. By contrast, I_{GE}^{θ} satisfies DEC, while I does not. Now consider the following three income distributions: $(p, x) = ((0.4, 0.6), (10, 50))$, $(q, y) = ((0.9, 0.1), (10, 50))$, and $(r, z) = ((0.65, 0.35), (10, 50))$. Let, moreover, $\kappa = \frac{I_{\text{GE}}^{-9}(p,x) - I_{\text{GE}}^{-9}(q,y)}{I_{\text{GE}}^{10}(q,y) - I_{\text{GE}}^{10}(p,x)} \approx 0.719$. We have

$$23.370 \approx I(r, z) = I(0.5(p, x) + 0.5(q, y)) < I(p, x) = I(q, y) \approx 270.061,$$

which implies that I violates QC, SQC, and CSQC. Furthermore, it can be shown that Eq. (4) holds for the chosen (p, x) and (q, y).

On its own, DEC implies a bias neither to quasi-concavity, nor to quasi-convexity: DEC implies instances of the weak versions of both quasi-concavity and quasi-convexity, and is (typically) incompatible with the strict versions of both. In order to see this, consider arbitrary income distributions $(p, x), (q, y) \in \mathcal{P}$ with $\mu(p, x) = \mu(q, y)$ and $(p, x) \preceq (q, y)$. It follows from DEC that

$$(p, x) \preceq \alpha(p, x) + (1 - \alpha)(q, y) \preceq (q, y) \qquad \text{for all } \alpha \in (0, 1). \text{[4]} \tag{5}$$

In other words, DEC implies instances of QC or QV in those cases in which the income distributions in the mixture have equal means. If $(p, x) \sim (q, y)$ (still with $\mu(p, x) = \mu(q, y)$), then DEC implies that the inequality relations in Eq. (5) hold with equivalence ($\sim$), thus giving rise to violations of both Eqs. (2) and (4). Hence, given the weak assumption—which would follow, e.g., from completeness and continuity—that at least one pair of income distributions $(p, x), (q, y) \in \mathcal{P}$ such that $\mu(p, x) = \mu(q, y)$ and $(p, x) \sim (q, y)$ exists, DEC is incompatible with both SQC and SQV.[5]

---

[4]This is obtained by letting (r, z) in the definition of DEC equal in turn (p, x) and (q, y).

[5]In a sense, CSQC is as far as one can go in the direction of SQC while still satisfying DEC. To see this, consider arbitrary income distributions $(p, x), (q, y) \in \mathcal{P}$ with $\mu(p, x) = \mu(q, y)$ and $(p, x) \preceq (q, y)$, i.e., income distributions for which SQC implies Eq. (2), while CSQC does not. Now, if $(p, x) \sim (q, y)$, then DEC is inconsistent with Eq. (2) which means that SQC goes too far, whereas if $(p, x) \prec (q, y)$, then DEC already implies Eq. (2) on its own.

To summarise, we have seen that TP and βINV are not sufficient for QC, SQC, or CSQC to be satisfied, and also that DEC typically rules out SQC. The following result says that each inequality quasi-ordering satisfying TP, βINV, and DEC must satisfy QC as well as CSQC.

Proposition 1. *Let $\preceq$ be an inequality quasi-ordering that satisfies TP, βINV, and DEC. Then, $\preceq$ satisfies QC and CSQC.*

Proposition 1 has implications that are relevant in the context of the study of the evolution of inequality during a process in which population gradually shifts from one constituent income distribution to another. We postpone the discussion of these implications until Section 5, but consider here relevant results concerning this context by Kakwani (1988) and Anand and Kanbur (1993) that are generalised in Proposition 1. Anand and Kanbur present results that imply that the inequality orderings represented by the following relative inequality measures satisfy CSQC: the first and second Theil inequality measures, the coefficient of variation, the entire class of Atkinson inequality measures, and the Gini index in the case of non-overlapping income distributions.[6] The same has been shown by Kakwani for the entire class of generalised entropy inequality measures, thus generalising the results pertaining to all measures considered by Anand and Kanbur except the Gini index.[7] Proposition 1 demonstrates that neither the demand that inequality be a relative concept, nor even completeness or continuity are essential in obtaining the result. Examples of absolute inequality measures covered by Proposition 1 are the variance and the entire class of Kolm inequality measures. Notable inequality measures that Proposition 1 does not deal with—because they do not satisfy DEC—are the Gini index in the general, possibly overlapping, case, as well as its rank-based generalisations. In the next section, we show that a similar result as Proposition 1 holds for a class of normative inequality measures that encompasses both the well known classes of decomposable normative inequality measures (the Atkinson and Kolm inequality measures) and the generalised Gini indices.

4. Normative inequality orderings

Normative inequality measures are based on a conception of social ethics, captured by a social welfare function $W : \mathcal{P} \rightarrow \mathbb{R}$.[8] We define the equally distributed equivalent income for an income distribution $(p, x) \in \mathcal{P}$ as the per capita income, $\xi(p, x)$, which, if distributed equally, yields the same level of social welfare as (p, x). That is, for each income distribution $(p, x) \in \mathcal{P}$, we have $\xi(p, x) = e$ where $e \in R_{++}$ is such that there is a $(q, y) \in \mathcal{P}$ for which $q_e = 1$ and $W(p, x) = W(q, y)$. It is common to define relative normative inequality measures using

$$I : \mathcal{P} \rightarrow \mathbb{R} : (p, x) \mapsto 1 - \frac{\xi(p, x)}{\mu(p, x)}, \tag{6}$$

and absolute normative inequality measures using

$$I : \mathcal{P} \rightarrow \mathbb{R} : (p, x) \mapsto \mu(p, x) - \xi(p, x). \tag{7}$$

---

[6]The logarithmic variance, also considered by Anand and Kanbur, can be added to the list. Anand and Kanbur borrow the result concerning this inequality measure from Robinson (1976). We do not consider the logarithmic variance as it does not satisfy TP.

[7]Kakwani (1988, pp. 210–213) mistakenly believes to have proven the result only for the generalised entropy inequality measures for which $\theta \geq 1$ and $\theta = 0$. However, he proves the result also for the entire Atkinson class, which is ordinally equivalent to the generalised entropy class in the case where $\theta < 1$. Therefore, the ordinal nature of the property CSQC implies that the result applies to the entire generalised entropy class.

[8]For an overview of the normative approach to inequality measurement, see Gajdos (2001).

The literature on inequality measurement has focused mainly on two particular social welfare functions: the social welfare function of the expected utility (EU) form on which among others the Atkinson and Kolm inequality measures are based, and the social welfare function of the Yaari (1987) form on which the generalised Gini indices are based. Both are special cases of the social welfare function of the rank-dependent expected utility (RDEU) form,

$$W : \mathcal{P} \rightarrow \mathbb{R} : (p,x) \mapsto \sum_{i=1}^{n} \pi_i(p) u(x_i), \qquad (8)$$

with $\pi_i(p) = \phi(p_i + p_{i+1} + \cdots + p_n) - \phi(p_{i+1} + p_{i+2} + \cdots + p_n)$ for all $i = 1, 2, \ldots, n-1$, and $\pi_n(p) = \phi(p_n)$. Furthermore, $\phi : [0, 1] \rightarrow [0, 1]$ is a continuous and strictly increasing function with $\phi(0) = 0$ and $\phi(1) = 1$, and $u : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and strictly increasing function. In the case where ϕ coincides with the identity function, we have $\pi_i(p) = p_i$ for all $i = 1, 2, \ldots, n$, and Eq. (8) reduces to the EU social welfare function. In the case where u coincides with the identity function, Eq. (8) reduces to the Yaari social welfare function.

Relative RDEU inequality measures, which we denote by $I_{\text{RDEU}}^{1,\varepsilon,\phi}$, are given by Eq. (6) with W as in Eq. (8), with

$$u : \mathbb{R} \rightarrow \mathbb{R} : t \mapsto \frac{1}{1-\varepsilon} t^{1-\varepsilon}, \quad \varepsilon \geq 0, \qquad (9)$$

and with ϕ a convex function. In order for TP to be satisfied, we assume, moreover, that either u is strictly concave (i.e., $\varepsilon > 0$), ϕ is strictly convex, or both.[9] Following Bossert and Pfingsten (1990), we obtain the relative, intermediate, and part of the absolute RDEU inequality measures as $I_{\text{RDEU}}^{\beta,\varepsilon,\phi}(p,x) = \frac{1}{\beta} I_{\text{RDEU}}^{1,\varepsilon,\phi}\left(p, x + \frac{1-\beta}{\beta}\right)$ for all $(p,x) \in \mathcal{P}$. Hence, we have

$$I_{\text{RDEU}}^{\beta,\varepsilon,\phi} : \mathcal{P} \rightarrow \mathbb{R} : (p,x) \mapsto \frac{1}{\beta} \left\{ 1 - \frac{\left[\sum_{i=1}^{n} \pi_i(p) \left(x_i + \frac{1-\beta}{\beta} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}}{\mu(p,x) + \frac{1-\beta}{\beta}} \right\}, \qquad (10)$$

where $0 < \beta \leq 1$ if $\varepsilon > 0$ and $0 \leq \beta \leq 1$ if $\varepsilon = 0$, where π is defined as above, and ϕ is strictly convex whenever $\varepsilon = 0$ and convex otherwise. The absolute RDEU inequality measures not given by $I_{\text{RDEU}}^{\beta,\varepsilon,\phi}$ are those for which u does not coincide with the identity function. These are obtained as Eq. (7) with W as in Eq. (8), and with

$$u : \mathbb{R} \rightarrow \mathbb{R} : t \mapsto -\exp(-\gamma t), \quad \gamma > 0, \qquad (11)$$

and are denoted by $I_{\text{RDEU}}^{0,\gamma,\phi}$. Hence, we have

$$I_{\text{RDEU}}^{0,\gamma,\phi} : \mathcal{P} \rightarrow \mathbb{R} : (p,x) \mapsto \mu(p,x) + \frac{1}{\gamma} \ln \left(\sum_{i=1}^{n} \pi_i(p) \exp(-\gamma x_i) \right), \qquad (12)$$

where $\gamma > 0$, π is defined as above, and ϕ is convex.

Several well known inequality measures belong to the class of RDEU inequality measures. For ϕ coinciding with the identity function, we obtain the class of EU inequality measures, with as special cases the Atkinson class (by letting, furthermore, $\beta = 1$ in $I_{\text{RDEU}}^{\beta,\varepsilon,\phi}$) and the Kolm class (given by $I_{\text{RDEU}}^{0,\gamma,\phi}$). For u coinciding with the identity function (i.e., $\varepsilon = 0$), we obtain the Yaari, or

---

[9] See Chew et al. (1987).

generalised Gini, indices.[10] A well known subclass of the generalised Gini indices is that of the S-Gini indices, for which $\phi : t \mapsto t^\rho$ with $\rho > 1$, which has as a notable special case the Gini index ($\rho = 2$). RDEU inequality measures for which neither ϕ nor u coincide with the identity function, and which, consequently, belong to neither the EU class nor the Yaari class, have been studied by Ebert (1988) and Chateauneuf et al. (2002), among others.

All RDEU inequality measures are consistent with TP and βINV. However, while all RDEU inequality measures also incorporate a weak decomposability idea as shown by Ebert (1988), only the inequality orderings corresponding to members of the EU subclass satisfy DEC. By consequence, the EU inequality measures are the only members of the RDEU class that are covered by Proposition 1. To prove that the inequality orderings representable by each of the remaining RDEU inequality measures also all satisfy the quasi-concavity properties of Proposition 1, we require a result that relates the convexity of the weighting function ϕ to the convexity of the RDEU social welfare function. A social welfare function W is said to be convex, for all $(p,x), (q,y) \in \mathcal{P}$,

$$W(\alpha(p,x) + (1-\alpha)(q,y)) \leq \alpha W(p,x) + (1-\alpha)W(q,y) \quad \text{for all } \alpha \in (0,\ 1). \tag{13}$$

The following lemma summarises the required relationship.

Lemma 2. *Let W be a social welfare function of the RDEU form, given by Eq. (8).*

  (i) *If ϕ is linear, then W is linear, i.e., Eq. (13) holds with equality for all $(p,x), (q,y) \in \mathcal{P}$.*
 (ii) *If ϕ is convex, then W is convex, i.e., Eq. (13) holds for all $(p,x), (q,y) \in \mathcal{P}$.*
(iii) *If ϕ is strictly convex, then W is strictly convex, i.e., Eq. (13) holds with strict inequality for all $(p,x), (q,y) \in \mathcal{P}$ with $(p,\ x) \neq (q,\ y)$.*

Note that we have $\alpha W(p,\ x) + (1-\alpha)W(q,\ y) \leq \min\{W(p,\ x),\ W(q,\ y)\}$ for all $(p,x), (q,y) \in \mathcal{P}$ and all $\alpha \in (0,\ 1)$. Using this observation together with Eq. (13), it can be seen that Lemma 2 has implications for the behaviour of RDEU social welfare functions with respect to mixing income distributions: a linear weighting function ϕ, as that of the EU social welfare function, corresponds to a neutral social welfare attitude to mixing, whereas a (strictly) convex weighting function ϕ, as that of the RDEU or Yaari social welfare functions, corresponds to a (strictly) negative attitude to mixing. Now, note that Proposition 1 can be interpreted as revealing that inequality orderings based on a social welfare function with a neutral attitude to mixing (the EU social welfare function) have a positive attitude to mixing as expressed by the properties QC and CSQC. Since social welfare and inequality are negatively related concepts (see Eqs. (6) and (7)), we would expect that if a neutral social welfare attitude to mixing translates into a positive inequality attitude to mixing, then a negative social welfare attitude to mixing should definitely translate into a positive inequality attitude to mixing. In other words, if inequality orderings based on an EU social welfare function satisfy the properties QC and CSQC, then this should be true a fortiori for inequality orderings based on an RDEU social welfare function. Proposition 2 confirms this intuition.

Proposition 2. *Let $\preceq$ be an RDEU inequality ordering, i.e., an inequality ordering representable by Eq. (10) or (12). Then, $\preceq$ satisfies QC and CSQC. If, in addition, the weighting function ϕ corresponding to $\preceq$ is strictly convex, then $\preceq$ satisfies SQC.*

Note that, since all generalised Gini indices have strictly convex weighting functions ϕ, the inequality orderings represented by these inequality measures all satisfy the strongest quasi-concavity property SQC.

---

[10]The absolute subclass of the generalised Gini indices is obtained by furthermore taking the limit $\beta \to 0$ in $I_{\text{RDEU}}^{\beta, \varepsilon, \phi}$.

5. Inequality and gradual population shifts

We now examine the implications of the properties QC and CSQC for the question of how inequality evolves during an adjustment process in which the population gradually shifts from one constituent income distribution to another over time. As discussed in Section 1, several empirical phenomena involve such an adjustment process. Suppose that the constituent income distributions are $(p, x) \in \mathcal{P}$ and $(q, y) \in \mathcal{P}$, and that population shifts from (q, y) to (p, x). Then, the overall income distribution is $\alpha(p, x) + (1-\alpha)(q, y)$ and α gradually rises over some interval $(\underline{\alpha}, \overline{\alpha}) \subseteq (0, 1)$. The question we are interested in is how inequality in the overall income distribution evolves as α rises over $(\underline{\alpha}, \overline{\alpha})$.

In the previous sections, we saw that all well known inequality concepts satisfy the properties QC and CSQC. As the next proposition shows, these properties reduce the number of allowed patterns, describing inequality evolution during the considered adjustment process, to only three: (i) an increasing pattern, (ii) a decreasing pattern, and (iii) an inverted-U pattern. In the case of the increasing pattern, inequality in the overall income distribution increases as α rises over $(\underline{\alpha}, \overline{\alpha})$, whereas, in the case of the decreasing pattern, inequality decreases. The inverted-U pattern implies that inequality increases in the early stages of the process and decreases afterwards— formally, there is some α^* in $(\underline{\alpha}, \overline{\alpha})$ such that inequality increases as long as α stays below α^* and decreases from as soon as α rises above α^*. The proposition focuses on CSQC, which has stronger implications than QC (given that $\mu(p, x) \neq \mu(q, y)$), and, for convenience, restricts attention to inequality orderings.

Proposition 3. *Let $\preceq$ be an inequality ordering that satisfies CSQC. Consider arbitrary $(p, x), (q, y) \in \mathcal{P}$ with $\mu(p, x) \neq \mu(q, y)$. Only the following three patterns, describing the evolution of inequality in $\alpha(p, x) + (1-\alpha)(q, y)$ as α rises over the interval $(\underline{\alpha}, \overline{\alpha}) \subseteq (0, 1)$, are possible.*

(i) *An increasing pattern, i.e., for all $\alpha, \alpha' \in (\underline{\alpha}, \overline{\alpha})$, we have that if $\alpha > \alpha'$, then*

$$\alpha'(p,x) + (1-\alpha')(q,y) \prec \alpha(p,x) + (1-\alpha)(q,y).$$

(ii) *A decreasing pattern, i.e., for all $\alpha, \alpha' \in (\underline{\alpha}, \overline{\alpha})$, we have that if $\alpha > \alpha'$, then*

$$\alpha(p,x) + (1-\alpha)(q,y) \prec \alpha'(p,x) + (1-\alpha')(q,y).$$

(iii) *An inverted-U pattern, i.e., there exists an $\alpha^* \in (\underline{\alpha}, \overline{\alpha})$ such that, for all $\alpha, \alpha' \in (\underline{\alpha}, \alpha^*]$, we have that if $\alpha > \alpha'$, then*
$$\alpha'(p,x) + (1-\alpha')(q,y) \prec \alpha(p,x) + (1-\alpha)(q,y),$$

and, for all $\alpha, \alpha' \in [\alpha^*, \overline{\alpha})$, we have that if $\alpha > \alpha'$, then
$$\alpha(p,x) + (1-\alpha)(q,y) \prec \alpha'(p,x) + (1-\alpha')(q,y).$$

A case of the gradual population shift process that is of theoretical interest is the simple one in which the two constituent income distributions (p, x) and (q, y) are both perfectly equal. If in (p, x) everyone has income $\hat{x}$ and in (q, y) everyone has income $\hat{y}$, then the overall income distribution can be written as $((\alpha, 1-\alpha), (\hat{x}, \hat{y}))$. We assume, furthermore, that $\hat{x} > \hat{y}$ and that α rises over $(0, 1)$. During this simple process, the relative group sizes of the "rich" and "poor"— those with incomes $\hat{x}$ and $\hat{y}$, respectively—change continuously. For this reason, this case has been regarded by Fields (1987, 1993), among others, as interesting for the theoretical question of

how to define the concept of inequality comparisons. Fields criticises the popular inequality measures—by which he means those studied by Anand and Kanbur (1993) (see Section 3 of this article)—because they all imply an inverted-U pattern of inequality in the simple gradual population shift process, while there are other patterns that would be at least as plausible in his opinion. The inverted-U pattern implies, loosely speaking, that income distributions with equally sized poor and rich groups are more unequal than income distributions with a small number of poor and a large number of rich or with a large number of poor and a small number of rich. In his own work, Fields defends the opposite view, the U pattern (inequality decreases in the early stages of the process and increases afterwards), which implies that situations with few poor and many rich or few rich and many poor are considered particularly unequal.[11]

It follows from the results of this article that Fields' critique applies not only to the inequality measures dealt with by Anand and Kanbur, but to all inequality measures commonly considered in the literature. Propositions 1 and 2 imply that the inequality orderings corresponding to all these inequality measures satisfy CSQC, and Proposition 3 implies that all continuous inequality orderings satisfying CSQC imply an inverted-U pattern in the simple gradual population shift process. The latter follows from the fact that if the two constituent income distributions are equally unequal, as is the case in the simple gradual population shift process,[12] then the patterns (i) and (ii) in Proposition 3 are only possible for noncontinuous inequality orderings since these patterns involve a discontinuity at $\alpha = 0$ or at $\alpha = 1$. Note, finally, that, as Example 1 shows, the fundamental axioms TP and βINV do not in general exclude the occurrence of a U pattern over part of the gradual shift process.

6. Conclusion

The literature on inequality measurement has focused exclusively on the specific strategy of supplementing the fundamental axioms, TP and βINV, with decomposability ideas, i.e., ideas concerning how changes in the inequality of constituent income distributions have to relate to changes in overall inequality—directly, in the form of the DEC axiom, or, indirectly, by basing inequality measures on an RDEU social welfare function, which incorporates a weak decomposability condition. It was demonstrated in this article that all inequality measures considered in the literature satisfy the quasi-concavity properties QC and CSQC. Moreover, it was shown that the latter property allows only three patterns describing how inequality evolves during a process in which population gradually shifts from one constituent income distribution to another.

On the one hand, the latter result reveals an attractive feature of CSQC: the property facilitates the study of empirical phenomena in which gradual population shifts occur. On the other hand, it

---

[11] Fields (1993) provides a justification for the U pattern on the basis of the notions "elitism of the rich" and "isolation of the poor." Loosely speaking, elitism of the rich says that, for relatively low values of α, decreases in α lead to greater inequality because the "rich" then attain a more elite position. Similarly, isolation of the poor says that, for relatively high values of α, increases in α cause inequality to increase because the "poor" then become more isolated. The simple case of the gradual population shift process has also been considered by Temkin (1986) and by Amiel and Cowell (1994). Using his own framework for inequality measurement, the philosopher Temkin gives justifications for the three patterns dealt with in Proposition 3 as well as for a pattern of constant inequality during the entire process. Amiel and Cowell provide questionnaire results showing that respondents support several patterns among which the U pattern proposed by Fields is quite popular. See also the discussion by Kolm (1999, pp. 36–38).
[12] At least, this would follow from βINV or from the commonsense assumption that all perfectly equal income distributions are equally equal.

may be argued that the three patterns allowed by CSQC are not the only plausible ones. If it is concluded that the other—non CSQC consistent—inequality views should also be expressible within a theory of inequality measurement, then our results show that one should focus on supplementing the fundamental axioms, TP and βINV, in alternative ways, rather than with decomposability ideas.

Appendix A

In the proofs, we usually abbreviate, for all (p,x), $(q,y) \in \mathcal{P}$ and all scalars α, the expression $\alpha(p,x) + (1-\alpha)(q,y)$ with $(\alpha; p, x; q, y)$.

Proof of Lemma 1. Consider an inequality quasi-ordering $\preceq$ that satisfies TP and βINV. Consider, moreover, any $(p,x) \in \mathcal{P}$, any scalar λ such that $(p,y) = (p, x + \lambda(\beta x + 1 - \beta)) \in \mathcal{P}$ and $(p,x) \neq (p,y)$, and any $\alpha \in (0,1)$. Note that we have $\lambda = \frac{\mu(p,y) - \mu(p,x)}{\beta\mu(p,x) + 1 - \beta}$ by definition. What has to be shown is that $(p,x) \prec (\alpha; p, x; p, y)$.

Consider $(p,z) = (p, x + \lambda'(\beta x + 1 - \beta))$, where $\lambda' = \frac{\mu(\alpha; p, x; p, y) - \mu(p,x)}{\beta\mu(p,x) + 1 - \beta}$. The choice of λ' ensures that $\mu(p,z) = \mu(\alpha; p, x; p, y)$. Since either $0 < x_i < z_i < y_i$ for all $i = 1, 2, \dots, n$, or $x_i > z_i > y_i > 0$ for all $i = 1, 2, \dots, n$, we have, furthermore, $(p,z) \in \mathcal{P}$. We now prove the claim that TP implies $(p,z) \prec (\alpha; p, x; p, y)$. Note that the supports of (p,x), (p,y), and (p,z) have the same number of elements. Now, clearly, to each element in the support of (p,x), say income level t, there corresponds one element in the support of (p,y) equal to $t + \lambda(\beta t + 1 - \beta)$. The frequency with which t appears in (p,x), say frequency s, is equal to the frequency with which $t + \lambda(\beta t + 1 - \beta)$ appears in (p,y). By consequence, in $(\alpha; p, x; p, y)$, there is, for each element in the support of (p,x), a pair of incomes such that the sum of frequencies is s and the mean income for the group of individuals with any of these two incomes is $\alpha t + (1-\alpha)[t + \lambda(\beta t + 1 - \beta)]$. Similarly, to each element in the support of (p,x), say t occurring with frequency s, there corresponds one income in the support of (p,z) equal to $t + \lambda'(\beta t + 1 - \beta)$ and occurring with frequency s. Now, we have $t + \lambda'(\beta t + 1 - \beta) = \alpha t + (1-\alpha)[t + \lambda(\beta t + 1 - \beta)]$. Therefore, $(\alpha; p, x; p, y)$ can be obtained from (p,z) by a sequence of mean preserving spreads and, hence, TP implies $(p,z) \prec (\alpha; p, x; p, y)$.

Since $(p,z) \prec (\alpha; p, x; p, y)$ by TP and $(p,x) \sim (p,z)$ by βINV, we obtain $(p,x) \prec (\alpha; p, x; p, y)$ using transitivity. $\square$

Proof of Proposition 1. Consider an inequality quasi-ordering $\preceq$ that satisfies TP, βINV, and DEC. Consider, moreover, any $(p,x), (q,y) \in \mathcal{P}$ such that $(p,x) \preceq (q,y)$ and any $\alpha \in (0,1)$. In the case where $\mu(p,x) = \mu(q,y)$, DEC already implies $(p,x) \preceq (\alpha; p, x; q, y)$. Therefore, we assume $\mu(p,x) \neq \mu(q,y)$ in what follows. What has to be shown is that $(p,x) \prec (\alpha; p, x; q, y)$.

Consider $(p,z) = (p, x + \lambda(\beta x + 1 - \beta))$, where $\lambda = \frac{\mu(q,y) - \mu(p,x)}{\beta\mu(p,x) + 1 - \beta}$. The choice of λ ensures that $\mu(p,z) = \mu(q,y)$. Two cases are possible: either (a) $(p,z) \in \mathcal{P}$, or (b) $(p,z) \notin \mathcal{P}$.

In case (a), we have $(p,x) \sim (p,z)$ by βINV. Using transitivity, we have $(p,z) \preceq (q,y)$ and, hence, $(\alpha; p, x; p, z) \preceq (\alpha; p, x; q, y)$ by DEC. Lemma 1 implies $(p,x) \prec (\alpha; p, x; p, z)$, and we obtain $(p,x) \prec (\alpha; p, x; q, y)$ using transitivity.

Case (b) occurs if and only if λ is such that in going from (p,x) to (p,z), nonpositive incomes get nonzero frequency (which is only possible if $\mu(p,x) > \mu(q,y)$). Consider $(p,x') = (p, x + \lambda'(\beta x + 1 - \beta))$ and $(q,y') = (q, y + \lambda'(\beta y + 1 - \beta))$ where λ' is any scalar such that $[x_1 + \lambda'(\beta x_1 + 1 - \beta)] + \lambda\{\beta[x_1 + \lambda'(\beta x_1 + 1 - \beta)] + 1 - \beta\} > 0$. We can then return to the beginning of this proof and prove the result for (p,x') and (q,y') without getting case (b). If the result is true for (p,x') and (q,y'), then it must be true for (p,x) and (q,y) as well by βINV and transitivity. $\square$

Proof of Lemma 2. First note that Eq. (8) can be rewritten as

$$W: \ \mathcal{P} \rightarrow \mathbb{R}: \ (p,x) \mapsto u(x_1) + \sum_{i=2}^{n} \phi \left(\sum_{j=i}^{n} p_j \right) [u(x_i) - u(x_{i-1})].$$

Consider any $(p,x), (q,y) \in \mathcal{P}$ and any scalar $\alpha \in (0,1)$. Define, furthermore, the ordered pairs (p',z) and (q',z) with $z = (z_1, z_2, \ldots, z_m)$ the vector that contains the components of both x and y arranged such that $0 < z_1 < z_2 < \cdots < z_m$. Moreover, $p' = (p'_1, p'_2, \ldots, p'_m)$ is a vector where, for all $i = 1, 2, \ldots, m$, $p'_i = p_{z_i}$ if z_i occurs in x and $p'_i = 0$ otherwise, and, similarly, $q' = (q'_1, q'_2, \ldots, q'_m)$ is a vector where, for all $i = 1, 2, \ldots, m$, $q'_i = q_{z_i}$ if z_i occurs in y and $q'_i = 0$ otherwise. We then have

$$W(\alpha(p,x) + (1-\alpha)(q,y)) = u(z_1) + \sum_{i=2}^{m} \phi \left(\sum_{j=i}^{m} \alpha p_j' + (1-\alpha)q_j' \right) [u(z_i) - u(z_{i-1})]$$

$$\leq \alpha \left\{ u(z_1) + \sum_{i=2}^{m} \phi \left(\sum_{j=i}^{m} p_j' \right) [u(z_i) - u(z_{i-1})] \right\}$$

$$+ (1-\alpha) \left\{ u(z_1) + \sum_{i=2}^{m} \phi \left(\sum_{j=i}^{m} q_j' \right) [u(z_i) - u(z_{i-1})] \right\}$$

$$= \alpha W(p,x) + (1-\alpha)W(q,y),$$

where the inequality follows from the convexity of ϕ. The inequality holds with equality if ϕ is linear, and holds strictly if ϕ is strictly convex and $(p,x) \neq (q,y)$ since the latter implies $p' \neq q'$. $\quad\square$

Proof of Proposition 2. Consider any $(p,x), (q,y) \in \mathcal{P}$ such that $(p,x) \preceq (q,y)$, and any scalar $\alpha \in (0,1)$. Since the case where $(p,x) = (q,y)$ is trivial, we assume $(p,x) \neq (q,y)$ in what follows.

We first consider the case where $\preceq$ is representable by Eq. (10). Defining the function W^{β} as $W^{\beta}(p,x) = W\left(p, x + \frac{1-\beta}{\beta}\right)$ for all $(p,x) \in \mathcal{P}$ with W as in Eq. (8) and u as in Eq. (9), we have

$$I_{\text{RDEU}}^{\beta,\varepsilon,\phi}(\alpha; p,x; q,y) = \frac{1}{\beta} \left\{ 1 - \frac{[(1-\varepsilon)W^{\beta}(\alpha; p,x; q,y)]^{\frac{1}{1-\varepsilon}}}{\mu(\alpha; p,x; q,y) + \frac{1-\beta}{\beta}} \right\}. \tag{14}$$

We have to show the following: (a) expression (14) is at least as great as (strictly greater than) $I_{\text{RDEU}}^{\beta,\varepsilon,\phi}(p,x)$ whenever ϕ is (strictly) convex, and (b) expression (14) is strictly greater than $I_{\text{RDEU}}^{\beta,\varepsilon,\phi}(p,x)$ whenever $\mu(p,x) \neq \mu(q,y)$.

First, consider

$$\frac{1}{\beta} \left\{ 1 - \frac{[(1-\varepsilon)(\alpha W^{\beta}(p,x) + (1-\alpha)W^{\beta}(q,y))]^{\frac{1}{1-\varepsilon}}}{\mu(\alpha; p,x; q,y) + \frac{1-\beta}{\beta}} \right\} \tag{15}$$

$$= \frac{1}{\beta} \left\{ 1 - \frac{1}{\mu(\alpha; p,x; q,y) + \frac{1-\beta}{\beta}} \left[\alpha \left(\mu(p,x) + \frac{1-\beta}{\beta} \right)^{1-\varepsilon} \left(\frac{[(1-\varepsilon)W^{\beta}(p,x)]^{\frac{1}{1-\varepsilon}}}{\mu(p,x) + \frac{1-\beta}{\beta}} \right)^{1-\varepsilon} \right. \right.$$

$$\left. \left. + (1-\alpha) \left(\mu(q,y) + \frac{1-\beta}{\beta} \right)^{1-\varepsilon} \left(\frac{[(1-\varepsilon)W^{\beta}(q,y)]^{\frac{1}{1-\varepsilon}}}{\mu(q,y) + \frac{1-\beta}{\beta}} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} \right\}$$

$$= \frac{1}{\beta} \left\{ 1 - \left[1 - \beta I_{\text{RDEU}}^{\beta,\varepsilon,\phi}(p,x) \right] A \right\},$$

where

$$A = \frac{\left[\alpha\left(\mu(p,x)+\frac{1-\beta}{\beta}\right)^{1-\varepsilon}+(1-\alpha)\left(\mu(q,y)+\frac{1-\beta}{\beta}\right)^{1-\varepsilon}\left(\frac{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(q,y)}{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}}{\mu(\alpha;p,x;q,y)+\frac{1-\beta}{\beta}} = B \times C,$$

where

$$B = \frac{\alpha\left(\mu(p,x)+\frac{1-\beta}{\beta}\right)+(1-\alpha)\left(\mu(q,y)+\frac{1-\beta}{\beta}\right)\left(\frac{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(q,y)}{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)}\right)}{\alpha\mu(p,x)+(1-\alpha)\mu(q,y)+\frac{1-\beta}{\beta}}$$

and

$$C = \frac{\left[\alpha\left(\mu(p,x)+\frac{1-\beta}{\beta}\right)^{1-\varepsilon}+(1-\alpha)\left(\mu(q,y)+\frac{1-\beta}{\beta}\right)^{1-\varepsilon}\left(\frac{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(q,y)}{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}}{\alpha\left(\mu(p,x)+\frac{1-\beta}{\beta}\right)+(1-\alpha)\left(\mu(q,y)+\frac{1-\beta}{\beta}\right)\left(\frac{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(q,y)}{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)}\right)}.$$

It is readily checked that $0 < B \leq 1$. Furthermore, if $\varepsilon = 0$, then $C = 1$, while if $\varepsilon > 0$, then

$$C = 1 - I^{1,\varepsilon,\iota}_{\mathrm{RDEU}}\left((\alpha,1-\alpha),\left[\mu(p,x)+\frac{1-\beta}{\beta},\left(\mu(q,y)+\frac{1-\beta}{\beta}\right)\left(\frac{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(q,y)}{1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)}\right)\right]\right),$$

where ι is the identity function. By consequence, we have $0 < C \leq 1$.

Second, notice that $\frac{1}{\beta}\left\{1-\left[1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)\right]BC\right\}\geq I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)$ since $0 < B \leq 1$ and $0 < C \leq 1$. Because, more-over, it follows from Lemma 2 that whenever ϕ is (strictly) convex, expression (14) is at least as great as (is strictly greater than) expression (15), (a) follows. The case in which ϕ is strictly convex has been dealt with, and since $\varepsilon = 0$ is only possible in that case, we assume $\varepsilon > 0$ in what follows. Notice that whenever $\mu(p,x) \neq \mu(q,y)$ and $I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x) = I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(q,y)$, we have $B = 1$ but $C < 1$ since $\varepsilon > 0$, so that $\frac{1}{\beta}\left\{1-\left[1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)\right]BC\right\}>I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)$, and whenever $\mu(p,x) \neq \mu(q,y)$ and $I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x) < I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(q,y)$, we have $B < 1$, so that, again, $\frac{1}{\beta}\left\{1-\left[1-\beta I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)\right]BC\right\}>I^{\beta,\varepsilon,\phi}_{\mathrm{RDEU}}(p,x)$. Combining this with the fact that convexity of ϕ implies that expression (14) is at least as great as expression (15), we find that (b) follows.

We now consider the second case where $\preceq$ is representable by Eq. (12). Using W as in Eq. (8) and u as in Eq. (11), we have

$$I^{0,\gamma,\phi}_{\mathrm{RDEU}}(\alpha;p,x;q,y) = \mu(\alpha;p,x;q,y) + \frac{1}{\gamma}\ln(-W(\alpha;p,x;q,y)). \tag{16}$$

The following has to be shown: (c) expression (16) is at least as great as (strictly greater than) $I^{0,\gamma,\phi}_{\mathrm{RDEU}}(p,x)$ whenever ϕ is (strictly) convex, and (d) expression (16) is strictly greater than $I^{0,\gamma,\phi}_{\mathrm{RDEU}}(p,x)$ whenever $\mu(p,x) \neq \mu(q,y)$.

Consider

$$\mu(\alpha;p,x;q,y) + \frac{1}{\gamma}\ln(-[\alpha W(p,x)+(1-\alpha)W(q,y)]), \tag{17}$$

and

$$\alpha\mu(p,x) + (1-\alpha)\mu(q,y) + \frac{1}{\gamma}\alpha\ln(-W(p,x)) + \frac{1}{\gamma}(1-\alpha)\ln(-W(q,y)) \tag{18}$$

$$= \alpha I_{\text{RDEU}}^{0,\gamma,\phi}(p,x) + (1-\alpha)I_{\text{RDEU}}^{0,\gamma,\phi}(q,y). \tag{19}$$

It follows from Lemma 2 that whenever ϕ is (strictly) convex, expression (16) is at least as great as (is strictly greater than) expression (17). Since, moreover, expression (17) is at least as great as expression (18) by concavity of the ln function, we have (c). In the case where $\mu(p, x) \neq \mu(q, y)$ and $I_{\text{RDEU}}^{0,\gamma,\phi}(p, x) = I_{\text{RDEU}}^{0,\gamma,\phi}(q, y)$, we have $W(p, x) \neq W(q, y)$ and, hence, expression (17) is strictly greater than expression (18) by strict concavity of the ln function. If $\mu(p, x) \neq \mu(q, y)$ and $I_{\text{RDEU}}^{0,\gamma,\phi}(p, x) \neq I_{\text{RDEU}}^{0,\gamma,\phi}(q, y)$, then expression (19) is strictly greater than $I_{\text{RDEU}}^{0,\gamma,\phi}(p, x)$. Hence, (d) follows. $\square$

Proof of Proposition 3. Consider an inequality ordering $\preceq$ that satisfies CSQC. Consider, moreover, any (p,x) $(q,y) \in \mathcal{P}$ with $\mu(p, x) \neq \mu(q, y)$. We have to show that only patterns (i), (ii), and (iii) are possible, as descriptions of the evolution of inequality in $(\alpha; p, x; q, y)$ as α rises over the interval $(\underline{\alpha}, \overline{\alpha}) \subseteq (0, 1)$.

Consider the following two subpatterns, both of which describe how inequality evolves in $(\alpha; p, x; q, y)$ as α rises over some subinterval $(\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}}) \subseteq (\underline{\alpha}, \overline{\alpha})$:

(a) A constant pattern over $(\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}})$, i.e., for all $\alpha, \alpha' \in (\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}})$, we have $(\alpha; p, x; q, y) \sim (\alpha'; p, x; q, y)$.
(b) A U pattern over $(\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}})$, i.e., there exists an $\alpha^* \in (\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}})$ such that, for all $\alpha, \alpha' \in (\widetilde{\underline{\alpha}}, \alpha^*]$, if $\alpha > \alpha'$, then $(\alpha; p, x; q, y) \prec (\alpha'; p, x; q, y)$, and, for all $\alpha, \alpha' \in [\alpha^*, \widetilde{\overline{\alpha}})$, if $\alpha > \alpha'$, then $(\alpha'; p, x; q, y) \prec (\alpha; p, x; q, y)$.

We first show by contradiction that neither subpattern (a) nor (b) can be the case for any subinterval $(\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}})$. Suppose, therefore, that (a) or (b) holds over some subinterval $(\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}}) \subseteq (\underline{\alpha}, \overline{\alpha})$. Both subpatterns imply that there exist some $\alpha, \alpha', \alpha'' \in (\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}})$ where $\alpha > \alpha' > \alpha''$ such that $(\alpha'; p, x; q, y) \preceq (\alpha; p, x; q, y)$ and $(\alpha'; p, x; q, y) \preceq (\alpha''; p, x; q, y)$. This is obvious in the case of (a), while in the case of (b) this can be seen by letting α' equal the α^* in the definition of (b). Note now that, for $\alpha''' = \frac{\alpha' - \alpha''}{\alpha - \alpha''}$, we have $(\alpha'''; (\alpha; p, x; q, y); (\alpha''; p, x; q, y)) = (\alpha'; p, x; q, y)$. By consequence, we obtain both $(\alpha'''; (\alpha; p, x; q, y); (\alpha''; p, x; q, y)) \preceq (\alpha; p, x; q, y)$ and $(\alpha'''; (\alpha; p, x; q, y); (\alpha''; p, x; q, y)) \preceq (\alpha''; p, x; q, y)$. Since, moreover, $\alpha''' \in (0, 1)$ and $\mu(\alpha; p, x; q, y) \neq \mu(\alpha''; p, x; q, y)$, we have a violation of CSQC.

Now, each pattern for which subpattern (a) is not the case for some $(\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}}) \subseteq (\underline{\alpha}, \overline{\alpha})$, is either pattern (i), (ii), (iii), or a pattern for which pattern (b) is the case for some $(\widetilde{\underline{\alpha}}, \widetilde{\overline{\alpha}})$. However, as we have seen, the latter is impossible. $\square$

References

Amiel, Y., Cowell, F.A., 1994. Inequality changes and income growth. In: Eichorn, W. (Ed.), Models and Measurement of Welfare and Inequality. Springer Verlag, Berlin, pp. 3–27.

Anand, S., Kanbur, S.M.R., 1993. The Kuznets process and the inequality–development relationship. Journal of Development Economics 40, 25–52.

Bossert, W., Pfingsten, A., 1990. Intermediate inequality: concepts, indices, and welfare implications. Mathematical Social Sciences 19, 117–134.

Chateauneuf, A., Gajdos, T., Wilthien, P.-H., 2002. The principle of strong diminishing transfer. Journal of Economic Theory 103, 311–333.

Chew, S.H., Karni, E., Safra, Z., 1987. Risk aversion in the theory of expected utility with rank dependent probabilities. Journal of Economic Theory 42, 370–381.

Cowell, F.A., 2000. Measurement of inequality. In: Atkinson, A.B., Bourguignon, F. (Eds.), Handbook of Income Distribution, vol. 1. Elsevier, Amsterdam, pp. 87–166.

Ebert, U., 1988. Measurement of inequality: an attempt at unification and generalization. Social Choice and Welfare 5, 147–169.

Fields, G.S., 1987. Measuring inequality change in an economy with income growth. Journal of Development Economics 26, 357–374.

Fields, G.S., 1993. Inequality in dual economy models. Economic Journal 103, 1228–1235.

Gajdos, T., 2001. Les fondements axiomatiques de la mesure des inégalités. Revue d'Economie Politique 5, 683–720.

Kakwani, N., 1988. Income inequality, welfare and poverty in a developing economy with applications to Sri Lanka. Social Choice and Welfare 5, 199–222.

Kolm, S.-C., 1999. The rational foundations of income inequality measurement. In: Silber, J. (Ed.), Handbook of Income Inequality Measurement. Kluwer, Dordrecht, pp. 19–94.

Kuznets, S., 1955. Economic growth and income inequality. American Economic Review 45, 1–28.

Robinson, S., 1976. A note on the U hypothesis relating income inequality and economic development. American Economic Review 66, 437–440.

Sen, A.K., Foster, J.E., 1997. On Economic Inequality, Expanded Edition. Clarendon Press, Oxford.

Temkin, L.S., 1986. Inequality. Philosophy and Public Affairs 15, 99–121.

Yaari, M.E., 1987. The dual theory of choice under risk. Econometrica 55, 95–115.