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Theory and Methodology

The reduced form of a game

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Abstract

The goal of this paper is twofold. Firstly a short proof of the unicity of the reduced form of a normal form game is provided, using a technique to reduce a game originally introduced by Mertens. Secondly a direct combinatorial-geometric interpretation of the reduced form is described. This description is then used to derive an algorithm for the calculation of the reduced form of a game. © 1998 Elsevier Science B.V.

1. Introduction

Usually the strategy space of the normal form of an extensive form game contains a number of 'duplicate' pure strategies. Such duplicate pure strategies arise when in the extensive form game a player has to specify his choices in a part of the decision tree that is not reached in the eventual play of the game (due to his own choices earlier in the tree). In the normal form of the game these duplicates lead to the same pay off for every player, no matter what the other players do. This specific property of duplicate strategies is usually referred to as payoff equivalence.

Before solving a game using the normal form, we are inclined to delete all but one of such payoff-equivalent pure strategies from the normal form, since the resulting game is easier to handle while the strategic possibilities of the players are not changed. Thus also the eventual solution of the game should not be altered by this deletion process. The final result of such a deletion process is referred to as the semi-reduced normal form of the (extensive form) game. The unicity of this semi-reduced normal form up to changes in the names of the pure strategies is intuitively clear since the elimination process only involves the preservation of exactly one element of each collection of payoff-equivalent pure strategies.

Kohlberg and Mertens (1986) systematically investigated as to what further extent payoff-equivalent strategies can be deleted from a normal form game. They argued that the process of deletion should not only concern the removal of duplicate pure strategies, but also in their opinion pure strategies that are payoff equivalent to other (possibly mixed) strategies can be deleted from the normal form without harming the eventual solution. Thus they introduced the reduced normal form as the

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game 'where all pure strategies that are (payoff equivalent with) convex combinations of other pure strategies have been deleted'.

Although this description of the reduced normal form appeals to the intuition it also has some drawbacks. Unlike with the earlier definition of the semi-reduced normal form, it is in this case not immediately clear that the process will lead to a (in some sense) uniquely defined game. [1] The main problem being the fact that the deletion process is necessarily performed successively, one after the other. Thus, the resulting game depends on the order in which pure strategies are deleted and a game may have different reduced normal forms. It is even not immediately clear that two exhaustive sequences of deletions require the same number of deletions. People who have tried to construct a direct proof of the existence of a unique reduced normal form in some sense may have noticed that a rigorous proof can get quite involved indeed.

Nevertheless, it will be shown that all reduced normal forms of a normal form game are identical up to what is called 'the relabeling of pure strategies' in Mertens (1987). The proof is based on the technique used by Mertens in his mimeo for the identification of games. This technique is in fact an elegant way to capture the process of deletion in mathematical terms, as well as the 'relabeling' of pure strategies. Given this technique the proof becomes fairly straightforward and quite short.

The second problem with the description by Kohlberg and Mertens is that it is still not clear *which* game will eventually come out of the process of deletion of pure strategies, even if the existence of a unique reduced normal form of a normal form game is taken for granted.

Concerning this problem, most people automatically feel that, although two reduced normal forms of a game may be different on a formal level as discussed above, they should be very much alike, simply because it must be possible to predict *beforehand* which pure strategies are going to be deleted from the original game in the above process. We will show that this is indeed the case. Given an arbitrary (normal form) game, the results of the first part are used to derive a direct combinatorial-geometric interpretation of the reduced normal form of that game. On one hand, this interpretation may serve as an alternative definition of the reduced normal form. On the other hand it accurately describes which pure strategies are superfluous and which are not. More precisely, exactly one pure strategy payoff equivalent with a given pure strategy remains in the reduced form if and only if the collection of strategies that are payoff equivalent with the given pure strategy is a face of the strategy space of the player involved. All other pure strategies vanish completely. Finally an algorithm based on this combinatorial-geometric interpretation is given for the actual computation of the reduced normal form.

1.1. Content of the paper

Section 2 is, save the preliminaries, concerned with the relation between the deletion of a pure strategy of a given game Γ and the notion of reductions of Γ. In Section 3 the unicity of the reduced form of Γ is proved using the language developed in Section 2. At the end of Section 3 a criterion is provided to check whether a given game actually is the reduced form of the game Γ. In Section 4 this criterion is used to show that a specific game constructed directly from Γ equals its reduced form. In Section 5 it is shown that this construction can be performed in finite time.

Notation. For a finite set T, $|T|$ denotes the number of elements of T. For a convex set C, $\text{ext}(C)$ denotes the set of extreme points of C and for a set D, $\text{ch}(D)$ denotes the convex hull of D.

2. Reduction and deletion of pure strategies

In this section we will establish the relation between the deletion of a pure strategy and the notion of a reduction of a game. (From now on we will omit the prefix 'normal form', since we will

[1] This problem was pointed out to us by an anonymous referee and the editor in charge of a previous paper.

work exclusively in the normal form context.) The latter notion of a reduction is based on a technique introduced by Mertens (1987) and this will, in Section 3, turn out to be the ideal tool for stating precisely in what sense reduced forms are identical. In order to put our arguments for these assertions on a sound basis we first need some notation.

For a natural number n, $N := \{1, \dots, n\}$. An (n-person) *game* is a pair $\Gamma = \langle A, u\rangle$ such that $A := \prod_i A_i$ is a product of n non-empty, finite sets and $u = (u_i)_{i \in N}$ is an n-tuple of functions $u_i : A \to \mathbb{R}$. Here A_i is the set of *pure strategies* of player i and u_i is his *payoff function.*

As usual, a game Γ will be identified with its mixed extension. For this game, the *mixed strategies* of player i are the elements of the set $\Delta(A_i)$ of probability distributions on A_i. By abuse of notation we will identify a pure strategy $a \in A_i$ with the mixed strategy in $\Delta(A_i)$ that puts all weight on a. So, A_i will simply be viewed as a subset of $\Delta(A_i)$. Also the pure strategy profiles will be denoted by $a \in A$. In case confusion might occur we will write $a_i \in A_i$ instead of simply $a \in A_i$. For a (mixed) strategy profile $x = (x_i)_{i \in N} \in \Delta_A := \prod_j \Delta(A_j)$, the (expected) payoff function of player i is defined by $u_i(x) := \sum_{a \in A} \prod_j x_{ja_j} u_i(a)$. Two strategies y_j and z_j of player j are called *payoff equivalent* if for all i and all $x_{-j} \in \prod_{h \neq j} \Delta(A_h)$

$$u_i(x_{-j}|y_j) = u_i(x_{-j}|z_j).$$

In the displayed strategy profile $(x_{-j}|y_j) \in \Delta_A$ player j uses the strategy $y_j \in \Delta(A_j)$ and his opponents use the strategies in $x_{-j} \in \prod_{i \neq j} \Delta(A_i)$.

Now let $\Gamma = \langle A, u\rangle$ be a game. In order to formalize the process of (successive) deletions as described by Kohlberg and Mertens, let $b \in A_j$ be a pure strategy that is payoff equivalent with some other (mixed) strategy $z_j \in \Delta(A_j)$. Then the game $\Gamma' = \langle A', u'\rangle$ induced by the deletion of b can be defined as follows: first take

$$A'_i := \begin{cases} A_j \setminus \{b\} & \text{if } i = j, \\ A_i & \text{else} \end{cases}$$

and then define u'_i as the restriction of u_i to $A' := \prod_i A'_i$.

Thus we can give a formal definition of a reduced form of Γ as follows. First check whether there is a pure strategy of some player that is payoff equivalent with some other strategy. If there is no such strategy, Γ is called *reduced.* If there are such strategies, pick one and delete it. This yields a game Γ' as previously described. Repeat the process using Γ' instead of Γ, etc., until finally (after a finite number of steps) a reduced game results. Such a game is called *a reduced form* of Γ. The question now is in what sense reduced forms of Γ are equal to each other. In order to give a precise meaning to this sense, and to get a short proof, we need another way to represent the deletion of a pure strategy, namely by means of so-called reduction maps. This representation was introduced by Mertens (1987) and can also be found in van Damme (1994).

A game $\Gamma' = \langle B, v\rangle$ is called a *reduction* of the game $\Gamma = \langle A, u\rangle$ if there is a map $f = (f_i)_{i \in N}$ from Δ_A to Δ_B such that for every $i \in N$:
(1) $f_i : \Delta(A_i) \to \Delta(B_i)$ is affine and onto,
(2) $u_i = v_i \circ f$.
The function $v_i \circ f$ denotes the composition of v_i and f. In this situation f is called a reduction map from Γ to Γ'. Note that each f_i preserves payoff equivalence, i.e., for all x_i and y_i in $\Delta(A_i)$, x_i is payoff equivalent with y_i if and only if $f_i(x_i)$ is payoff equivalent with $f_i(y_i)$.

Roughly speaking, reducing a game captures both the idea of deletion of a pure strategy and the 'relabeling' of strategies. This specific combination makes it an ideal tool to tackle the problem at hand. However, first we need to establish the connection between deleting a single pure strategy and reducing a game.

Lemma 1. *Let Γ' be the game induced by the deletion of a pure strategy b of player j payoff equivalent with some other strategy $z_j \in \Delta(A_j)$. Then Γ' is a reduction of Γ.*

Proof. We have to show that there is a reduction map $f = (f_i)_{i \in N}$ from Γ to Γ'. Obviously for $i \neq j$ we can choose f_i to be the identity $id_i : \Delta(A_i) \to \Delta(A_i)$. For j, we define $f_j : \Delta(A_j) \to \Delta(A'_j)$ as follows. For $x_j \in \Delta(A_j)$ and $a \in A'_j$,

$$f_j(x_j)_a := x_{ja} + z_{ja}(1 - z_{jb})^{-1} x_{jb}.$$

Note that this definition makes sense, since the assumption that $z_j \neq b$ implies that $1 - z_{jb}$ is larger than zero. It is straightforward to check that $f_j(x_j) \in \Delta(A'_j)$ and that f_j is affine and onto. The fact that $u_i = u'_i \circ f$ can be seen as follows. First note that both u_i and $u'_i \circ f$ are multi-affine maps on Δ_A. So we only have to prove that they coincide on the set A of pure strategy profiles. Now note that, for each $a \in A$ with $a_j \neq b$, we have that $f_j(a_j) = a_j$. Furthermore, u'_i is the restriction of u_i to the set A' of pure strategy profiles $a \in A$ with $a_j \neq b$. So in this case it is clear that $u_i(a) = (u'_i \circ f)(a)$. Now take a pure strategy profile $c \in A$ with $c_j = b$. Then for each $i \in N$,

$$\begin{aligned}(u'_i \circ f)(c) &= \sum_{a \in A'} \prod_h f_h(c_h)_{a_h} u'_i(a) \\ &= \sum_{a_j \in A'_j} z_{ja_j}(1 - z_{jb})^{-1} u_i(c_{-j}|a_j) \\ &= (1 - z_{jb})^{-1}\{u_i(c_{-j}|z_j) - z_{jb}u_i(c_{-j}|b)\} \\ &= (1 - z_{jb})^{-1}\{u_i(c_{-j}|b) - z_{jb}u_i(c_{-j}|b)\} \\ &= u_i(c).\end{aligned}$$

The penultimate equality follows from the fact that b and z_j are payoff- equivalent. □

Now we can also capture successive deletions of pure strategies in terms of reduction maps. Suppose that a map f is a reduction map from a game Γ to a game Γ' and that g is a reduction map from Γ' to Γ''. Then it is easy to check that the composition $g \circ f$ of f and g is a reduction map from Γ to Γ''. Thus it follows from Lemma 1 that any game Γ' obtained from Γ by the successive (not necessarily exhaustive) deletion of pure strategies is a reduction of Γ. So, if we have a way to identify two reduced games that are both reductions of Γ, we also have a way to identify two reduced forms of Γ.

3. Uniqueness of the reduced form

After thus having translated the process of deletion of strategies in terms of reductions of Γ, we can again use reduction maps to describe in what way two reduced forms of Γ are identical. Two games $\Gamma^* = \langle B, v\rangle$ and $\Gamma^{**} = \langle C, w\rangle$ are called *isomorphic* if there is a reduction map f from Γ^* to Γ^{**} that is also one-to-one. It is equivalent to require that each f_i induces a one-to-one and onto function between B_i and C_i. The well-known phrase 'the reduced form is determined up to the relabeling of pure strategies' refers to the latter property of isomorphic games. In the proof of the isomorphy of two reduced forms of Γ we will need the following well-known lemma. For a proof we refer to Lemma 1 of Vermeulen and Jansen (1996).

Lemma 2. *Let f be an affine and onto map from a polytope P to a polytope Q. Then* ext(Q) *is a subset of f*(ext(P)).

Now suppose that f is a reduction map from Γ to Γ^* and that g is a reduction map from Γ to Γ^{**}.

Theorem 1. *If both Γ^* and Γ^{**} are reduced forms of Γ, then Γ^* and Γ^{**} are isomorphic.*

Proof. (a) In this part we will only use the fact that Γ^{**} is a reduced form of Γ. We will first construct a reduction map h from Γ^* to Γ^{**}. To this end, note that for a player i, f_i is an affine onto map from $\Delta(A_i)$ to $\Delta(B_i)$. So $B_i \subset f_i(A_i)$ by the previous lemma. Then there must exist a map $s_i : B_i \to A_i$ with $(f_i \circ s_i)(b) = b$ for all $b \in B_i$. Let $t_i : \Delta(B_i) \to \Delta(A_i)$ be the affine extension of s_i. Then it is easy to check that $f_i \circ t_i$ equals the identity id_i on $\Delta(B_i)$. So, if we write $t := (t_i)_{i \in N}$, then $f \circ t$ equals the identity on Δ_B.

Now define $h : \Delta_B \to \Delta_A$ as $h := (h_i)_{i \in N}$ with $h_i := g_i \circ t_i$. Clearly, h_i is an affine map. Furthermore, for all $i \in N$ and $y \in \Delta_B$,

$$\begin{aligned}v_i(y) &= v_i((f \circ t)(y)) = (v_i \circ f \circ t)(y) \\ &= (u_i \circ t)(y) = (w_i \circ g \circ t)(y) = (w_i \circ h)(y)\end{aligned}$$

because $f \circ t$ is the identity and $v_i \circ f = u_i = w_i \circ g$. So we only need to check that h_i is onto. To this end, take a pure strategy $c \in C_i$. Again by the previous lemma we know that there exists a pure strategy $a \in A_i$ with $g_i(a) = c$. Write $x_i := (t_i \circ f_i)(a) \in t_i(\Delta(B_i))$. Then

$$f_i(x_i) = (f_i \circ t_i \circ f_i)(a) = (id_i \circ f_i)(a) = f_i(a).$$

So, since the strategies $f_i(x_i)$ and $f_i(a)$ are identical, they are certainly payoff equivalent. Then x_i and a must also be payoff equivalent, since f_i preserves payoff equivalence. Thus, since g_i also preserves payoff equivalence, $g_i(x_i)$ and $g_i(a) = c$ are payoff equivalent. Hence, c and

$$(h_i \circ f_i)(a) = (g_i \circ t_i \circ f_i)(a) = g_i(x_i)$$

must also be payoff equivalent. However, since Γ^{**} is a reduced form of Γ, it is certainly a reduced game. So, since c is a pure strategy, we get that $(h_i \circ f_i)(a) = c$. Hence, $C_i \subset h_i(\Delta(A_i))$ and h_i must be onto since it is affine.
(b) Now since both Γ^* and Γ^{**} are reduced forms of Γ, part (a) yields a reduction map h from Γ^* to Γ^{**} and a reduction map h' from Γ^{**} to Γ^*. It is sufficient to prove that the onto map h is also one-to-one. By Lemma 2 we know that both $C_i \subset h_i(B_i)$ and $B_i \subset h'_i(C_i)$ hold. This however is only possible if $|C_i| = |B_i|$. Hence, h_i must be one-to-one. □

Conclusion. Thus we can interpret the reduced form of a game Γ as follows. First note that *any* two reduced games obtained from Γ by the exhaustive successive deletion of pure strategies are isomorphic, and isomorphy induces an equivalence relation on the class of all normal form games. So, all reduced games that can be obtained from Γ by successive deletions are contained in the same equivalence class. Hence, the reduced form of the game Γ can formally be seen as the equivalence class that contains all such reduced games. Practically speaking, any game in this equivalence class can be called the reduced form of Γ and then this game is said to be unique up to isomorphisms. Hence we have the following theorem.

Theorem 2. *A game Γ^* is the reduced form of Γ if and only if*
(1) Γ^* is a reduced game and
(2) Γ^* is a reduction of Γ.

4. Construction of the reduced form

In this section we will show that for any game $\Gamma = \langle A, u\rangle$ the reduced form Γ^* of Γ can be obtained directly from the game Γ by the identification of the strategies within certain payoff-equivalence classes. First we will formally define this game Γ^*. To that purpose consider the equivalence classes corresponding to the relation of payoff equivalency in the strategy space $\Delta(A_i)$ of player i. Let $\mathscr{E}_i$ denote the finite collection of those equivalence classes, say $E_1, \dots, E_S$, in $\Delta(A_i)$ that contain some pure strategy in A_i. Let $\mathscr{F}_i$ be the collection of those sets in $\mathscr{E}_i$ that are a face of $\Delta(A_i)$ and write $\mathscr{F} := \prod_{i\in N} \mathscr{F}_i$. Then, since for each player i and every $E = (E_h)_{h\in N} \in \mathscr{F}$ the payoff function u_i is constant on the subset $\prod_h E_h$ of A, we can define $u_i^* : \mathscr{F} \to \mathbb{R}$ by

$$u_i^*(E) := u_i\left(\prod_h E_h\right).$$

So at least $\Gamma^* := \langle \mathscr{F}, u^*\rangle$ is a well-defined object. However, in order to show that Γ^* is indeed a game, we need to know that $\mathscr{F}_i$ is not empty for each player i. In other words, we need to show that at least one of these equivalence classes is such a face. In order to prove this, define

$$B_i := \{a \in A_i \mid a \in E_s \text{ for some } E_s \in \mathscr{F}_i\}.$$

Furthermore, let E_s^* be the collection of pure strategies contained in the equivalence class E_s and, for a strategy $x_i \in \Delta(A_i)$ of player i, let $C(x_i) := \{a \in A_i \mid x_{ia} > 0\}$ be the *carrier* of x_i. First we need to show the following lemma.

Lemma 3. *If E_s is not a face of $\Delta(A_i)$, then there is a strategy $z(s)_i \in E_s$ with $C(z(s)_i) \subset B_i$.*

Proof. Suppose that we can prove the following proposition: for every subset $\mathscr{G}_i$ of $\mathscr{E}_i$ with $\mathscr{G}_i \cap \mathscr{F}_i = \emptyset$ we have: for every $E_s \in \mathscr{G}_i$ there is a strategy $z(s)_i \in E_s$ whose carrier has an empty intersection with every E_t^* for which $E_t \in \mathscr{G}_i$.
Then this is in particular true for $\mathscr{G}_i = \mathscr{E}_i \backslash \mathscr{F}_i$. Thus, for every $E_s \notin \mathscr{F}_i$ we get a strategy $z(s)_i \in E_s$ whose carrier has an empty intersection with every E_t^* for which $E_t \notin \mathscr{F}_i$, which means exactly that $C(z(s)_i)$ is a subset of B_i.

So, we have to show that the proposition $P(k)$: for every subset $\mathscr{G}_i$ of $\mathscr{E}_i$ with $|\mathscr{G}_i| = k$ and $\mathscr{G}_i \cap \mathscr{F}_i = \emptyset$ we have: for every $E_s \in \mathscr{G}_i$ there is a strategy $z(s)_i \in E_s$ whose carrier has an empty intersection with every E_t^* for which $E_t \in \mathscr{G}_i$ holds for every natural number k. We will prove this by induction over k. To do this we need the following.

(a) Assume that E_s is not a face of $\Delta(A_i)$. Then there is a strategy $y(s)_i \in E_s$ whose carrier is not contained in E_s^*. Now let $z(s)_i$ be the strategy obtained by normalizing the non-zero vector $\sum_{a \notin E_s^*} y(s)_{ia}a$. Then it is straightforward to check that

$$z(s)_i = \left[\sum_{a \notin E_s^*} y(s)_{ia}\right]^{-1} \left\{ y(s)_i - \sum_{a \in E_s^*} y(s)_{ia} a \right\}$$

is an affine combination of the payoff-equivalent strategies $y(s)_i$ and a with $a \in E_s^*$. Therefore $z(s)_i$ is also payoff equivalent with these strategies and hence $z(s)_i \in E_s$. Furthermore it is clear by construction that the carrier of $z(s)_i$ is a subset of the carrier of $y(s)_i$ and that it has an empty intersection with E_s^*.

(b) Now we can show $P(1)$ as follows. Note that $P(1)$ is equivalent with: for every $E_s \notin \mathscr{F}_i$ there is a strategy $z(s)_i \in E_s$ whose carrier has an empty intersection with E_s^*. This however is a direct consequence of part (a). So, we only need to prove the induction step. To this end, assume that $P(k)$ is true. We will show $P(k+1)$.

Assume that there is a subset $\mathscr{H}_i$ of $\mathscr{E}_i$ with $|\mathscr{H}_i| = k+1$ and $\mathscr{H}_i \cap \mathscr{F}_i$ is empty. Take an $E_s \in \mathscr{H}_i$. Since $k+1 \geq 2$, we can also take an $E_r \in \mathscr{H}_i$ with $r \neq s$. Then both $\mathscr{G}_i := \mathscr{H}_i \backslash \{E_r\}$ and $\mathscr{G}'_i := \mathscr{H}_i \backslash \{E_s\}$ satisfy the conditions of $P(k)$. So, there are strategies, let us call them $x(s)_i \in E_s$ and $x(r)_i \in E_r$, with

$$C(x(s)_i) \cap \bigcup_{E_t \in \mathscr{G}_i} E_t^* = \emptyset,$$

$$C(x(r)_i) \cap \bigcup_{E_t \in \mathscr{G}'_i} E_t^* = \emptyset.$$

If also $C(x(s)_i) \cap E_r^*$ is the empty set, then it is clear that the carrier of $x(s)_i$ has an empty intersection with every E_t^* for which $E_t \in \mathscr{H}_i$ and we have a proof of the statement for $k+1$. So, assume that this is not the case, which implies that $\sum_{a \in E_r^*} x(s)_{ia} > 0$. Define the strategy $y(s)_i \in E_s$ by

$$y(s)_i := x(s)_i + \sum_{a \in E_r^*} x(s)_{ia}[x(r)_i - a].$$

Now suppose that the carrier of $y(s)_i$ is a subset of E_s^*. Since $\sum_{a \in E_r^*} x(s)_{ia} > 0$ by assumption and the carrier of $x(r)_i$ has an empty intersection with E_r^*, it follows directly from the definition of $y(s)_i$ that the carrier of $x(r)_i$ is a subset of the carrier of $y(s)_i$. So, the carrier of $x(r)_i$ must also be a subset of E_s^*. This would imply that $x(r)_i$ is an element of E_s, which is impossible since $x(r)_i$ is an element of E_r and $E_r \neq E_s$.

Thus we know that the carrier of $y(s)_i \in E_s$ is not contained in E_s^*. So, we can apply the construction described in part (a) to $y(s)_i$ to obtain a strategy $z(s)_i \in E_s$ whose carrier is contained in the carrier of $y(s)_i$ and has an empty intersection with E_s^*. Now note that the carrier of $y(s)_i$ has an empty intersection with E_r^* and every E_t^* with $E_t \in \mathscr{G}_i \cap \mathscr{G}'_i = \mathscr{H}_i \backslash \{E_s, E_r\}$. Hence, the carrier of $z(s)_i \in E_s$ has an empty intersection with every E_t^* for which $E_t \in \mathscr{H}_i$. This concludes the proof of the induction step. □

Now it is easy to show that $\mathscr{F}_i$ is not empty. Suppose that it is empty. Then none of the elements of the non-empty set $\mathscr{E}_i$ is a face of $\Delta(A_i)$. So, we can take an $E_s \in \mathscr{E}_i$ that is not a face of $\Delta(A_i)$ and by Lemma 3 there is a strategy $z(s)_i \in E_s$ such that

$$C(z(s)_i) \cap A_i = C(z(s)_i) \cap \bigcup_{t \in S} E_t^* = \emptyset.$$

Since this is impossible, we know that $\mathscr{F}_i$ is not empty. Hence, $\Gamma^* := \langle \mathscr{F}, u^* \rangle$ is indeed a game. Finally we will show that Γ^* is indeed the reduced form of Γ. So, by Theorem 2, we need to prove that Γ^* is a reduction of Γ and that Γ^* is a reduced game. First we will prove that Γ^* is a reduction of Γ. It is convenient to split this proof into two parts. Consider the game $\Gamma' = \langle B, u' \rangle$ wherein $u'_i : \to \mathbb{R}$ is the restriction of u_i to the subset $B := \prod_{i \in N} B_i$ of A.

Lemma 4. *The game Γ' is a reduction of Γ.*

Proof. We will show that there exists a reduction map f from Γ to Γ'.

To this end, take a player i and a pure strategy $a \notin B_i$. Then the equivalence class in $\mathscr{E}_i$ that contains a is not an element of $\mathscr{F}_i$. So we can use Lemma 3 and choose a strategy $z(a)_i$ that is payoff equivalent with a while $C(z(a)_i) \subset B_i$. (Obviously we can coordinate these choices in such a way that $z(a)_i = z(b)_i$ whenever a is payoff equivalent with b, but this is not necessary for our argument.) We introduce the map $f_i : \Delta(A_i) \to \Delta(B_i)$ with, for every $x_i \in \Delta(A_i)$ and every $b \in B_i$,

$$f_i(x_i)_b := x_{ib} + \sum_{a \notin B_i} x_{ia} z(a)_{ib}.$$

Note that $f_i(x_i)$ is an element of $\Delta(B_i)$ because of the fact that $C(z(a)_i) \subset B_i$ for every $z(a)_i$. Furthermore, it is easily verified that f_i is affine and onto.

So, we only have to show that $u_i = u_i' \circ f$. The exact proof of this, although not difficult, is a bit messy. Therefore we will only present one step of the proof. Take $i, j \in N$ and $x \in \Delta_A$. For $b = (b_i)_{i \in N} \in B$, write $b_{-i} = (b_j)_{j \neq i}$ and $\prod_{b_{-i}} = \prod_{h \neq i} f_h(x_h)_{b_h}$. Then

$$\begin{aligned}(u_j' \circ f)(x) &= \sum_{b \in B} f_i(x_i)_{b_i} \prod_{b_{-i}} u_j'(b) \\ &= \sum_{b_{-i} \in B_{-i}} \prod_{b_{-i}} \sum_{b_i \in B_i} f_i(x_i)_{b_i} u_j(b_{-i}|b_i).\end{aligned}$$

Now take a fixed $b_{-i} \in B_{-i}$. Then we can compute that

$$\begin{aligned}&\sum_{b_i \in B_i} f_i(x_i)_{b_i} u_j(b_{-i}|b_i) \\ &= \sum_{b_i \in B_i} \Big[x_{ib_i} + \sum_{a \notin B_i} x_{ia} z(a)_{ib_i}\Big] u_j(b_{-i}|b_i) \\ &= \sum_{b_i \in B_i} x_{ib_i} u_j(b_{-i}|b_i) + \sum_{a \notin B_i} x_{ia} \sum_{b_i \in B_i} z(a)_{ib_i} u_j(b_{-i}|b_i) \\ &= \sum_{a \in B_i} x_{ia} u_j(b_{-i}|a) + \sum_{a \notin B_i} x_{ia} u_j(b_{-i}|\, z(a)_i) \\ &= \sum_{a \in B_i} x_{ia} u_j(b_{-i}|a) + \sum_{a \notin B_i} x_{ia} u_j(b_{-i}|a) = u_j(b_{-i}|x_i).\end{aligned}$$

The third equality follows from the fact that the carrier of $z(a)_i$ is contained in B_i and the fourth one from the payoff equivalence of a and $z(a)_i$. Now the substitution of the result of the second displayed computation into the first one yields

$$(u_j' \circ f)(x) = \sum_{b \in B} \prod_{b_{-i}} u_j(b_{-i}|\, x_i).$$

Thus, repetition of this computation eventually yields the equality $(u_j' \circ f)(x) = u_j(x)$. Hence, f is a reduction map from Γ to Γ'. □

Secondly, we have the following lemma.

Lemma 5. *The game Γ^* is a reduction of the game Γ'.*

Proof. Define for player i the map $\pi_i : \Delta(B_i) \to \Delta(\mathscr{F}_i)$ by, for $y_i \in \Delta(B_i)$ and $E \in \mathscr{F}_i$,

$$\pi_i(y_i)_E := \sum_{a \in E} y_{ia}$$

and $\pi := (\pi_i)_{i \in N}$. It is to be shown that π is a reduction map from Γ' to Γ^*. Evidently π_i is an affine map onto $\Delta(\mathscr{F}_i)$ for every player i. So it remains to be shown that π preserves pay offs. However, there is a simple argument why $u_j' = u_j^* \circ \pi$. Take a pure strategy profile $(b_i)_{i \in N} \in B$. Let $E(b_i) \in \mathscr{F}_i$ denote the unique equivalence class that contains b_i. Then

$$\begin{aligned}u_j^*(\pi_i(b_i)_{i \in N}) = u_j^*(E(b_i)_{i \in N}) = u_j\left(\prod_i E(b_i)\right) \\ = u_j((b_i)_{i \in N}) = u_j'((b_i)_{i \in N}).\end{aligned}$$

So, both $u_j^* \circ \pi$ and u_j' are multi-affine maps from Δ_B to $\Delta_{\mathscr{F}}$ that agree on the set of extreme points of Δ_B. Then they are necessarily identical, which completes the proof. □

The last two lemmas together show that $\pi \circ f$ is a reduction map from Γ to Γ^*. So, Γ^* is a reduction of Γ and, by Theorem 2, the only thing left to show is the following theorem.

Theorem 3. *Γ^* is a reduced game.*

Proof. Suppose that it is not. Then for some player j there must be a pure strategy $E \in \mathscr{F}_j$ that is payoff equivalent with a strategy $z_j \in \Delta(\mathscr{F}_j)$, while $z_j \neq E$. Furthermore, since $\pi \circ f$ is a reduction map from Γ to Γ^* it is certainly onto. In particular, there is a strategy $x_j \in \Delta(A_j)$ with $(\pi_j \circ f_j)(x_j) = z_j$. It is also easily checked that $(\pi_j \circ f_j)(b) = E$ given a pure strategy $b \in E$. Then the fact that $\pi_j \circ f_j$ preserves payoff equivalence implies that x_j is payoff equivalent with $b \in E$. So, x_j must also be an element of E. However, E itself is an element of $\mathscr{F}_j$, which means that E is a face of $\Delta(A_j)$. Therefore, x_j must be a convex combination of the pure strategies $b \in E^*$ in E. Since for all these strategies $b \in E^*$ it can easily be checked that $(\pi_j \circ f_j)(b) = E$, it follows from the affinity of $\pi_j \circ f_j$ that also $(\pi_j \circ f_j)(x_j) = E$. This however contradicts the assumption that $(\pi_j \circ f_j)(x_j) = z_j \neq E$. Hence, Γ^* must be a reduced game. □

5. Computational aspects

The actual computation of the reduced form $\Gamma^* = \langle \mathscr{F}, u^* \rangle$ of the game $\Gamma = \langle A, u \rangle$ can be done in finite time. To see this we will first argue that for each player i the set $\mathscr{F}_i$ can be computed in finite time. To this end, take a pure strategy $b \in A_i$ and let $E_s \in \mathscr{E}_i$ be the unique equivalence class that contains b. Write $A_{-i} := \prod_{h \neq i} A_h$. Then E_s is exactly the set of points $x \in \mathbb{R}^{A_i}$ that satisfy the finite system of linear (in)equalities

$$\text{for all } j \in N \text{ and for all } a_{-i} \in A_{-i}: \quad u_j(a_{-i}|x_i) = u_j(a_{-i}|b)$$
$$\text{for all } a \in A_i : x_{ia} \geqslant 0, \quad \sum_{a \in A_i} x_{ia} = 1.$$

Thus we have a polyhedral description of the polytope E_s and the set $\text{ext}(E_s)$ of extreme points of E_s can be calculated in finite time. Now note that

$$E_s \in \mathscr{F}_i \quad \text{if and only if} \quad \text{ext}(E_s) \subset A_i$$

and that the second condition can also be checked in finite time. So, since there are only finitely many elements $b \in A_i$ and every element of $\mathscr{E}_i$ occurs at least once in the above procedure when b ranges through A_i, we have a method to check within finite time exactly which elements of $\mathscr{E}_i$ are also elements of $\mathscr{F}_i$. Now select exactly one element in each set $\text{ext}(E_s)$ for which $E_s \in \mathscr{F}_i$. This selection yields a subset C_i of A_i. Write $C := \prod_i C_i$ and let v_i be the restriction of u_i to C. Since both the selection process and the evaluation of u_i on C can also be done in finite time, we can construct the game $\Gamma' = \langle C, v \rangle$ from Γ in finite time. Finally note that Γ' and Γ^* are isomorphic. Hence, Γ' is the reduced form of Γ and can be derived from Γ in a finite number of steps.

Example. For the strategy space Δ_4 of the second player of the 2×4-bimatrix game

$$\Gamma = \begin{bmatrix} 0,1 & 0,2 & 0,-1 & 0,0 \\ 0,-1 & 0,-2 & 0,1 & 0,0 \end{bmatrix},$$

there are four equivalence classes containing a pure strategy:

$$E_1 = \text{ch}\{e_1, \tfrac{2}{3}e_2 + \tfrac{1}{3}e_3, \tfrac{1}{2}e_2 + \tfrac{1}{2}e_4\},$$
$$E_2 = \{e_2\}, \quad E_3 = \{e_3\},$$
$$E_4 = \text{ch}\{e_4, \tfrac{1}{3}e_2 + \tfrac{2}{3}e_3, \tfrac{1}{2}e_1 + \tfrac{1}{2}e_3\}.$$

Clearly $\mathscr{F}_2 = \{E_2, E_3\}$. Hence, the reduced game of Γ is the game

$$\begin{bmatrix} 0,2 & 0,-1 \\ 0,-2 & 0,1 \end{bmatrix}$$

obtained by deleting the first and last column.

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