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Nash Refinement of Equilibria[1]

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Abstract. A method for choosing equilibria in strategic form games is proposed and axiomatically characterized. The method as well as the axioms are inspired by the Nash bargaining theory. The method can be applied to existing refinements of Nash equilibrium (e.g., perfect equilibrium) and also to other equilibrium concepts, like correlated equilibrium.

Key Words. Noncooperative games, Nash equilibrium, Nash bargaining solution, refinements.

1. Introduction

The multiplicity of Nash equilibria in strategic form games has been a problem since the introduction of the concept by Nash (Ref. 1). Many refinements of Nash equilibria have since then been proposed; see Van Damme (Ref. 2) for a survey. Furthermore, methods of equilibrium selection were developed by Harsanyi and Selten (Ref. 3) and also by Güth and Kalkofen (Ref. 4). As it refers to a selection from a correspondence, the expression "equilibrium selection" refers to methods that always lead to a unique equilibrium; see Güth and Kalkofen, Ref. 4, p. 13. This paper will be concerned with correspondences assigning to a strategic form game a nonempty subset of the set of lotteries over all equilibria of a certain kind. Therefore, it deals with refinement of equilibrium, rather than selection. A correspondence as meant here will be called simply a "solution."

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Instead of proposing another refinement of Nash equilibrium, the paper introduces a general method of choosing equilibria from classes of equilibria associated with various specifications of the admissible joint strategies of the players. These specifications may lead to well-known concepts like Nash equilibrium or correlated equilibrium (cf. Aumann, Refs. 5–6), but in principle also to other kinds of equilibrium, as long as the specification of the admissible joint strategies is symmetric with respect to the players, closed within the set of probability distributions over the players' pure action combinations, and consistent with respect to subgames. Furthermore, the method may also be applied to refinements of equilibrium sets that satisfy certain minimal requirements. Examples of Nash equilibrium refinements are: perfect equilibrium (Selten, Ref. 7), proper equilibrium (Myerson, Ref. 8), and persistent equilibrium (Kalai and Samet, Ref. 9). The application to the set of perfect Nash equilibria is discussed in detail in Section 4.

In order for the method to be applied to some equilibrium concept, in each game a so-called disagreement payoff vector has to be chosen. This is done by specifying a disagreement map, satisfying certain conditions. For instance, in an earlier version of this paper (Peters and Vrieze, Ref. 10), the method was developed for the class of Nash equilibria, with the vector of the players' maximin payoffs as disagreement vector.

For a given disagreement map and a given equilibrium concept, the paper focuses on the solution assigning those lotteries between equilibria that maximize the product of the payoff gains of the players relative to their disagreement payoffs, and so clearly is inspired by the Nash bargaining solution (Nash, Ref. 11). This so-called Nash solution will be characterized in a way that is closely related to the Nash characterization of the Nash bargaining solution. However, the axioms used relate to the strategic form game, in contrast with the approach in Harsanyi and Selten (Ref. 12, p. 98).

It should be stressed that application of the method proposed in this paper does not entail that the noncooperative game is changed into a cooperative one, nor that it is solved by cooperative principles. Rather, the presumption is that an equilibrium can be self-enforcing and that the problem is not with the notion of equilibrium itself, but with the multiplicity of equilibria. Therefore in this paper, a cooperative part is added to the normal form game: the players agree (perhaps with binding force) on a method to guide them in their playing the game. However, the recommendation of this method is not binding, but nevertheless important, since it may serve as a focal point. Thus ultimately, the players still play the game noncooperatively, according to the specification of the admissible joint strategies. If fully binding agreements on the playing of the game were possible, then there would be no need to consider only equilibria, and a method like the Nash solution could as well be applied to the whole cooperative payoff space.

As stated, a solution chooses lotteries between equilibria. The reason for modeling a solution like this is partly technical (it leads to convex payoff sets) and partly intuitive. Consider for instance the battle of the sexes (see also Section 5),

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{bmatrix}, \end{array}$$

and the Nash equilibrium. The Nash solution with the maximin payoff vector as disagreement vector prescribes each of the two pure Nash equilibria (T, L) and (B, R) with probability 1/2. This seems not only fair, but also helps the players in reaching expected payoffs that Pareto-dominate the symmetric mixed Nash equilibrium payoffs. The Pareto optimality property seems to be compelling for a cooperative recommendation method. However, a method recommending both (T, L) and (B, R) would not be very helpful. In analogy with the well-known interpretation of mixed strategies embodying the players uncertainty as to what their opponents will do, a lottery on Nash equilibria can be interpreted as the players uncertainty as to the final recommendation, rather than as a public lottery performed by some mediating institution. In the case of correlated equilibria, such a lottery is again a correlated equilibrium.

The present paper takes the belief that an equilibrium or refinement can be self-enforcing as a starting point and tries to present a general (i.e., applicable to a whole class of games) method to select from a possible multiplicity of equilibria. It should be noted that Tedeschi (Ref. 13) proposed independently a refinement for correlated equilibrium that is related to our method applied to the correlated equilibria case.

The organization of the paper is as follows. Section 2 describes the basics of the model, and Section 3 introduces and discusses the axioms. In Section 4, the characterization result is stated and proved. Section 5 concludes.

2. Basics of the Model

For a nonempty set A, $\mathscr{L}(A)$ denotes its lottery set, i.e., the set of probability distributions on A with finite support. Let $N = \{1, 2, \ldots, n\}$ denote the set of players.

In order to define a game in strategic form, we first introduce the concept of a joint strategy specification for N. This is a correspondence T assigning, to each n-tuple $(S_1, \ldots, S_n)$ of finite sets, a subset of the lottery set $\mathscr{L}(\times_{i=1}^n S_i)$ and satisfying conditions (T1)–(T3) below. Elements of $T(S_1, \ldots, S_n)$ are admissible joint strategies. The probability assigned by $\sigma \in T(S_1, \ldots, S_n)$ to the element $(s_1, \ldots, s_n)$ of $\times_{i=1}^n S_i$ is denoted by $\sigma(s_1, \ldots, s_n)$. The set S_i is the action set of player i, i.e., the set of pure strategies available to player i.

In what follows, some notations will be useful. For

$$s = (s_1, s_2, \ldots, s_n) \in \mathop{\times}_{i=1}^{n} S_i,$$

s^{-i} denotes the vector obtained by deleting the ith component of s. The set S^{-i} is defined similarly, i.e.,

$$S^{-i} := \mathop{\times}_{k \neq i} S_k.$$

The meaning of other notations in this spirit, like $s^{-i,j}$, $S^{-i,j}$, $(s^{-i}, \hat{s}_i)$, etc., should be clear. In particular, $(s^{-i,j}, v, w)$ has v and w as ith and jth entries, respectively.

The conditions on a joint strategy specification T are as follows:

(T1) $T(S_1, \ldots, S_n)$ is a closed subset of $\mathscr{L}(\times_{i=1}^n S_i)$;

(T2) $T(S_1, \ldots, S_n)$ is symmetric; i.e., if $S_i = S_j$, then for all $\hat{s}_i \in S_i$, $\hat{s}_j \in S_j$, and $\sigma \in T(S_1, \ldots, S_n)$, there exists a $\hat{\sigma} \in T(S_1, \ldots, S_n)$ such that $\sigma(s^{-i,j}, \hat{s}_i, \hat{s}_j) = \hat{\sigma}(s^{-i,j}, \hat{s}_j, \hat{s}_i)$ for every $s^{-i,j} \in S^{-i,j}$;

(T3) T is consistent; i.e., if $S_i' \subset S_i$ for every $i \in N$, then extending each $\sigma \in T(S_1', \ldots, S_n')$ to $\times_{i=1}^n S_i$, by assigning the probability 0 to action combinations not in $\times_{i=1}^n S_i'$, gives an element of $T(S_1, \ldots, S_n)$.

Well-known examples of specifications $T(S_1, \ldots, S_n)$ are, with some abuse of notation, $\mathscr{L}(\times_{i=1}^n S_i)$, $\times_{i=1}^n \mathscr{L}(S_i)$, and $\times_{i=1}^n S_i$, i.e., the sets of correlated, mixed, and pure joint strategies, respectively.

A strategic form game on N has the form

$$\Gamma = \langle S_1, \ldots, S_n, T, K_1, \ldots, K_n \rangle,$$

where the sets S_i are the players' action sets, T is a joint strategy specification, and $K_i: \mathscr{L}(\times_{i=1}^n S_i) \to \mathbb{R}$ is the Von Neumann–Morgenstern payoff function of player i, assigning expected payoffs in particular to elements σ of $T(S_1, \ldots, S_n)$; so, the functions K_i are supposed to be determined by their values on $\times_{i=1}^n S_i$.

Next, the concept of equilibrium will be defined. For that purpose, we need some notations. For $\sigma \in T(S_1, \ldots, S_n)$, $i \in N$, and $\hat{s}_i \in S_i$ such that $\sigma(t^{-i}, \hat{s}_i) > 0$ for some $t^{-i} \in S^{-i}$, let

$$\sigma(s^{-i} \mid \hat{s}_i) := \sigma(s^{-i}, \hat{s}_i) / \sum_{t^{-i} \in S^{-i}} \sigma(t^{-i}, \hat{s}_i)$$

be the marginal probability of $s^{-i} \in S^{-i}$, given $\hat{s}_i$. Let further

$$\mu_i(\sigma) := \{s_i \in S_i : \exists s^{-i} \in S^{-i}, \text{ with } \sigma(s^{-i}, s_i) > 0\}$$

be the support of σ for player i. When judging the quality of an action $\hat{s}_i$, player i compares the associated expected payoff to the expected payoff from a switch to a different action s_i, conditionally on the other players playing according to $\sigma(\cdot \mid \hat{s}_i)$. Hence, player i compares the numbers

$$K_i(\sigma(\cdot \mid \hat{s}_i), s_i) := \sum_{s^{-i} \in S^{-i}} K_i(s^{-i}, s_i)\sigma(s^{-i} \mid \hat{s}_i), \qquad s_i \in S_i.$$

We are now in a position to define the equilibrium concept. A strategy $\sigma \in T(S_1, \ldots, S_n)$ is said to be an equilibrium if, for each player i and each $\hat{s}_i \in \mu_i(\sigma)$,

$$K_i(\sigma(\cdot \mid \hat{s}_i), \hat{s}_i) \geq K_i(\sigma(\cdot \mid \hat{s}_i), s_i), \qquad \text{for each } s_i \in S_i.$$

It can be verified easily that this definition coincides with the idea of a correlated equilibrium or a Nash equilibrium in pure or mixed strategies, in the respective associated cases.

From now on, the joint strategy specification T will be fixed. $\mathscr{G}^N$ will denote the collection of all games with N players. $E(\Gamma)$ will denote the set of equilibria of Γ. For

$$T(S_1, \ldots, S_n) = \bigtimes_{i=1}^{n} \mathscr{L}(S_i),$$

the nonemptiness of $E(\Gamma)$ was already established by Nash (Ref. 1), while for

$$T(S_1, \ldots, S_n) = \mathscr{L}\left(\bigtimes_{i=1}^{n} S_i\right),$$

the nonemptiness of $E(\Gamma)$ is proved in Aumann (Ref. 4).

In general, $E(\Gamma)$ contains more than one element, and different elements of $E(\Gamma)$ give rise to different payoff vectors. In the next section, we present a procedure which will turn out to assign a unique payoff vector to each game. This solution procedure can not only be applied directly to $E(\Gamma)$, but also to refinements of $E(\Gamma)$.

In order to introduce this procedure, the concept of cyclic symmetry of a game needs to be defined. In this concept, cyclic permutations of the player set or subsets of the player set play a role. For $\emptyset \neq L \subset N$, a permutation π of N is called L-cyclic if π is $N \backslash L$-invariant [i.e., $\pi(i) = i$, for every $i \notin L$] and the restriction of π to L is cyclic of order $|L|$. Recall that this last property means that

$$\pi^{|L|}(i) = i \text{ and } L = \{\pi^1(i), \ldots, \pi^{|L|}(i)\}, \qquad \text{for every } i \in L.$$

Here, $\pi^k(i) = \pi \circ \cdots \circ \pi(i)$, π applied k times. As an example, suppose that $L = \{1, 2, 3, 4\}$; then, π with $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 4$, and $\pi(4) = 1$ is an L-cyclic permutation. For $\emptyset \neq L \subset N$ and an L-cyclic permutation π of N, a game Γ is called π-cyclic symmetric if the following two conditions hold:

(CS1) $S_i = S_j$, for all $i, j \in L$;

(CS2) $K_i(s) = K_{\pi^k(i)}(\pi^k(s))$, for every $i \in N$, every $s \in S_1 \times \cdots \times S_N$, and every $k = 1, \ldots, |L| - 1$, where $(\pi^k(s))_j := s_{(\pi^k)^{-1}(j)}$, for every $j \in N$.

Observe that cyclic symmetry is a weak kind of symmetry. Usually for $\emptyset \neq L \subset N$, a game is called L-symmetric if (CS1) is satisfied and, for every $N \backslash L$-invariant permutation π, it holds that

$$K_i(s) = K_{\pi(i)}(\pi(s)), \qquad \text{for all } i \in N.$$

Hence, an L-symmetric game in the usual sense is also π-cyclic symmetric; but except for $|L| = 1$ and $|L| = 2$, the converse is not true.

With π an L-cyclic permutation of N, a subset $V \subset T(S_1, \ldots, S_n)$ is called π-cyclic symmetric if, for every $\sigma \in V$ and $k = 1, 2, \ldots, |L| - 1$, also $\sigma_k \in V$, where σ_k is defined by

$$\sigma_k(\pi^k(s)) := \sigma(s), \qquad \text{for all } s \in S_1 \times \cdots \times S_n.$$

Observe that, for a π-cyclic symmetric game Γ, also $E(\Gamma)$ is π-cyclic symmetric.

A game

$$\Gamma' = \langle S'_1, \ldots, S'_n, T, K'_1, \ldots, K'_n \rangle$$

is a subgame of Γ if $S'_i \subset S_i$ and K'_i is the restriction of K_i to S'_i, for every $i \in N$.

Let D be a map which assigns to every game a vector in the payoff outcome space with the following properties:

(D1) $D_i(\Gamma) \leq K_i(\sigma)$, for all $\sigma \in E(\Gamma)$ and $i \in N$;

(D2) for all $\Gamma, \Gamma' \in \mathscr{G}^N$, with Γ' a subgame of Γ for which, for all $i \in N$ and for all $s \in S'_1 \times \cdots \times S'_{i-1} \times S_i \backslash S'_i \times S'_{i+1} \times \cdots \times S'_n$, it holds that $K_i(s) = D_i(\Gamma')$, we have $D(\Gamma) = D(\Gamma')$;

(D3) if Γ is π-cyclic symmetric for an L-cyclic permutation π, then $D_i(\Gamma) = D_j(\Gamma)$ for all $i, j \in L$.

D is called a disagreement map, and $D(\Gamma)$ is called a disagreement point.

An example of D in the Nash-equilibria case [the case where $T(S_1, \ldots, S_n) = \times_{i=1}^n \mathscr{L}(S_i)$] is the value vector defined by

$$v_i(\Gamma) = \min_{\sigma^{-i} \in \times_{j \neq i} \mathscr{L}(S_j)} \max_{s_i \in S_i} K_i(\sigma^{-i}, s_i).$$

Evidently, being the payoff that player i can guarantee for himself, $v_i(\Gamma)$ is less than or equal to $K_i(\sigma)$ for all $\sigma \in E(\Gamma)$, while properties (D2) and (D3) can easily be checked to hold for v.

For a given disagreement map D, a solution is a correspondence Φ assigning to each $\Gamma \in \mathscr{G}^N$ a nonempty subset of $\mathscr{L}(E(\Gamma))$, the set of lotteries on $E(\Gamma)$. If player i has payoff function K_i, then $K_i(\mu)$ denotes i's expected payoff from the lottery $\mu \in \mathscr{L}(E(\Gamma))$.

The Nash solution Σ is defined as follows. For every $\Gamma \in \mathscr{G}^N$, define $N(\Gamma)$ by

$$N(\Gamma) := \{i \in N\colon K_i(\sigma) > D_i(\Gamma), \text{ for some } \sigma \in E(\Gamma)\}.$$

Then, if $N(\Gamma) \neq \varnothing$, for every $\bar{\mu} \in \mathscr{L}(E(\Gamma))$, let $\bar{\mu} \in \Sigma(\Gamma)$ if and only if $\bar{\mu}$ maximizes the product $\prod_{i \in N(\Gamma)} [K_i(\mu) - D_i(\Gamma)]$, over all $\mu \in \mathscr{L}(E(\Gamma))$. Otherwise, if $N(\Gamma) = \varnothing$, for every $\bar{\mu} \in \mathscr{L}(E(\Gamma))$, let $\bar{\mu} \in \Sigma(\Gamma)$ if and only if

$$K(\bar{\mu}) := [K_1(\bar{\mu}), \ldots, K_n(\bar{\mu})] = D(\Gamma).$$

Observe the similarity between this definition and the definition of the Nash bargaining solution; cf. Nash (Ref. 11).

3. Axioms

This section is concerned with the formulation and discussion of the axioms that will be used in the characterization of the Nash solution. Throughout, a fixed joint strategy specification T and disagreement map D will be assumed. Let Φ be a solution.

(A1) Pareto Optimality. For all $\Gamma \in \mathscr{G}^N$, $\mu \in \Phi(\Gamma)$, $\nu \in \mathscr{L}(E(\Gamma))$, if $K(\nu) \geq K(\mu)$, then $K(\nu) = K(\mu)$.

Note that Pareto optimality is required with respect to $\mathscr{L}(E(\Gamma))$; as is well known, equilibria that are efficient with respect to the total strategy space may fail to exist.

(A2) Payoff Representation Invariance. For all $\Gamma \in \mathscr{G}^N$ and all $a, b \in \mathbb{R}^n$, with a strictly positive,

$$\Phi(\Gamma) = \Phi(a\Gamma + b),$$

with

$$a\Gamma + b := \langle S_1, \ldots, S_n, T, a_1K_1 + b_1, \ldots, a_nK_n + b_n \rangle.$$

The Von Neumann–Morgenstern payoff functions K_i are unique only up to positive affine transformations. This fact is reflected by the payoff representation invariance axiom. Alternatively, applying positive affine transformations to the payoffs in a game does not affect the players strategic possibilities.

(A3) Payoff Completeness. For all $\Gamma \in \mathscr{G}^N$, $\mu \in \Phi(\Gamma)$, $\nu \in \mathscr{L}(E(\Gamma))$, if $K(\nu) = K(\mu)$, then $\nu \in \Phi(\Gamma)$.

Payoff completeness requires a solution not to discriminate between lotteries of equilibria with the same expected payoffs.

(A4) Cyclic Symmetry. For every $\Gamma \in \mathscr{G}^N$ and L-cyclic permutation π, if Γ is π-cyclic symmetric, then $K_i(\mu) = K_j(\mu)$, for all $\mu \in \Phi(\Gamma)$ and all $i, j \in L$.

If Γ is π-cyclic symmetric, then there is no apparent way to distinguish between the players in L, and so the solution should not make such a distinction. Observe that, if Φ happens to be payoff unique [i.e., $K(\mu) = K(\nu)$ for all μ, $\nu \in \Phi(\Gamma)$], then cyclic symmetry would be implied by the standard anonymity property.

The final axiom is given below.

(A5) Multilateral Reduction Independence. Consider all games $\Gamma, \Gamma' \in \mathscr{G}^N$ such that:

(M1) $\Gamma' = \langle S'_1, \ldots, S'_n, T, K'_1, \ldots, K'_n \rangle$ is a subgame of

$$\Gamma = \langle S_1, \ldots, S_n, T, K_1, \ldots, K_n \rangle;$$

(M2) $D(\Gamma') = D(\Gamma)$;

(M3) for all $i \in N$ and for all $s \in S'_1 \times \cdots \times S'_{i-1} \times S_i \backslash S'_i \times S'_{i+1} \times \cdots \times S'_n$, $K_i(s) \leq D_i(\Gamma)$.

If $\Phi(\Gamma) \cap \mathscr{L}(E(\Gamma')) \neq \varnothing$, then $\Phi(\Gamma') \subset \Phi(\Gamma)$.

In words: Assume that a game Γ' arises from a game Γ by reducing the strategy sets in such a way that (i) the disagreement point does not change and that (ii) for any player, deviating to a strategy only feasible in the larger game yields him at most his disagreement payoff if all other players stick to their strategies in the reduced sets. Then, the solution in the smaller game Γ' should be a subset of the solution in the original game Γ, provided this is feasible.

Note that condition (M3) describes a Nash equilibrium-like situation with respect to omitting strategies. One can imagine a bargaining process resulting in each player promising not to use certain of his strategies. If the conditions of multilateral reduction independence are fulfilled, then these promises are self-enforcing.

Also note that, by (T3), (D1), (M2), (M3),

$$E(\Gamma') \subset E(\Gamma). \tag{1}$$

A stronger version of the multilateral reduction independence axiom can be obtained by replacing (M3) by (1). This version could also be used in the characterization result in the next section. However, condition (M3) has a more intuitive strategic interpretation, which we prefer. Consistently with what we wrote in Section 1, we view the solution Φ as a recommendation method on which the players have reached an agreement, possibly after having discussed its characterizing properties. In particular, they may agree on a property like multilateral reduction independence on the basis of the given strategic interpretation.

The multilateral reduction independence axiom is clearly related to Nash's independence of irrelevant alternatives (Nash, Ref. 11) and to Aumann's version of that axiom for correspondences (Aumann, Ref. 14). It is formulated in terms of the underlying strategic game, contrary to Axiom 6 in Harsanyi and Selten (Ref. 12, p. 99).

Of the axioms proposed, the only one that does not have an immediate equivalent in the Nash (Ref. 11) original formulation is that of payoff completeness. This is not surprising, since the Nash bargaining problem is modeled in the utility space (i.e., payoff space), and an underlying set of alternatives does not play an explicit role.

4. Characterization of the Nash Solution

As before, $\mathscr{G}^N$ is the class of games on N for some fixed pair T, D. The characterization theorem of the Nash solution is given below.

Theorem 4.1. A solution Φ on $\mathscr{G}^N$ satisfies Pareto optimality, payoff representation invariance, payoff completeness, cyclic symmetry, and multilateral reduction independence if and only if Φ is the Nash solution Σ.

In the following lemma the "if" part of Theorem 4.1 is checked.

Lemma 4.1. The Nash solution Σ satisfies the five axioms mentioned in Theorem 4.1.

Proof. Pareto optimality, payoff representation invariance, and payoff completeness of the Nash solution Σ are straightforward. Cyclic symmetry of Σ follows from the fact that $E(\Gamma)$ is π-cyclic symmetric whenever Γ has this property, payoff uniqueness of Σ [i.e., $K(\mu) = K(\nu)$ for all $\mu, \nu \in \Sigma(\Gamma)$] and anonymity of Σ [i.e., the vector $K(\mu)$ permutes with every permutation of the player set, for every $\mu \in \Sigma(\Gamma)$].

Finally, multilateral reduction independence of Σ follows with the aid of (1). □

The proof of the "only if" part will be based on a cyclic symmetrization method, which in the two-person case is closely related to a symmetrization method introduced by Griesmer et al. (Ref. 15). To

$$\Gamma = \langle S_1, \ldots, S_n, T, K_1, \ldots, K_n \rangle,$$

we associate the game

$$\bar{\Gamma} = \langle S, \ldots, S, T, \bar{K}_1, \ldots, \bar{K}_n \rangle$$

in the following way:

(i) $S = S_1 \cup S_2 \cup \cdots \cup S_n$.
The original action sets are supposed to be disjoint.

(ii) Let π be the shift-permutation

$$\pi(1) = 2, \qquad \pi(2) = 3, \ldots, \pi(n-1) = n, \qquad \pi(n) = 1.$$

If $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_n) \in S \times \cdots \times S$ and there is a $k \in \{1, 2, \ldots, n\}$ such that

$$\bar{s}_i \in S_{\pi^k(i)}, \qquad \text{for each } i \in N,$$

then

$$\bar{K}_i(\bar{s}_1, \ldots, \bar{s}_n) := K_{\pi^k(i)}(\pi^k(\bar{s})), \qquad \text{for each } i \in N.$$

Note that $\pi^k(s)$ was defined in Section 2; see condition (CS2).

(iii) If $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_n) \in S \times \cdots \times S$ and a k as in (ii) does not exist, then

$$\bar{K}_i(\bar{s}) := D_i(\Gamma), \qquad \text{for each } i \in N.$$

Observe that (ii) above applies to $\bar{s}$ if and only if

$$\bar{s} \equiv (s_{k+1}, s_{k+2}, \ldots, s_n, s_1, \ldots, s_k), \qquad \text{for some } k \in \{1, 2, \ldots, n\},$$

with

$$s_i \in S_i, \qquad \text{for each } i \in N.$$

In that case,

$$\bar{K}_i(s_{k+1}, s_{k+2}, \ldots, s_n, s_1, \ldots, s_k) = K_{i+k(\text{mod } n)}(s_1, s_2, \ldots, s_n).$$

It can easily be checked that the game $\bar{\Gamma}$ is π-cyclic symmetric.

Recall that the map T is defined for all n-tuples of finite sets, in particular also for $(S, \ldots, S)$ as above.

For the two-person case, a game Γ written as a bimatrix $[K_1, K_2]$ gives as cyclic symmetrization the game $\bar{\Gamma}$ defined as

$$\begin{bmatrix} \hat{D}_1(\Gamma), \hat{D}_2(\Gamma) & K_1, K_2 \\ K_2^t, K_1^t & \hat{D}_1(\Gamma), \hat{D}_2(\Gamma) \end{bmatrix},$$

where $\hat{D}_k(\Gamma)$ denotes a square matrix of appropriate size with each entry equal to $D_k(\Gamma)$ and the superscript t denotes the transposed matrix.

In the three-person case, the cyclic symmetrization of a game Γ leads to

$$\begin{array}{cc} & \text{pl.2} \\ \text{pl.1} & \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & K_2, K_3, K_1 \\ \cdot & \cdot & \cdot \end{bmatrix} \\ & \bar{s}_3 \in S_1 \end{array} \quad \begin{array}{cc} & \text{pl.2} \\ \text{pl.1} & \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & K_3, K_1, K_2 & \cdot \end{bmatrix} \\ & \bar{s}_3 \in S_2 \end{array} \quad \begin{array}{cc} & \text{pl.2} \\ \text{pl.1} & \begin{bmatrix} K_1, K_2, K_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \\ & \bar{s}_3 \in S_3 \end{array},$$

where the first block row of each matrix corresponds to actions of player 1 from S_1, the second one to actions from S_2, and the third one to actions from S_3. The three-block columns correspond in a similar way to S_1, S_2, S_3 for player 2. Observe that the first matrix is $|S_1|$ layers deep, each one corresponding with a certain action of player 3 in $\bar{\Gamma}$ from S_1. Analogous structures hold for the second and third matrix. Each dot consists of a block of appropriate size with each element equal to $[D_1(\Gamma), D_2(\Gamma), D_3(\Gamma)]$.

Similarly as above, cyclic symmetrization can be defined for subsets $L \subset N$, for instance for $L = \{1, 2, \ldots, l\}$,

$$\bar{\Gamma} = \langle \underbrace{S, \ldots, S}_{l \text{ times}}, S_{l+1}, \ldots, S_n, T, \bar{K}_1, \ldots, \bar{K}_n \rangle,$$

with $S = \bigcup_{k=1}^{l} S_k$ and $\bar{K}_i$, $i = 1, 2, \ldots, n$, defined analogously as above, using the L-cyclic shift permutation.

In the sequel, we need some additional notation. Let $\bar{\sigma} \in T(S, \ldots, S)$ in $\bar{\Gamma}$, and let π be a permutation of N. Define the number

$$\bar{\sigma}_\pi(S_{\pi(1)}, \ldots, S_{\pi(n)}) := \sum_{\bar{s}_1 \in S_{\pi(1)}} \cdots \sum_{\bar{s}_n \in S_{\pi(n)}} \bar{\sigma}(\bar{s}_1, \ldots, \bar{s}_n),$$

i.e., as the total probability mass assigned by $\bar{\sigma}$ to the subgame in $\bar{\Gamma}$ arising from the permutation π. If this number is positive, define the joint strategy $\bar{\sigma}_\pi \in T(S_{\pi(1)}, \ldots, S_{\pi(n)})$ in the game $\langle S_{\pi(1)}, \ldots, S_{\pi(n)}, T, K_{\pi(1)}, \ldots, K_{\pi(n)} \rangle$ by

$$\bar{\sigma}_\pi(s_1, \ldots, s_n) := \bar{\sigma}(s_1, \ldots, s_n) / \bar{\sigma}_\pi(S_{\pi(1)}, \ldots, S_{\pi(n)}).$$

The next lemma paves the way for the proof of the "only if" part of Theorem 4.1. It states that an equilibrium in $\bar{\Gamma}$ induces equilibria in the n subgames

$$\pi^k(\Gamma) := \langle S_{k+1}, \ldots, S_n, S_1, \ldots, S_k, T, K_{k+1}, \ldots, K_n, K_1, \ldots, K_t \rangle.$$

Lemma 4.2. Let Γ and $\bar{\Gamma}$ be as above, with additionally $D_i(\Gamma) = D_j(\Gamma)$, for all $i, j \in N$. Let π be the shift permutation. Then:

(i) $\bar{\Gamma}$ is π-cyclic symmetric;
(ii) $D(\bar{\Gamma}) = D(\Gamma)$;
(iii) for every $\bar{\sigma} \in E(\bar{\Gamma})$, if $\sigma_{\pi^k}(S_{\pi^k(1)}, \ldots, S_{\pi^k(n)}) > 0$, then $\bar{\sigma}_{\pi^k} \in E(\pi^k(\Gamma))$.

Proof.

(i) This is obvious from the definition of $\bar{\Gamma}$.

(ii) Consider the subgame $\langle S_1, S_2, \ldots, S_n, T, K_1, \ldots, K_n \rangle$ of $\bar{\Gamma}$, which is identical to the original game Γ. By definition of $\bar{\Gamma}$, we have that, for each $(\bar{s}_1, \ldots, \bar{s}_n) \in S_1 \times \cdots \times S_{i-1} \times S \backslash S_i \times S_{i+1} \times \cdots \times S_n$,

$$K_i(\bar{s}_1, \ldots, \bar{s}_n) = D_i(\Gamma).$$

Hence, property (D2) of D implies $D(\Gamma) = D(\bar{\Gamma})$.

(iii) Let $\bar{\sigma} \in E(\bar{\Gamma})$, and let $\bar{\sigma}_{\pi^k}(S_{k+1}, \ldots, S_n, S_1, \ldots, S_k) > 0$ for some $k \in \{1, 2, \ldots, n\}$. Let for player $i - k (\mathrm{mod}\, n)$, $\hat{s}_i \in S_i$, with $\hat{s}_i \in \mu_{i-k(\mathrm{mod}\, n)}(\bar{\sigma})$. Then, by definition of equilibrium,

$$\bar{K}_i(\bar{\sigma}(\hat{s}_i), \hat{s}_i) \geq \bar{K}_i(\bar{\sigma}(\hat{s}_i), s_i), \qquad \text{for all } s_i \in S_i.$$

Observe that

$$\begin{aligned} \bar{K}_i(\bar{\sigma}(\hat{s}_i), s_i) = &\sum_{s^{-i} \in S^{-i}} \bar{K}_i(s^{-i}, s_i) \bar{\sigma}(s^{-i} \mid \hat{s}_i) \cdot \sum_{t^{-i} \in S^{-i}} \bar{\sigma}(t^{-i}, \hat{s}_i) \\ &+ \left[1 - \sum_{t^{-i} \in S^{-i}} \bar{\sigma}(t^{-i}, \hat{s}_i)\right] D_{i-k(\mathrm{mod}\, n)}(\Gamma) \end{aligned}$$

$$= K_i(\sigma_{\pi^k}(\hat{s}_i), s_i) \cdot \sum_{t^{-i} \in S^{-i}} \bar{\sigma}(t^{-i}, \hat{s}_i) + \left[1 - \sum_{t^{-i} \in S^i} \bar{\sigma}(t^{-i}, \hat{s}_i)\right] D_{i-k(\text{mod } n)}(\Gamma).$$

On the right-hand side of this equality, only the first term depends on s_i. Hence from

$$\bar{K}_i(\bar{\sigma}(\hat{s}_i), \hat{s}_i) \geq \bar{K}_i(\bar{\sigma}(\hat{s}_i), s_i),$$

we can conclude that

$$K_i(\sigma_{\pi^k}(\hat{s}_i), \hat{s}_i) \geq K_i(\sigma_{\pi^k}(\hat{s}_i), s_i),$$

which yields $\sigma_{\pi^k} \in E(\pi^k(\Gamma))$. □

It should be noted that part (iii) of Lemma 4.2 for $n = 2$ is related to parts (iii) and (iv) of Theorem 4.1 in Jansen et al. (Ref. 16).

Next, the "only if" part of Theorem 4.1 can be proved.

Lemma 4.3. Let the solution Φ on $\mathcal{G}^N$ satisfy the five axioms mentioned in Theorem 4.1. Then, Φ equals the Nash solution Σ.

Proof. Let $\Gamma \in \mathcal{G}^N$, and let $N(\Gamma)$ be as in the definition of the Nash solution. If $N(\Gamma) = \varnothing$, then

$$\{K(\mu): \mu \in \mathcal{L}(E(\Gamma))\} = \{D(\Gamma)\},$$

so

$$\Phi(\Gamma) \subset \Sigma(\Gamma) = \mathcal{L}(E(\Gamma)).$$

From this and the payoff completeness of Φ,

$$\Phi(\Gamma) = \Sigma(\Gamma).$$

If $|N(\Gamma)| = 1$, then $\Phi(\Gamma) = \Sigma(\Gamma)$ follows straightforwardly by Pareto optimality and payoff completeness. From now on, let

$$N(\Gamma) = \{1, \ldots, l\}, \qquad \text{with } l \geq 2.$$

In view of the payoff representation invariance, it is w.l.o.g. to assume

$$D(\Gamma) = (0, \ldots, 0)$$

and

$$\Sigma(\Gamma) = \{\mu \in \mathcal{L}(E(\Gamma)): K_i(\mu) = 1, 1 \leq i \leq l;\ K_i(\mu) = 0, l+1 \leq i \leq n\}.$$

The definition of the Nash solution then implies that, for all $\mu \in \mathscr{L}(E(\Gamma))$,

$$K(\mu) \in \text{conv}(\{le^i: 1 \leq i \leq l\} \cup \{(0, \ldots, 0)\}), \tag{2}$$

where e^i is the ith unit vector in $\mathbb{R}^n$.

Let $\bar{\Gamma}$ be the π-cyclic symmetric game associated with Γ, where π is the $N(\Gamma)$-cyclic shift permutation, as defined above. Let $\sigma \in E(\Gamma)$. Then, σ can be extended to $\bar{\Gamma}$ in the obvious way by assigning probability 0 to the added action combinations. So, $E(\Gamma) \subset E(\bar{\Gamma})$. Let $\bar{\sigma} \in E(\bar{\Gamma})$. Then by Lemma 4.2 applied to Γ and $\bar{\Gamma}$, we obtain [recall that $D(\Gamma) = 0$]

$$\bar{K}(\bar{\sigma}) = [\bar{K}_1(\bar{\sigma}), \ldots, \bar{K}_n(\bar{\sigma})] = \sum_{k=1}^{l} \lambda_k [K_{\pi^k(1)}(\bar{\sigma}_{\pi^k}), \ldots, K_{\pi^k(n)}(\bar{\sigma}_{\pi^k})],$$

with

$$\lambda_k \geq 0 \quad \text{and} \quad \sum_{k=1}^{l} \lambda_k \in [0, 1].$$

Since for each $k = 1, \ldots, l$ we have

$$[K_{\pi^k(1)}(\bar{\sigma}_{\pi^k}), \ldots, K_{\pi^k(n)}(\bar{\sigma}_{\pi^k})] \in \text{conv}(\{le^i: 1 \leq i \leq l\} \cup \{(0, 0, \ldots, 0)\}),$$

by (2), it follows that

$$\bar{K}(\bar{\sigma}) \in \text{conv}(\{le^i: 1 \leq i \leq l\} \cup \{(0, 0, \ldots, 0)\}).$$

Therefore,

$$\{K(\bar{\sigma}): \bar{\sigma} \in E(\bar{\Gamma})\} \subset \text{conv}(\{le^i: 1 \leq i \leq l\} \cup \{(0, 0, \ldots, 0)\}).$$

$\bar{\Gamma}$ is π-cyclic symmetric, so for all $\sigma \in \Phi(\bar{\Gamma})$ we have

$$K_i(\sigma) = K_j(\sigma), \qquad i, j \in \{1, \ldots, l\},$$

By Pareto optimality, we then have $\sigma \in \Phi(\bar{\Gamma})$ only if

$$K_i(\sigma) = K_j(\sigma) = 1, \qquad \text{for all } i, j \in \{1, \ldots, l\},$$

since there exists a $\sigma \in \mathscr{L}(E(\Gamma)) \subset \mathscr{L}(E(\bar{\Gamma}))$ with

$$K_i(\sigma) = 1, \qquad 1 \leq i \leq l,$$

$$K_i(\sigma) = 0, \qquad \text{otherwise}.$$

Therefore, by payoff completeness, we have

$$\Phi(\bar{\Gamma}) = \{\sigma \in \mathscr{L}(E(\bar{\Gamma})): K_i(\sigma) = 1, i \leq 1 \leq l;\ K_i(\sigma) = 0, \text{otherwise}\}.$$

In the formulation of the multilateral reduction independence axiom, the roles of $\Gamma, \Gamma', S'_1, \ldots, S'_n$ can be played by $\bar{\Gamma}, \Gamma, S_1, \ldots, S_n$, respectively. Condition (M3) of the axiom follows from the definition of $\bar{\Gamma}$. Also,

$$\Phi(\bar{\Gamma}) \cap \mathscr{L}(E(\Gamma)) \neq \varnothing;$$

hence,

$$\Phi(\Gamma) \subset \Phi(\bar{\Gamma}),$$

by this axiom. Again using the payoff completeness axiom, this implies that

$$\Phi(\Gamma) = \{\sigma \in \mathscr{L}(E(\Gamma)) \colon K_i(\sigma) = 1, 1 \leq i \leq l;\ K_i(\sigma) = 0, \text{otherwise}\}.$$

Therefore,

$$\Phi(\Gamma) = \Sigma(\Gamma). \qquad \square$$

Evidently, Theorem 4.1 follows from Lemmas 4.1 and 4.3.

The Nash proof of the characterization of the Nash bargaining solution uses a triangular bargaining problem [e.g., as in (2)] as a symmetric superset of the given bargaining problem, to which in particular the bargaining axioms of Pareto optimality, symmetry, and independence of irrelevant alternatives can be applied. In the proof of Theorem 4.1, more specifically of Lemma 4.3, the image of $\mathscr{L}(E(\Gamma))$ in payoff space is rotated with respect to the straight line $x_1 = x_2 = \cdots = x_n$, and the union is taken. This leads to the cyclic-symmetric image of $\mathscr{L}(E(\bar{\Gamma}))$. This approach could be applied to the Nash original characterization as well; however, the converse is not apparent, because it is not clear how, for a given game Γ', a related game Γ (larger in the sense formulated in the multilateral reduction independence axiom) can be constructed that has a triangular image in payoff space.

Next, we show that Theorem 4.1 is tight in the sense that none of the five axioms can be dispensed with. For each of the axioms, we will describe a solution violating that axiom while satisfying the four other axioms.

(i) Pareto Optimality. Let $\Phi(\Gamma) := \{\mu \in \mathscr{L}(E(\Gamma)) \colon K(\mu) = D(\Gamma)\}$, if this set is nonempty; let $\Phi(\Gamma) := \Sigma(\Gamma)$, otherwise.

(ii) Payoff Representation Invariance. Let Φ assign to Γ the set of all elements in $\mathscr{L}(E(\Gamma))$ with lexicographically maximal payoffs.

(iii) Payoff Completeness. Let $\Phi(\Gamma) := \Sigma(\Gamma)$, if $\Sigma(\Gamma) \cap E(\Gamma) = \varnothing$; let $\Phi(\Gamma) := \Sigma(\Gamma) \cap E(\Gamma)$, otherwise.

(iv) Cyclic Symmetry. Let Φ assign to Γ the set of all elements of $\mathscr{L}(E(\Gamma))$ that maximize successively the payoffs to the players in some given order.

(v) Multilateral Reduction Independence. Let $n = 2$ and

$$u(\Gamma) := \left[\max_{\sigma \in E(\Gamma)} K_1(\sigma), \max_{\sigma \in E(\Gamma)} K_2(\sigma)\right].$$

Let Φ assign to Γ the set of all members of $\mathscr{L}(E(\Gamma))$ that maximize the players' payoffs restricted to the line segment connecting $D(\Gamma)$ and $u(\Gamma)$.

The proofs of these statements are left to the reader. In particular, to check the multilateral reduction independence, it is usually convenient to use inclusion (1).

Finally, we substantiate our claim, made in the introduction, that our method and in particular Theorem 4.1 may be applied to refinements of Nash equilibrium, by showing this for perfect equilibria (Selten, Ref. 7). Thus, let $PE(\Gamma)$ denote the set of perfect Nash equilibria in a game Γ. This set is π-cyclic symmetric whenever Γ is. In (D1), in the definition of a solution and in particular of the Nash solution, and in the definitions of the axioms, replace $E(\Gamma)$ by $PE(\Gamma)$. In the definition of multilateral reduction independence, replace (M3) by the condition $PE(\Gamma') \subset PE(\Gamma)$. It can be chekced, but is not completely trivial, that with these modifications Lemmas 4.1, 4.2, and 4.3 still hold true, with $E(\bar{\Gamma})$ replaced by $PE(\bar{\Gamma})$ and $E(\pi^k(\Gamma))$ by $PE(\pi^k(\Gamma))$ in Lemma 4.2, and with $E(\Gamma)$ and $E(\bar{\Gamma})$ replaced by $PE(\Gamma)$ and $PE(\bar{\Gamma})$, respectively, in Lemma 4.3. Consequently, Theorem 4.1 presents a characterization for the case of perfect Nash equilibria as well. See also Example 5.4 in the next section.

5. Conclusions

In this paper, we have introduced and axiomatically characterized a general method of equilibrium refinement. The adjective "general" refers here to the fact that the method may be applied to several equilibrium concepts and refinements thereof and to a whole class of games. Since in many games, existing refinements do not lead to unique equilibria, the method described may well serve as an aid to arrive at a unique equilibrium, at least at unique equilibrium payoffs.

The paper will be concluded by some examples.

Example 5.1. If the set

$$K := \{K(\mu) = (K_1(\mu), \ldots, K_n(\mu)) \colon \mu \in \mathscr{L}(E(\Gamma))\}$$

has a unique Pareto optimal point, then the Nash solution consists of those $\mu \in \mathscr{L}(E(\Gamma))$ which yield this point as payoff outcome. If in particular, E is the set of all Nash equilibria and a certain equilibrium point payoff dominates all other equilibria, then this equilibrium point will be assigned by the Nash solution.

Example 5.2. A generic 2×2 bimatrix game possesses 1 or 3 equilibrium points. In the latter case, two of these are pure and diagonal-wise situated, while the third one is completely mixed. Let

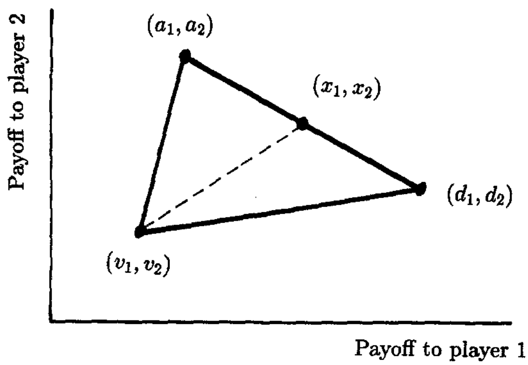

Fig. 1. Description of Example 5.2.

$$\begin{bmatrix} a_1, a_2 & b_1, b_2 \\ c_1, c_2 & d_1, d_2 \end{bmatrix}$$

be a generic bimatrix game with three equilibria. Without loss of generality, assume that the two pure equilibria are on the same diagonal, with equilibrium payoffs (a_1, a_2) and (d_1, d_2). It is well known that the equilibrium payoffs of the mixed equilibrium equal (v_1, v_2), where v_i is the maximin value of player i, so

$$v_1 = (a_1 d_1 - b_1 c_1)/(a_1 + d_1 - b_1 - c_1),$$
$$v_2 = (a_2 d_2 - b_2 c_2)/(a_2 + d_2 - b_2 - c_2).$$

Suppose that the Nash solution is applied to the set of Nash equilibria E, and let D assign the individual maxmin values. With

$$\lambda = (d_1 - v_1)/2(d_1 - a_1) - (d_2 - v_2)/2(a_2 - d_2),$$

it can be checked that the Nash solution payoffs equal (x_1, x_2), where

(i) $(x_1, x_2) = (a_1, a_2)$, if $\lambda \geq 1$,
(ii) $(x_1, x_2) = (d_1, d_2)$, if $\lambda \leq 0$,
(iii) $(x_1, x_2) = \lambda(a_1, a_2) + (1 - \lambda)(d_1, d_2)$, if $0 \leq \lambda \leq 1$ (see Fig. 1).

Notice that case (iii) corresponds to the correlated equilibrium where with probability λ both players choose their first action and with probability $1 - \lambda$ both players choose their second action. Applying this result to the battle of the sexes,

$$\begin{bmatrix} 2, 1 & 0, 0 \\ 0, 0 & 1, 2 \end{bmatrix},$$

it follows that the Nash solution corresponds to playing cell (2, 1) with probability 1/2 and cell (1, 2) with probability 1/2.

Example 5.3. Let E be the set of Nash equilibria. It can easily be checked that any extreme point of

$$K = \{K(\mu) = (K_1(\mu), \ldots, K_n(\mu)): \mu \in \mathscr{L}(E(\Gamma))\}$$

can be associated with an extreme point of a maximal Nash subset; see Nash (Ref. 1) or Heuer and Millham (Ref. 17). Hence, the Nash solution can always be written as a possibly nonunique convex combination of extreme points of maximal Nash subsets.

Example 5.4. Consider the bimatrix game

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} 8, 0 & 5, 0 \\ 0, 0 & 0, -1 \\ 7, 0 & 5, 2 \end{bmatrix} \end{array}.$$

Let the disagreement point be given by the individual maxmin values, i.e., the point $(5, 0)$. Applying the Nash solution for the case of Nash equilibrium, we obtain the outcome $(6.5, 0.5)$, which is reached by playing (T, L) and (B, R) each with probability 1/2. If we restrict attention to perfect Nash equilibria, the Nash solution prescribes (T, L) with probability 1. By deleting the second row of player 1, so that the game becomes

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} 8, 0 & 5, 0 \\ 7, 0 & 5, 2 \end{bmatrix} \end{array},$$

the Nash solution is not affected if lotteries between all Nash equilibria are considered. However, the pair (T, L) is no longer perfect, and when restricted to perfect Nash equilibria, the Nash solution prescribes (T, R), the sole remaining perfect Nash equilibrium. Observe that this does not violate the multilateral reduction independence axiom.

References

1. NASH, J. F., *Noncooperative Games*, Annals of Mathematics, Vol. 54, pp. 286–295, 1951.
2. VAN DAMME, E. E. C., *Stability and Perfection of Nash Equilibria*, Springer Verlag, Berlin, Germany, 1987.
3. HARSANYI, J. C., and SELTEN, R., *A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information*, Management Science, Vol. 18, pp. 80–106, 1972.

4. GUTH, W., and KALKOFEN, B., *Unique Solutions for Strategic Games: Equilibrium Selection Based on Resistance Avoidance*, Springer Verlag, Berlin, Germany, 1989.
5. AUMANN, R. J., *Subjectivity and Correlation in Randomized Strategies*, Journal of Mathematical Economics, Vol. 1, pp. 67–96, 1974.
6. AUMANN, R. J., *Correlated Equilibrium as an Expression of Bayesian Rationality*, Econometrica, Vol. 55, pp. 1–18, 1987.
7. SELTEN, R., *Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games*, International Journal of Game Theory, Vol. 4, pp. 25–55, 1975.
8. MYERSON, R. B., *Refinements of the Nash Equilibrium Concept*, International Journal of Game Theory, Vol. 7, pp. 73–80, 1978.
9. KALAI, E., and SAMET, D., *Persistent Equilibria in Strategic Games*, International Journal of Game Theory, Vol. 13, pp. 129–144, 1984.
10. PETERS, H., and VRIEZE, K., *Nash Equilibria*, Report M91-02, University of Limburg, 1991.
11. NASH, J. F., *The Bargaining Problem*, Econometrica, Vol. 18, pp. 155–162, 1950.
12. HARSANYI, J. C., and SELTEN, R., *A General Theory of Equilibrium Selection in Games*, MIT Press, Cambridge, Massachusetts, 1988.
13. TEDESCHI, P., *Bargained-Correlated Equilibria*, Instituto di Economia Politica, Università Commerciale Luigi Bocconi, Milano, Italy, 1990.
14. AUMANN, R. J., *An Axiomatization of the Nontransferable Utility Value*, Econometrica, Vol. 53, pp. 599–612, 1985.
15. GRIESMER, J. H., HOFMANN, A. J., and ROBINSON, A., *On Symmetric Bimatrix Games*, Research Paper RC-959, IBM Watson Research Center, Yorktown Heights, New York, 1963.
16. JANSEN, M. J. M., POTTERS, J. A. M., and TIJS, S. H., *Symmetrizations of Two-Person Games*, Methods of Operations Research, Vol. 54, pp. 385–402, 1986.
17. HEUER, G. A., and MILLHAM, C. B., *On Nash Subsets and Mobility Chains in Bimatrix Games*, Naval Research Logistics Quarterly, Vol. 23, pp. 311–319, 1976.