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Ordinality of solutions of noncooperative games

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Abstract

In this paper we reformulate Mertens' definition of ordinality for solutions defined on the class of strategic form games. Using this reformulation, the relations between (strong) invariance, abr-invariance and ordinality can easily be described. This results in a short proof of a theorem of Mertens. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

In game theory, the invariance of solutions under certain 'irrelevant' changes of a game is an important issue. Well-known examples of this general notion of invariance are symmetry, anonymity and independence of irrelevant alternatives. The strength of these types of invariance in cooperative game theory, bargaining theory and social choice theory is clear since they are frequently being used in the axiomatic characterization of solutions in these areas.

1.1. Historical background

Thompson (1952) and Dalkey (1953) introduced a number of types of invariance of solutions for non-cooperative games. They described four ways to change

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the tree of an extensive form game without changing the strategic possibilities of the players and showed that the Nash equilibrium concept was invariant under these transformations. Kohlberg and Mertens (1986) further developed the ideas of Dalkey and Thompson. They introduced two more such 'irrelevant' ways to change an extensive form game and showed that two extensive form games can thus be transformed into one another if and only if these games have the same reduced normal form. In this context, the reduced normal form of an extensive form game is the game that is obtained by taking the normal form of the extensive form game and then successively deleting those pure strategies that are payoff-equivalent with some other mixed strategy. This observation led Kohlberg and Mertens to the following formulation of invariance: *solutions (for either extensive form games or normal form games) should depend only on the reduced normal form of a game*. Subsequently they carried this argument even further and stated that *one should treat mixed strategies just like pure strategies*. With this, they meant that one should not only identify pure strategies with payoff-equivalent mixed strategies, but simply identify any pair of payoff-equivalent mixed strategies.

However, these formulations of invariance still had some loose ends. First of all, it is not clear why a solution should *only* depend on the reduced game that results after identifying payoff-equivalent (mixed) strategies. There might be other, more rigorous but still natural, identifications of games. The second problem is related to the way Kohlberg and Mertens interpreted solutions. In their terminology, in contrast to earlier definitions, a solution is a rule that assigns to each game a *collection of subsets* of the (mixed) strategy space of the game. These subsets are called solution sets. Now, even if two games yield the same game after identification of certain strategies, it is not clear how the solution sets of the original two games should be compared with each other.

1.2. Mertens' approach

These two problems—(1) how to compare games and (2) how to compare solution sets of 'identical' games—were addressed in (Mertens, 1987). We will briefly discuss his ideas. For a more detailed motivation of his approach we refer to Mertens (1987).

Concerning the first (main) problem, he argued as follows. The only relevant information about a game is the choice correspondence, i.e., set of optimal choices of a player against each system of beliefs this player can have about what his opponents will do. Moreover, a solution should also be stable against any identification of (mixed) strategies that respects the structure of this choice correspondence (e.g., identification of payoff-equivalent strategies is supposed to 'respect the structure'). As an answer to the question what this set of optimal choices, the choice set, should actually be, Mertens writes:

> Still, there remain at least two different possible interpretations for the choice set: either the set of all best replies, or the set of all admissible (in the sense of individual decision theory) best replies. We will to some extent pursue both avenues in parallel, but when they start to diverge, we will abandon the first track, because the other is ultimately the only good one...

Although Mertens did indeed investigate a number of alternative definitions of the choice set, and even of admissibility, he also proved that his notion of (i,α)-admissibility eventually yields the strongest results. He consequently stated that this type of admissibility seemed to be preferable. In this paper, we will adopt this view and only analyse the case where the players only have independent beliefs (the 'i' in (i,α) refers to independence of beliefs) and only play α-admissible best replies to such beliefs.

As a next step to a solution of the first problem, he developed an intricate mechanism to compare the (i,α)-admissible best reply correspondences of two given games with each other. Since this mechanism is the main subject of analysis in this paper, we will give a loose description of how it works. Let us mention here that we will deviate slightly from the terminology used by Mertens (1987). This is mainly due to the fact that we felt that his use of the notions of 'correspondence' and 'isomorphism' was unconventional and therefore a bit confusing. Instead we will use the more neutral terms 'relations' and 'permissible relations', respectively.

Consider two games with the same player set and, for each player, a (for the moment arbitrary) relation between his (mixed) strategies in the one game and those in the other game. In our context, a relation is a subset of the product of the respective strategy spaces of a given player in these two games. We implicitly assume that every strategy in the one game is related to at least one strategy in the other game and vice versa. Now we focus our attention on one of the players. Given some (arbitrary) belief [2] this player has about what his opponents will do in the first game, the weights that this belief puts on the strategy profiles of his opponents can be transferred [3] to the strategy profiles of the second game using the relations between the respective strategy spaces of the opponents in these games. Now suppose that these relations are such that, for any belief this player can have in the first game, his choice set (i.e., his set of (i,α)-admissible best replies) in this game corresponds exactly, via *his* relation, to the choice set in the second game against any belief that can be obtained by the transfer of weights as mentioned above. Then, Mertens argued, this player would make exactly the same

[2] Formally a belief is a probability distribution over the set of mixed strategy profiles of the opponents.

[3] How this transfer works precisely is not easy to explain and elaborated in an example in Section 4. Formally, both strategy profiles should be the marginal distributions of a finite distribution over the relation.

choices in both games (according to his relation), provided that he believes that his opponents will make the same choices in both games (according to *their* relations). So, if these relations are such that the above holds for all players (in which case we will say that the relations are *permissible*), then any reasonable solution should assign 'the same' solution sets to both games.

This brings us to the second problem: given this identification of games using permissible relations between the strategy sets of two games, how do we identify solution sets a given solution assigns to the two games? The answer is simple, once we have these relations. A solution set in the first game is 'the same' as a solution set in the second game if every strategy profile in the first solution set is related to a strategy profile in the second solution set and vice versa. The solution is called *compatible* with these relations if for every solution set of the first game, and any pair of related strategy profiles for which the first profile is an element of this solution set, there is a solution set of the second game containing the second strategy profile, while it is also 'the same' as the first solution set. (Other definitions using this identification are also conceivable, e.g., we might say that the solution is compatible with these relations if *any* set of strategy profiles of the second game that is 'the same' as a solution set of the first game is also a solution set. However, this would for instance rule out point-valued solutions. Therefore, Mertens chose to use the above definition.) Finally, Mertens called a solution (i, α)-*ordinal* if it is compatible with any maximal permissible relation between two games.

1.3. Discussion

Apart from the development of this systematic approach to tackle the two previous problems, there are at least two other facets of the paper of Mertens that need to be mentioned. First, Mertens shared the conviction of Kohlberg and Mertens that any reasonable definition of invariance should treat mixed strategies just like pure strategies. As a result of this conviction he regarded the class of *strategic form games* (i.e., the class of games where each player has a polytope as strategy space and where every payoff function is affine w.r.t. each player) as the natural domain for a solution since this is the smallest class of games containing all normal form games that is closed under the identification of arbitrary payoff-equivalent strategies.

Secondly, Mertens designed the abstract definition of ordinality in order to capture the essence of our intuitive understanding of what the invariance of a solution should be. For this reason, he wanted his definition of ordinality not to depend on any additional (convex, or even topological) structure of the strategy spaces since such additions are not relevant for the general notion of invariance. Therefore, Mertens could not use the conventional definition of admissible best replies for his purposes, since this conventional definition involves completely mixed strategies, a notion that depends entirely on the convexity of the strategy spaces. It forced him to make a sharp distinction between the actual strategies the players can use and the beliefs of a player about what his opponents will do. To be sure that his definition of ordinality did indeed not depend on any structure of the strategy space, he maintained to make this distinction in every step he took. In his paper it for instance plays a prominent role in the definition of (i,α)-admissibility and (therefore also) in the definition of permissible relations.

The major drawback of this approach, although it is a very general and elegant way to deal with the problems at hand, is that it yields a definition of ordinality that is, to put it in Mertens' own words, 'highly abstract and apparently unmanageable'. There are, however, two results in his paper where Mertens tries to get insight in this highly abstract definition. First of all he analyzes the structure of permissible relations. To this end, he introduces an equivalence relation on the strategy space of each strategic form game, thus splitting the strategy space of the game under consideration into equivalence classes. Then he shows that a permissible relation between a game and itself induces a one-to-one and onto function between the equivalence classes of the game. Secondly, he analyzes the connection between ordinality and the notions of invariance [4] and admissible best reply invariance. [5] To be precise, he proves that a solution that is both invariant and admissible best reply invariant is also ordinal. Furthermore he argues that ordinality is equivalent with invariance and admissible best reply invariance for point-valued solutions as well as for solutions whose solution sets consist of unions of entire equivalence classes. Our example in Section 7, however, shows that an ordinal solution need neither be invariant nor admissible best reply invariant. So, for an arbitrary solution, being ordinal is certainly *not* equivalent with being both invariant and admissible best reply invariant. This only holds for very specific types of solutions as said above, e.g., when we only consider one-point solutions or equivalence-class-valued solutions. Furthermore, there are a number of frequently used solutions, such as persistency or strategic stability, that are not of such a type. All these together show that it is still not entirely clear what ordinality actually means for a number of well-known solutions, apart from the highly abstract definition provided by Mertens.

1.4. Aim of the paper

The main goal of this paper is to get more insight in what Mertens' notion of ordinality entails. Specifically, we derive an equivalent definition of ordinality in the context of strategic form games. The advantage of the equivalent definition is twofold. First, it is stated in terms of the familiar notion of admissibility instead of (i,α)-admissibility. Secondly, there is no need to consider the relation between

[4] A solution is invariant if it assigns the same solution sets to games that have the same reduced normal form.

[5] A solution is abr-invariant if it assigns the same solution sets to games that have the same admissible best reply correspondence.

belief systems induced by the relation between actual strategies. As a result of this, the 'alternative' definition of ordinality is easier to understand and to work with than the original one. For example, Theorem 2 of Mertens can now be proven fairly quick as we will show in Sections 6 and 8. The price we pay for this simplification is an additional convexity requirement on the relations between the strategy spaces of games.

Furthermore, we provide an example that shows to what extent ordinality and the related notions of invariance and abr-invariance differ from each other.

1.5. Organization of the paper

After some preliminary work in Section 2, our definition of ordinality is presented in Section 3. Since the original definition is still not very well understood, we go through the notions necessary for our definition one by one, and try to give some intuition for each of them.

Then, in Section 4, we give Mertens' original definition of (i,α)-ordinality and explain the link of each part of this definition with the corresponding part in our definition.

In Section 5 we show that both definitions are equivalent. (Although we work in the specific context of (a subclass of) strategic form games when proving the equivalence, this is not very restrictive. Mertens never meant his definition to be used outside the scope of strategic form games. His motivation for stating his definitions as general as possible was that he wanted to stress the fact that his definition did not depend on any particular structure.)

Finally, a new type of invariance, called *strong invariance* is introduced in Section 6. It is shown that this notion is equivalent with invariance and admissible best reply invariance. Also, in Sections 7 and 8, the relations between ordinality and invariance, admissible best reply invariance, and strong invariance are investigated. In Section 7, we present an example of a solution for strategic form games that is ordinal, but neither invariant nor admissible best reply invariant, and therefore certainly not strongly invariant. Section 8 provides a short proof of the fact that strong invariance implies ordinality. Thus, Sections 6 and 8 combine to a compact proof of Theorem 2 of Mertens (1987).

Notation For $n \in \mathbb{N} := \{1,2,\dots\}$, $\mathbb{R}^n$ is the vector space of n-tuples of real numbers and $\Delta_n := \{p \in \mathbb{R}^n | p_i \geq 0 \text{ for all } i \text{ and } \sum_i p_i = 1\}$. For a set $S \subset \mathbb{R}^n$, conv(S) is the convex hull of S. For an $s \in S$, δ_s is the function on S defined by

$$\delta_s(t) := \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases}.$$

If $C \subset \mathbb{R}^n$ is a convex set, relint(C) is the relative interior of C and ext(C) denotes the set of extreme points of C. For two sets S and T, proj$_S$: $S \times T \to S$ is the map defined by proj$_S(s,t) = s$. The map proj$_T$ is defined similarly.

2. Preliminaries

A *general form game* is a tuple $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where $N = \{1, 2, \ldots, n\}$ is the set of players, X_i is the *strategy set* of player i and $u_i : \Pi_{i \in N} X_i \to \mathbb{R}$ is the *payoff function* of player i. Such a game is also denoted by $\langle X, u \rangle$.

A *strategic form game* is a general form game for which each strategy set is a polytope [6] (contained in some Euklidean space) and each payoff function is multi-affine. Such a game is denoted by $\langle P, u \rangle$. The best reply correspondence of player j is denoted by $\beta_j : P \to P_j$.

Following Mertens (1987) we call a strategy $p_j \in P_j$ of a strategic form game $\langle P, u \rangle$ an *admissible best reply* of player j against $q \in P$ if there exists a sequence $(q^t)_{t \in \mathbb{N}}$ in relint(P) converging to q such that $p_j \in \beta_j(q^t)$ for all t. For a $q \in P$, $\beta_j^a(q)$ denotes the set of admissible best replies of player j against q and $\beta^a(q) := \Pi_i \beta_i^a(q)$.

A *solution* for general (strategic) form games is a map σ that assigns to each general (strategic) form game $G = \langle X, u \rangle$ a collection $\sigma(G)$ of subsets of X. The elements of $\sigma(G)$ are called *solution sets*.

3. Ordinality for strategic form games

In this section, we will give a reformulation of Mertens' original definition of (i, α)-ordinality for strategic form games. This definition will be entirely in terms of (well known) notions concerning strategic form games and therefore easier to understand than the more general one of Mertens. In Section 5, we will show that for strategic form games both definitions are equivalent.

Since ordinality is a type of invariance—roughly meaning that the solution sets of comparable games should be related—we first need to explain when two games with the same players are comparable and how solution sets of such games should be related. This will be elaborated in four steps.

(1) First we describe how to compare strategy profiles of two games $G = \langle P, u \rangle$ and $H = \langle Q, v \rangle$.

A non-empty subset $\rho = \Pi_i \rho_i$ of the product $P \times Q$ is called a *relation* between P and Q if, for any i, ρ_i is a subset of $P_i \times Q_i$ and $\text{proj}_{P_i}(\rho_i) = P_i$ and $\text{proj}_{Q_i}(\rho_i) = Q_i$.

The interpretation of '$(p_i, q_i) \in \rho_i$' is simply that the strategy $p_i \in P_i$ can be compared with the strategy $q_i \in Q_i$ and vice versa. Furthermore, any strategy in P_i can be compared with at least one strategy in Q_i and vice versa.

[6] The fact that the strategy spaces of strategic form games are polytopes is crucial for the equivalence of our definition with the one of Mertens. The equivalence hinges on a property, reflected in Lemma 6, that is very specific for polytopes.

(2) Next we explain when the two games G and H are comparable.

Clearly not every relation between P and Q is a sensible one since related profiles should in some sense be strategically equivalent. Therefore, we incorporate the admissible best replies into the definition of comparability of two games. Furthermore, we will consider only convex relations.

Definition 1 A convex relation ρ is called admissible-best-reply preserving (abr-preserving for short) if $p \in \beta^a(p')$, $q \in \beta^a(q')$ for all pairs $(p,q),(p',q') \in \rho$.

Definition 1 expresses the fact that ρ is supposed to respect the structure of the choice correspondence. After all, this is the kind of identification of games against which we want solutions to be stable.

If there exists at least one maximal (with respect to inclusion) abr-preserving convex relation between the (strategy sets of) two games, then these games are called *comparable*.

(3) Now we describe how the solution sets of two comparable games should be related.

Let ρ be a maximal abr-preserving convex relation between G and H. We say that a set $A \subset P$ is *related* to a set $B \subset Q$ if there exists a set $C \subset \rho$ with $\mathrm{proj}_P(C) = A$ and $\mathrm{proj}_Q(C) = B$.

This condition says the following: *two solution sets are related if any strategy profile in the one solution set can be compared with some strategy profile in the other set*.

(4) Finally, let σ be a solution. We say that σ is *compatible* with a maximal abr-preserving convex relation ρ between the strategic form games G and H if for every $A \in \sigma(G)$ and $(x,y) \in \rho$ with $x \in A$ there exists a $B \in \sigma(H)$ related to A that contains y.

This condition says the following: *if a solution set of the one game contains a strategy profile that can be compared with a strategy profile of the other game, then this other game has a related solution set containing this profile.*

Definition 2 A solution σ for strategic form games is called *ordinal* if σ is compatible with any maximal abr-preserving convex relation between two strategic form games. [7]

Summarizing this section: the ordinality of a solution σ for strategic form games can be checked in three steps:

1. take two (arbitrary) strategic form games $\langle P,u \rangle$ and $\langle Q,v \rangle$

[7] If ρ is a maximal permissible relation between two strategic form games, then $\rho' := \{(q,p) | (p,q) \in \rho\}$ is also a maximal permissible relation between the same two games. This shows that in the definition of a compatible solution, the roles of G and H, x and y and A and B are symmetric and may therefore be exchanged simultaneously. In other words: a solution is compatible with ρ iff it is compatible with ρ'. Hence ordinality is a symmetric notion.

2. determine all abr-preserving convex relations between P and Q; find the maximal elements [8] of the relations found in the foregoing step

3. check whether σ is compatible with all the relations found in step (2).

4. Mertens' definition of ordinality

In this section we give Mertens' original definition of ordinality. Although Mertens meant his definition to be used in the strategic form context, he defined it in a very general setting (because he wanted to stress the independence of his definition from any additional structure in the strategic form context). We will, for ease of exposition, stick to the general framework. Since the definition does need some preliminary work we subdivided this section into subsections, each subsection corresponding to one step in our own definition.

4.1. (Admissible best replies against) beliefs

First, we describe how Mertens defines, for a general form game, (admissible) best replies of a player against the belief of this player concerning the behavior of his opponents.

For a (strategy) set S, $\tilde{S}$ is the collection of functions $f\colon S \to [0,1]$ such that its *support* $\mathrm{supp}(f) := \{s \in S | f(s) \neq 0\}$ is a finite set and $\Sigma_{s \in \mathrm{supp}(f)}(s) = 1$. Usually an element of $\tilde{S}$ is called a (finite) *belief* over S. Note that $\tilde{S}$ is a subset of the normed [9] vector space consisting of all real-valued functions on S with finite support.

Following Mertens we suppose that player j of the game G

1. chooses an element [10] $f_{-j} \in \tilde{X}_{-j} := \Pi_{i \neq j} \tilde{X}_i$, where an $f_i \in \tilde{X}_i$ represents the belief of player j concerning the behavior of player i

2. chooses a strategy $x_j \in X_j$ maximizing his expected payoff given his beliefs summarized in f_{-j}.

[8] For two given games the number of maximal elements is finite; moreover these relations can be characterized using identifications of certain equivalence classes in the polytopes involved.

[9] For a finite belief $f, \|f\|_1 := \Sigma_{s \in \mathrm{supp}(f)} |f(s)|$.

[10] We will frequently abuse the notation $\tilde{X}$ when X is a product set $\Pi_i X_i$. In that case $\tilde{X}$ will be $\Pi_i \tilde{X}_i$ instead of $\Pi_i X_i^{\sim}$. Although the latter can also be used (Mertens), it yields a weaker notion of ordinality.

Note that the expected payoff to player j corresponding to a strategy x_j given the belief vector f_{-j} is equal to

$$U_j(x_j|f_{-j}) := \sum_{y_{-j} \in \mathrm{supp}(f_{-j})} \prod_{i \neq j} f_i(y_i) u_j(x_j|y_{-j}).$$

A strategy $x_j \in X_j$ is called a *best reply of player j* against $f \in \tilde{X} := \Pi_i \tilde{X}_i$—denoted as $x_j \in B_j(f)$—if

$$U_j(x_j|f_{-j}) \geq U_j(z_j|f_{-j}) \quad \text{for all } z_j \in X_j.$$

Mertens calls a strategy $x_j \in X_j$ an (i,α)-*admissible best reply* of player j against $f \in \tilde{X}$ if for any finite set $\Pi_i C_i \subset X$ there exists a sequence $(f^t)_{t \in \mathbb{N}}$ in $\tilde{X}$ converging to f (w.r.t. the $\| \|_1$-norm) such that for all t

1. x_j is a best reply against f^t

2. $C_i \subset \mathrm{supp}(f_i^t)$ for all $i \neq j$.

The set of (i,α)-admissible best replies of player j against f is denoted by $B_j^{i,\alpha}(f)$ and $B^{i,\alpha}(f) := \Pi_k B_k^{i,\alpha}(f)$.

Note that the second condition implies that the sequence $(f^t)_{t \in \mathbb{N}}$ can be chosen to have an arbitrarily large support. Compared with the definition of admissibility in the context of games in strategic form, this condition replaces the statement that the sequence should be completely mixed. In fact, in Theorem 1 we will show that for such games (i,α)-admissibility and admissibility are related in a very natural manner.

4.2. Mertens' original definition

Now we are prepared to formulate Mertens' original definition. This will be done by describing, for each of the four steps we used in Section 3, the way Mertens has coped with them.

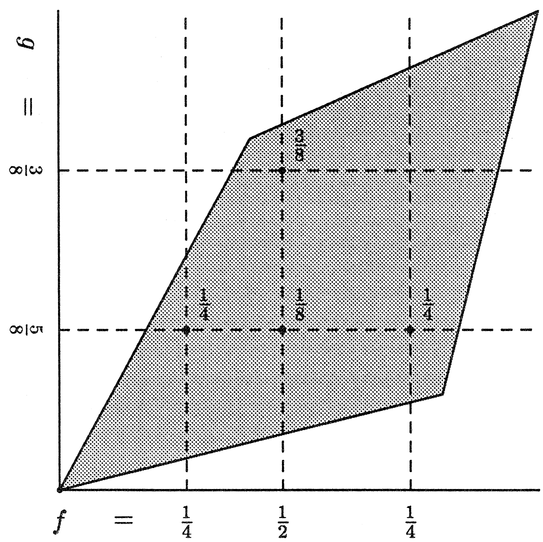
(1) Mertens also uses relations [11] as a tool to compare strategies and strategy profiles.

Since Mertens also wants to compare *beliefs* over strategies, he introduces, for a relation ρ between (strategy) sets S and T, the subset $\tilde{\rho}$ of $\tilde{S} \times \tilde{T}$ consisting of those pairs $(f, g) \in \tilde{S} \times \tilde{T}$ for which an $h \in \tilde{\rho}$ exists such that f and g are the marginals of h.

Intuitively, the relation $\tilde{\rho}$ links a pair of beliefs whenever it is possible to redistribute the weights of the first belief to the weights of the second belief via ρ.

[11] Mertens uses the term 'correspondence' instead of 'relation'.

Note that the convexity requirement we imposed in (2) of Section 3 is in fact a consequence of Mertens' requirement on $\tilde{\rho}$.



The following picture is to clarify the above construction. Imagine that the horizontal axis depicts the set S, while the vertical axis represents T. The shaded area indicates the relation ρ, a subset of the Cartesian product $S \times T$. Now the fractions in the shaded area indicate the probability distribution h. The weights are obviously supposed to be put on the dots next to it (so h is indeed carried by ρ). Clearly f and g are the marginals of h.

(2) In order to define the comparability of two general form games, say $G = \langle X,u \rangle$ and $H = \langle Y,v \rangle$, Mertens proposes to incorporate the admissible best replies into the definition of comparability of two games.

Following Mertens, we call a relation $\rho = \prod_i \rho_i$ between X and Y *permissible* if [12] for all $(x,y) \in \rho$ and all $(f,g) \in \tilde{\rho} = \prod_i \tilde{\rho}_i$

$$x \in B^{i,\alpha}(f) \Leftrightarrow y \in B^{i,\alpha}(g).$$

If ρ is moreover maximal with respect to inclusion, it is called *maximal permissible*.

Using this terminology the identification of games used by Mertens can be defined as follows. Two games are *comparable* if there exists at least one maximal permissible relation between (the strategy sets of) these games.

[12] To simplify notation, we write $(x,y) \in \Pi_i \rho_i$ instead of $((x_1, y_1),\dots,(x_n, y_n)) \in \Pi_i \rho_i$ and $(f,g) \in \Pi_i \tilde{\rho}_i$ instead of $((f_1,g_1),\dots,(f_n,g_n)) \in \prod_i \tilde{\rho}_i$.

(3) The way Mertens considers two solution sets of two comparable games as related given a maximal permissible relation is the same as described in (3) of Section 3.

(4) The compatibility of a maximal permissible relation is defined analogously to (4) of Section 3.

Definition 3 A solution σ for general form games is called (i,α)-*ordinal* (*ordinal* for short) if σ is compatible with any maximal permissible relation ρ between two general form games.

Summarizing, in order to check the ordinality of a solution σ for general form games the following three steps are needed:

1. take two (arbitrary) general form games $\langle X,u\rangle$ and $\langle Y, v\rangle$

2′. determine all permissible relations between X and Y; take the maximal ones

3. check whether σ is compatible with all the relations found in step 2′ (i.e., the maximal ones).

5. Equivalence of both definitions for strategic form games

In this section we show that for strategic form games (i,α)-ordinality [13] is equivalent with ordinality as defined in Section 3. Crucial in our proof is the fact that for a strategic form game (i,α)-admissibility can be redefined in terms of admissible best replies without using beliefs over relations. Theorem 1 is, in fact, the mathematical formalization of this fact.

In order to prove Theorem 1 we need to establish a link between beliefs over a polytope, say P, and elements of that polytope. To that end we define the affine (projection) mapping $\pi:\tilde{P}\to P$ by

$$\pi(f) := \sum_{p\in \mathrm{supp}(f)} f(p)p \quad \left(f\in\tilde{P}\right).$$

It is straightforward to show that this mapping is Lipschitz continuous.

Now let $\langle P,u\rangle$ be a strategic form game. If $f\in\tilde{P}$ represents a collection of beliefs, one for each player, then $\pi(f) := (\pi(f_1),\dots,\pi(f_n))$ is a strategy profile in P. The fact that, for all j, the payoff functions U_j and u_j are componentwise

[13] A solution for strategic form games is called (i,α)-ordinal if the requirement in Definition 3 holds for all strategic form games.

linear implies that the best replies against a collection f of beliefs coincide with the best replies against the strategy profile $\pi(f)$. Formally:

Lemma 1 *For all* $f \in \tilde{P}$ *and all* j, $B_j(f) = \beta_j(\pi(f))$.

The next theorem states that we have an even stronger result: the (i,α)-admissible best replies against a collection f of beliefs coincide with the admissible best replies against the strategy profile $\pi(f)$.

Theorem 1 *For all* $f \in \tilde{P}$, $B^{i,\alpha}(f) = \beta^{a}(\pi(f))$.

Proof. Let $f \in \tilde{P}$, $p \in P$ and $j \in N$. We will show that $p_j \in B_j^{i,\alpha}(f) \Leftrightarrow p_j \in \beta_j^{a}(\pi(f))$.

(a) Suppose that $p_j \in B_j^{i,\alpha}(f)$. Then we can find a sequence $(f^t)_{t \in \mathbb{N}}$ in $\tilde{P}$ converging to f such that for all t, (1) p_j is a best reply against f^t; and (2) $\text{ext}(P_i) \subset supp(f_i^t)$ for all $i \neq j$.

Now take $q^t := \pi(f^t)$. Then $q^t \in \text{relint}(P)$ and, by the continuity of π, $q^t \to \pi(f)$ as $t \to \infty$. Furthermore, by Lemma 1, $p_j \in B_j(f^t) = \beta_j(\pi(f^t)) = \beta_j(q^t)$ for all t. So p^j is an admissible best reply against $\pi(f)$.

(b) Suppose that $p_j \in \beta_j^{a}(\pi(f))$. Then we can find a sequence $(q^t)_{t \in \mathbb{N}}$ in $\text{relint}(P)$ converging to $\pi(f)$ such that $p_j \in \beta_j(q^t)$ for all t.

Let, for each i, C_i be a finite subset of P_i and $g_i \in \tilde{P}_i$ some function with $\text{supp}(g_i) = C_i$.

By Lemma 6 in Appendix A, there exist sequences $(r^t)_{t \in \mathbb{N}}$ in P and $(\varepsilon_t)_{t \in \mathbb{N}}$ in (0,1) such that $q^t = (1-\varepsilon_t)\,\pi(f) + \varepsilon_t r^t$ for all t and $\varepsilon_t \to 0$ as $t \to \infty$. Since $q^t \in \text{relint}(P)$, we can find an $s^t \in P$ and a number $\lambda_t \in (0,1)$ such that $q^t = (1-\lambda_t)\,\pi(g) + \lambda_t s^t$. Now we consider for $i \in N$ and $t \in \mathbb{N}$

$$f_i^t := (1-\varepsilon_t)\left[(1-\varepsilon_t) f_i + \varepsilon_t \delta_{r_i^t}\right] + \varepsilon_t\left[(1-\lambda_t) g_i + \lambda_t \delta_{s_i^t}\right] \in \tilde{P}_i.$$

Obviously, $f_i^t \to f_i$ as $t \to \infty$ and, for all t, $\pi(f_i^t) = q_i^t$ and $C_i = \text{supp}(g_i) \subset \text{supp}(f_i^t)$. Further,

$$p_j \in \beta_j(q^t) = \beta_j(\pi(f^t)) = B_j(f^t)$$

for all t. So p_j is an (i,α)-admissible best reply against f. □

Using the previous theorem we will now show that (i,α)-ordinality can also be defined in terms of admissible best replies without using beliefs over relations. For this, we first need to get some insight in the connection between a relation between two polytopes and the set of beliefs over it.

For a relation ρ between two polytopes P and Q, we introduce the subset

$$\pi(\tilde{\rho}) := \{(\pi(f),\pi(g)) | (f,g) \in \tilde{\rho}\}$$

of $P \times Q$. Since for a pair $(p, q) \in \rho$, $\delta_{(p,q)}$ is an element of $\widetilde{P \times Q}$ with support contained in ρ and δ_p and δ_q as marginals, the pair (δ_p, δ_q) is contained in $\tilde{\rho}$.

Hence $(p,q) = (\pi(\delta_p), \pi(\delta_q)) \in \pi(\tilde{\rho})$ and we have a proof of the first part of the following lemma. The proof of part (2) is quite straightforward and (3) is an immediate consequence of (2) and the fact that π is affine.

Lemma 2 *Let ρ be a relation between two polytopes P and Q. Then* (1) $\rho \subset \pi(\tilde{\rho})$, (2) $\tilde{\rho}$ *is convex,* (3) $\pi(\tilde{\rho})$ *is convex.*

With the help of this result we can show that the collection of projections of beliefs over a relation is just the convex hull of that relation.

Theorem 2 *If ρ is a relation between two polytopes P and Q, then*

$$\pi(\tilde{\rho}) = \mathrm{conv}(\rho).$$

Proof. (a) First we show that $\rho = \pi(\tilde{\rho})$ if ρ is convex. In view of (1) of the foregoing lemma, we only need to prove that $\pi(\tilde{\rho}) \subset \rho$. Let $(p,q) \in \pi(\tilde{\rho})$. Then there exists a pair $(f,g) \in \tilde{\rho}$ such that $\pi(f) = p$ and $\pi(g) = q$. Furthermore there is an $h \in \widetilde{P \times Q}$ with f and g as marginals and $\mathrm{supp}(h) \subset \rho$. Then

$$p = \pi(f) = \sum_{x \in supp(f)} f(x)x = \sum_{x \in supp(f)} \left(\sum_{\substack{y \in Q \\ (x,y) \in \mathrm{supp}(h)}} h(x,y) \right) x$$

$$= \sum_{(x,y) \in \mathrm{supp}(h)} h(x,y)x$$

and, similarly, $q = \sum_{(x,y) \in \mathrm{supp}(h)} h(x,y)y$. Hence, $(p,q) = \sum_{(x,y) \in \mathrm{supp}(h)} h(x,y)(x,y)$ is a convex combination of points contained in $\mathrm{supp}(h) \subset \rho$. So $(p,q) \in \mathrm{conv}(\rho) = \rho$.

(b) Since $\rho \subset \mathrm{conv}(\rho)$, we have $\pi(\tilde{\rho}) \subset \pi(\widetilde{\mathrm{conv}(\rho)})$. Hence, by part (a) and (1) and (3) of the foregoing lemma

$$\mathrm{conv}(\rho) \subset \pi(\tilde{\rho}) \subset \pi\left(\widetilde{\mathrm{conv}(\rho)}\right) = \mathrm{conv}(\rho).$$

This implies that $\pi(\tilde{\rho}) = \mathrm{conv}(\rho)$. □

Consider two strategic form games $\Gamma = \langle P,u \rangle$ and $\Gamma' = \langle Q,v \rangle$ with the same set of players. We will show that the permissibility of a relation between P and Q can be checked without using beliefs.

Lemma 3 *A relation ρ between P and Q is permissible if and only if for all $(p,q) \in \rho$ and all $(p',q') \in \pi(\tilde{\rho})$*

$$p \in \beta^a(p') \Leftrightarrow q \in \beta^a(q').$$

Proof. (a) Suppose that ρ is permissible and let $(p,q) \in \rho$ and $(p',q') \in \pi(\tilde{\rho})$. Then there is a pair $(f',g') \in \tilde{\rho}$ with $(\pi(f'),\pi(g')) = (p', q')$. Hence, by Theorem 1 and the permissibility of ρ,

$$p \in \beta^a(p') \Leftrightarrow p \in \beta^a(\pi(f')) \Leftrightarrow p \in B^{i,\alpha}(f') \Leftrightarrow q \in B^{i,\alpha}(g') \Leftrightarrow q \in \beta^a(q').$$

(b) Suppose that the equivalence described in the lemma holds for all $(p,q) \in \rho$ and all $(p',q') \in \pi(\tilde{\rho})$. Take a pair $(p,q) \in \rho$ and a pair $(f,g) \in \tilde{\rho}$. Then $(\pi(f),\pi(g)) \in \pi(\tilde{\rho})$. In combination with Theorem 1 this leads to

$$p \in B^{i,\alpha}(f) \Leftrightarrow p \in \beta^a(\pi(f)) \Leftrightarrow q \in \beta^a(\pi(g)) \Leftrightarrow q \in B^{i,\alpha}(g).$$

That is: ρ is permissible. □

For a convex relation ρ between P and Q, Theorem 2 implies the equality $\pi = \pi(\tilde{\rho})$. So Lemma 3 states that a convex relation between P and Q is permissible if and only if it is abr-preserving. Now we have a characterization of permissibility of a convex relation directly in terms of β^a.

So in order to show that β^a can be used as the basic notion for the definition of ordinality we only need to show that a maximal permissible relation is convex. This is done in

Lemma 4 *If ρ is a permissible relation between P and Q, then $\pi(\tilde{\rho})$ is also a permissible relation between P and Q.*

Proof. Let ρ be a permissible relation between P and Q. Since $\pi(\tilde{\rho})$ is a convex relation, $\pi(\tilde{\rho})$ is permissible if and only if the equivalence $p \in \beta^a(p') \Leftrightarrow q \in \beta^a(q')$ holds for all pairs $(p,q),(p',q') \in \pi(\tilde{\rho})$. By Lemma 3, however, we know that this equivalence is correct if $(p,q) \in \rho$. Now take pairs $(p,q),(p',q') \in \pi(\tilde{\rho})$ and suppose that $p \in \beta^{\mathrm{a}}(p')$.

Let j be fixed. Since $\pi(\tilde{\rho}_j) = \mathrm{conv}(\rho_j)$, we can find pairs $(p_j^1, q_j^1), \dots, (p_j^K, q_j^K)$ in ρ_j and positive numbers $\lambda_1, \dots, \lambda_K$ summing up to 1 such that $(p_j, q_j) = \sum_{k=1}^{K} \lambda_k \, (p_j^k, q_j^k)$.

Since $p_j \in \beta_j^a(p')$, there exists a sequence $(p^t)_{t \in \mathbb{N}}$ in relint(P) converging to p' such that $p_j \in \beta_j(p^t)$ for all t. By Lemma 8 in Appendix A, there exists a sequence $(q^t)_{t \in \mathbb{N}}$ in relint(Q) converging to q' such that $(p^t,q^t) \in \pi(\tilde{\rho})$ for all t.

Fix a $t \in \mathbb{N}$. Since $p_j \in \beta_j(p^t)$, we find that $p_j^k \in \beta_j(p^t) = \beta_j^a(p^t)$ for all $k = 1, 2, \dots, K$. So $q_j^k \in \beta_j^a(q^t) = \beta_j(q^t)$ for all k. Hence, $q_j \in \beta_j(q^t)$.

Since this relation holds for any t, q_j is an admissible best reply against q'. So $q \in \beta^a(q')$, because the foregoing holds for all j. By symmetry the proof is complete. □

Now let ρ be a maximal permissible relation between P and Q. Because $\rho \subset \pi(\tilde{\rho})$ and $\pi(\tilde{\rho})$ is permissible, we obtain the equality $\rho = \pi(\tilde{\rho})$. So ρ is convex.

So now we have shown that a relation ρ is maximal permissible if and only if it is maximal abr-preserving. Hence, a solution σ for strategic form games is compatible with all maximal permissible relations (i.e., σ is (i,α)-ordinal) if and only if it is compatible with all maximal abr-preserving convex relations (i.e., σ is ordinal). This establishes our main goal of this section: a solution is (i,α)-ordinal if and only if it is ordinal.

6. Some other invariance concepts for solutions of games in strategic form

In this section, we will investigate three invariance concepts for solutions of strategic form games. Two of these concepts—invariance and abr-invariance—were introduced by Mertens, while the third one, called strong invariance, is new. Our purpose is to show that a solution is strongly invariant if and only if it is an invariant and abr-invariant solution.

Within the framework of ordinality the strategy sets of the same player of two strategic form games (with the same set of players) are compared by means of a relation between these strategy sets. For a number of other invariance concepts these strategy sets are compared by means of an affine and surjective mapping between these sets. In order to make this precise, we need a number of concepts to be introduced now.

A *reduction* from a strategic form game $\Gamma = \langle P,u \rangle$ onto a strategic form game $\Gamma' = \langle Q,v \rangle$ is a mapping $f = (f_1, \dots, f_n)$ such that $f_i : P_i \to Q_i$ is affine and surjective for all i.

A reduction f from Γ onto Γ' *preserves payoffs* if $u_i = v_i \circ f$ for all i. In this situation we write $\Gamma \overset{\text{pay}}{\to}_f \Gamma'$. A reduction f from Γ onto Γ' *preserves admissible best replies* if for all $p,\ p' \in P$

$$p \in \beta^a(p') \Leftrightarrow f(p) \in \beta^a(f(p')).$$

In this situation we write $\Gamma \overset{\text{abr}}{\to}_f \Gamma'$.

Definition 4 A solution σ for strategic form games is called *invariant* if for all triplets $(\Gamma,\ \Gamma',\ f)$, with $\Gamma \overset{\text{pay}}{\to}_f \Gamma'$

(1) $\sigma(\Gamma') = \{f(S) | S \in \sigma(\Gamma)\}$,

(2) $f^{-1}(T) = \cup \{S \in \sigma(\Gamma) | f(S) = T\}$ for all $T \in \sigma(\Gamma')$.

If (1) and (2) are satisfied for all triplets $(\Gamma,\ \Gamma',\ f)$, with $\Gamma \overset{\text{abr}}{\to}_f \Gamma'$, then the solution is called *strongly invariant*.

A solution is abr-invariant if $\sigma(\Gamma) = \sigma(\Gamma^*)$ for all pairs $(\Gamma,\ \Gamma^*)$ with $\Gamma \overset{\text{abr}}{\to}_{\text{id}} \Gamma^*$. Here id is the identity between the (necessarily identical) strategy sets of Γ and Γ^*.

Since a reduction preserving payoffs also preserves admissible best replies, a strongly invariant solution is also invariant. Obviously, a strongly invariant solution is abr-invariant. Moreover

Theorem 5 *A solution σ for strategic form games is strongly invariant if and only if σ is invariant and abr-invariant.*

Proof. Suppose that σ is an invariant and abr-invariant solution for strategic form games. Let f be a reduction from a strategic form game $\Gamma = \langle P, u \rangle$ onto the strategic form game $\Gamma' = \langle Q, v \rangle$ that preserves admissible best replies.

We introduce the strategic form game $\Gamma^* = \langle P, w \rangle$, where $w_i := v_i \circ f$ is a multi-affine mapping on P. Then obviously f is a reduction from Γ^* onto Γ' that preserves payoffs.

Next we will show that $\Gamma \overset{\text{abr}}{\underset{\text{id}}{\rightarrow}} \Gamma^*$. Using the fact that the reduction f (from Γ onto Γ' and from Γ^* onto Γ') preserves admissible best replies, we find that for all $p,\ p' \in P$

$$p \in \beta_\Gamma^a(p') \Leftrightarrow f(p) \in \beta_{\Gamma'}^a(f(p')) \Leftrightarrow p \in \beta_{\Gamma^*}^a(p').$$

Hence $\sigma(\Gamma) = \sigma(\Gamma^*)$, which implies that $\sigma(\Gamma') = \{f(S) | S \in \sigma(\Gamma^*)\} = \{f(S) | S \in \sigma(\Gamma)\}$. So σ satisfies property (1) of Definition 4.

Finally, let $T \in \sigma(\Gamma')$ and suppose that $x \in f^{-1}(T)$. Since $\Gamma^* \overset{\text{pay}}{\underset{f}{\rightarrow}} \Gamma'$ and σ is invariant, there must be an $S \in \sigma(\Gamma^*) = \sigma(\Gamma)$ containing x such that $f(S) = T$. Hence, $f^{-1}(T) \subset \cup\{S \in \sigma(\Gamma) | f(S) = T\}$. Since the other inclusion is trivial, property (2) of Definition 4 has been proved. So σ is strongly invariant. □

7. An example

In this section, we give an example of an ordinal solution for strategic form games that is *not* (abr)-invariant and hence not strongly invariant.

Let σ be the solution that assigns to a strategic form game $\langle P,u \rangle$ the collection of all non-empty subsets of P. There is however one exception: σ assigns to the 2×2-bimatrix game

$$\Gamma^* = \begin{bmatrix} (0,0) & (0,0) \\ (0,0) & (0,0) \end{bmatrix}$$

the set $\{\Delta_2 \times \Delta_2\}$.

Next we consider the game

$$\Gamma^{**} = \begin{bmatrix} (1,1) & (1,1) \\ (1,1) & (1,1) \end{bmatrix}$$

and note that the map *id*: $\Delta_2 \times \Delta_2 \rightarrow \Delta_2 \times \Delta_2$ is a reduction from Γ^* onto Γ^{**} that preserves admissible best replies. Because every proper subset $S \neq \phi$ of $\Delta_2 \times \Delta_2$ is contained in $\sigma(\Gamma^{**})$ but not in $\sigma(\Gamma^*)$, σ is not abr-invariant.

Note that σ is also not invariant: if

$$\Gamma^{***} = \begin{bmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{bmatrix},$$

then $\Gamma^{***} \to_{\pi}^{\text{pay}} \Gamma^{*}$, but $\{\pi(T) | T \in \sigma(\Gamma^{***})\} = 2^{\Delta_2 \times \Delta_2} \setminus \{\phi\} \neq \sigma(\Gamma^{*})$. Hence, according to Theorem 5, it is certainly not strongly invariant. This shows that Theorem 2 of Mertens (1987) only works one way.

In order to show that σ is ordinal, let ρ be a maximal permissible relation between two strategic form games $\Gamma = \langle P,u \rangle$ and $\Gamma' = \langle Q,v \rangle$. We will prove that σ is compatible with ρ. We distinguish two cases.

(1) Suppose that $\Gamma' \neq \Gamma^{*}$. Let A be a solution set for Γ containing p^{*} and let q^{*} be an element of Q with $(p^{*}, q^{*}) \in \rho$. Since ρ is a relation, for each $p \in A$ we can choose an $h(p) \in Q$ with $(p, h(p)) \in \rho$. For convenience, we choose $h(p^{*}) = q^{*}$. Now let $C := \{(p, h(p)) | p \in A\} \subset \rho$. Then $B := \pi_Q(C)$ is a solution set for Γ' containing q^{*}. Since $\pi_P(C) = A$, σ is compatible with ρ.

(2) Suppose that $\Gamma' = \Gamma^{*}$. We will need three steps to prove the compatibility of σ.

(2a) First we will show that $p \in \beta_{\Gamma}^{a}(p')$ for all $p, p' \in P$. Since ρ is a relation, we can find $q, q' \in \Delta_2 \times \Delta_2$ such that (p,q) and (p', q') are elements of ρ. By the permissibility of ρ, $p \in \beta_{\Gamma}^{a}(p') \Leftrightarrow q \in \beta_{\Gamma^{*}}^{a}(q') = \Delta_2 \times \Delta_2$.

(2b) Let ρ^{*} be the convex relation $P \times (\Delta_2 \times \Delta_2)$. In view of part (2a), ρ^{*} is permissible. Since ρ is maximal, $\rho = \rho^{*}$.

(2c) Now let A be a solution set for Γ containing p^{*} and let q^{*} be an element of $\Delta_2 \times \Delta_2$ with $(p^{*}, q^{*}) \in \rho$. Then by taking $C := A \times (\Delta_2 \times \Delta_2) \subset \rho$ and $B := \Delta_2 \times \Delta_2 \in \sigma(\Gamma^{*})$, one can show that σ is compatible with ρ.

8. Strong invariance and ordinality

In this section, we investigate, for solutions of strategic form games, the relation between strong invariance and ordinality. We will use the reformulation of ordinality as described at the end of Section 5.

So, let $\Gamma = \langle P,u \rangle$ and $\Gamma' = \langle Q,v \rangle$ be two strategic form games and suppose that $(p^{*},q^{*}) \in \rho$, where ρ is an abr-preserving convex relation between P and Q.

Lemma 5 *There exists a strategic form game $\Gamma^{*} = \langle R,w \rangle$ such that $(p^{*},q^{*}) \in R \subset \rho$,*

$$\Gamma^{*} \overset{\text{abr}}{\to}_{\text{proj}_P} \Gamma$$

and

$$\Gamma^{*} \overset{\text{abr}}{\to}_{\text{proj}_Q} \Gamma'.$$

Proof. We start for a given j with the construction of R_j. Since ρ_j is a relation between P_j and Q_j, we can choose for each $d_j \in \text{ext}(P_j)$ a $d_j' \in Q_j$ with $(d_j,$

$d_j') \in \rho_j$ and similarly for each $e_j \in \mathrm{ext}(Q_j)$ an $e_j' \in P_j$ with $(e_j', e_j) \in \rho_j$. Let E_j be the set consisting of all pairs obtained in this way plus the pair (p_j^*, q_j^*).

Now we define $R_j := \mathrm{conv}(E_j)$ and $w_j := u_j \circ \mathrm{proj}_P$. By the convexity of ρ_j, $(p_j^*, q_j^*) \in R_j \subset \rho_j$. Since $\mathrm{proj}_P\colon R \to P$ is affine and surjective, $\Gamma^* \overset{\mathrm{pay}}{\to}_{\mathrm{proj}_P} \Gamma$.

So $\Gamma^* \overset{\mathrm{abr}}{\to}_{\mathrm{proj}_P} \Gamma$.

In order to prove that $\Gamma^* \overset{\mathrm{abr}}{\to}_{\mathrm{proj}_Q} \Gamma'$, take $(p, p'), (q, q') \in R \subset \rho$. Then the fact that $\Gamma^* \overset{\mathrm{abr}}{\to}_{\mathrm{proj}_P} \Gamma$ implies that $(p,q) \in \beta_{\Gamma^*}^a(p',q') \Leftrightarrow p \in \beta_\Gamma^a(p')$. Since ρ preserves admissible best replies, this implies $(p,q) \in \beta_{\Gamma^*}^a(p',q') \Leftrightarrow p \in \beta_\Gamma^a(p') \Leftrightarrow q \in \beta_{\Gamma'}^a(q') \Leftrightarrow \mathrm{proj}_Q(p,q) \in \beta_{\Gamma'}^a(\mathrm{proj}_Q(p', q'))$. □

Theorem 6 *A solution σ for strategic form games is strongly invariant if and only if σ is compatible with any abr-preserving convex relation between two strategic form games.*

Proof. (a) Suppose that σ is compatible with any abr-preserving convex relation between two strategic form games.

Take a reduction f from the strategic form game $\Gamma = \langle P, u \rangle$ onto the strategic form game $\Gamma' = \langle Q, v \rangle$ that preserves admissible best replies. Then $\rho := \{(p, f(p)) | p \in P\}$ is a convex relation between P and Q that preserves admissible best replies.

(a1) Let $S \in \sigma(\Gamma)$. Take a $p \in S$. Then $(p, f(p)) \in \rho$ and the compatibility of σ with respect to ρ implies the existence of a $B \in \sigma(\Gamma')$ containing $f(p)$ and a $C \subset \rho$ with $\mathrm{proj}_P(C) = S$ and $\mathrm{proj}_Q(C) = B$. Now the fact that $C \subset \rho$ implies that $B = f(S)$. Hence, $\{f(S) | S \in \sigma(\Gamma)\} \subset \sigma(\Gamma')$.

(a2) Let $T \in \sigma(\Gamma')$. In order to prove that $f^{-1}(T) \subset \cup\{S \in \sigma(\Gamma) | f(S) = T\}$, take a $p \in f^{-1}(T)$.

As we noticed in [6], the definition of compatibility is symmetric with respect to the two games involved. Hence, there exist an $A \in \sigma(\Gamma)$ containing p and a $C \subset \rho$ with $\mathrm{proj}_P(C) = A$ and $\mathrm{proj}_Q(C) = T$. Again we find that $T = f(A)$ which implies that $p \in \cup\{S \in \sigma(\Gamma) | f(S) = T\}$.

In view of the relations proved in (a1) and (a2), σ satisfies conditions (1) and (2) of Definition 4, i.e. σ is strongly invariant.

(b) Suppose that σ is strongly invariant. Suppose that ρ is an abr-preserving convex relation between the strategic form games Γ and Γ'. In order to show that σ is compatible with ρ, take an $A \in \sigma(\Gamma)$ and a profile $p \in A$ such that $(p,q) \in \rho$ for some strategy profile q for the game Γ'. By the foregoing lemma, there exist a game $\Gamma^* = \langle R, w \rangle$ with $(p,q) \in R \subset \rho$ and $\Gamma^* \overset{\mathrm{abr}}{\to}_{\mathrm{proj}_P} \Gamma$ and $\Gamma^* \overset{\mathrm{abr}}{\to}_{\mathrm{proj}_Q} \Gamma'$.

Since $p \in A$, $(p,q) \in \mathrm{proj}_P^{-1}(A)$. Since σ is strongly invariant and

$$\Gamma^* \overset{\text{abr}}{\rightarrow}_{\mathrm{proj}_P} \Gamma,$$

there is a $C \in \sigma(\Gamma^*)$ containing (p,q) with $f(C) = A$.

The strong invariance of σ and the fact that $\Gamma^* \overset{\text{abr}}{\rightarrow}_{\mathrm{proj}_P} \Gamma'$ then implies that $B := g(C) \in \sigma(\Gamma')$. Furthermore, $q = \mathrm{proj}_Q(p,q) \in \mathrm{proj}_Q(C) = B$.Hence σ is compatible with ρ. □

This result, together with the observations at the end of Section 5 and Theorem 5, yields

Corollary (Mertens) *A strongly invariant solution for strategic form games is ordinal.*

Appendix A

Lemma 6 *Let P be a polytope. If $p \in P$ and $(p^t)_{t \in \mathbb{N}}$ is a sequence in P converging to p, then there exist sequences $(r^t)_{t \in \mathbb{N}}$ in P and $(\varepsilon_t)_{t \in \mathbb{N}}$ in (0,1) such that $p^t = (1-\varepsilon_t)p + \varepsilon_t\ r^t$ for all t and*

$$\lim_{t \to \infty} \varepsilon_t = 0.$$

Proof. Without loss of generality we assume that $p^t \neq p$ for all t.

(a) Suppose that P is given by the linear system $Ax \geq b$. Then for $t \in \mathbb{N}$ we consider the number

$$\lambda_t := \max\left\{ \frac{b_i - (Ap^t)_i}{(Ap)_i - (Ap^t)_i} \middle| (Ap)_i > (Ap^t)_i \right\} \leq 0.$$

It is easy to show that $\lambda_t < 0$ for large t and that $r^t := (1-\lambda_t)p^t + \lambda_t p \in P$.

(b) Furthermore we choose for each t a natural number $i(t)$ such that $(Ap)_{i(t)} > (Ap^t)_{i(t)}$ and

$$\lambda_t = \frac{b_{i(t)} - (Ap^t)_{i(t)}}{(Ap)_{i(t)} - (Ap^t)_{i(t)}}.$$

Now we take $\varepsilon_t := 1/(1-\lambda_t) = [(Ap)_{i(t)} - (Ap^t)_{i(t)}]/[(Ap)_{i(t)} - b_{i(t)}] \in (0,1)$. Since there is a positive number η with $(Ap)_{i(t)} - b_{i(t)} \geq \eta$ for large t, the fact that $p^t \to p$ as $t \to \infty$ guarantees that $\varepsilon_t \to 0$ as $t \to \infty$. Finally, $(1-\varepsilon_t)p + \varepsilon_t r^t = p^t$ for all t. □

Lemma 7 *Let ρ be a relation between two polytopes P and Q. If $p \in$ relint(P), then there exists a $q \in$ relint(Q) such that $(p,q) \in \pi(\tilde{\rho})$.*

Proof. Since ρ is a relation between P and Q, for any $d \in \text{ext}(P)$ a $d' \in Q$ exists with $(d, d') \in \rho$ and similarly for each $e \in \text{ext}(Q)$ an $e' \in P$ with $(e', e) \in \rho$. Since $p \in \text{relint}(P)$, there are positive numbers λ_d, μ_e summing up to one such that

$$p = \sum_{d \in \text{ext}(P)} \lambda_d d + \sum_{e \in \text{ext}(Q)} \mu_e e'.$$

Now

$$q := \sum_{d \in ext(P)} \lambda_d d' + \sum_{e \in \text{ext}(Q)} \mu_e e \in \text{relint}(Q).$$

Furthermore, by Theorem 2,

$$(p,q) = \sum_{d \in \text{ext}(P)} \lambda_d (d,d') + \sum_{e \in \text{ext}(Q)} \mu_e (e',e) \in \text{conv}(\rho) = \pi(\tilde{\rho}). \qquad \square$$

Lemma 8 *Let ρ be a relation between two polytopes P and Q. If $(p,q) \in \pi(\tilde{\rho})$ and $(p^t)_{t \in \mathbb{N}}$ is a sequence in relint(P) converging to p, then there exists a sequence $(q^t)_{t \in \mathbb{N}}$ in relint(Q) converging to q such that $(p^t, q^t) \in \pi(\tilde{\rho})$ for all t.*

Proof. By Lemma 6 there exist sequences $(r^t)_{t \in \mathbb{N}}$ in P and $(\varepsilon_t)_{t \in \mathbb{N}}$ in $(0,1)$ such that $p^t = (1-\varepsilon_t) p + \varepsilon_t r^t$ for all t and

$$\lim_{t \to \infty} \varepsilon_t = 0.$$

Since $\pi(\tilde{\rho})$ is a relation between P and Q, we can find, for all t, an $s^t \in Q$ such that $(r^t, s^t) \in \pi(\tilde{\rho})$. By the foregoing lemma there is, for all t, a $\tilde{q}^t$ in relint(Q) such that $(p^t, \tilde{q}^t) \in \pi(\tilde{\rho})$ for all t. Now we take, for $t \in \mathbb{N}$,

$$q^t := (1-\varepsilon_t)\left[(1-\varepsilon_t) q + \varepsilon_t s^t\right] + \varepsilon_t \tilde{q}^t \in \text{relint}(Q).$$

In view of the boundedness of Q, $q^t \to q$ as $t \to \infty$. Furthermore, the equality

$$\begin{aligned}(p^t, q^t) &= \left((1-\varepsilon_t) p^t + \varepsilon_t p^t, q^t\right) \\ &= \left((1-\varepsilon_t)\left[(1-\varepsilon_t) p + \varepsilon_t r^t\right] + \varepsilon_t p^t, q^t\right) \\ &= \Big((1-\varepsilon_t)\left[(1-\varepsilon_t) p + \varepsilon_t r^t\right] \\ &\quad + \varepsilon_t p^t, (1-\varepsilon_t)\left[(1-\varepsilon_t) q + \varepsilon_t s^t\right] + \varepsilon_t \tilde{q}^t\Big) \\ &= (1-\varepsilon_t)^2 (p,q) + \varepsilon_t (1-\varepsilon_t)(r^t, s^t) + \varepsilon_t \left(p^t, \tilde{q}^t\right)\end{aligned}$$

shows that (p^t, q^t) is, for all t, a convex combination of elements of $\pi(\tilde{\rho})$. Hence, by (3) of Lemma 3, $(p^t,q^t) \in \pi(\tilde{\rho})$ for all t. □

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